TWO-STEP NILPOTENT MONODROMY OF LOCAL SYSTEMS ON SPECIAL VARIETIES

by

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Abstract. — Let X be a smooth complex quasi-projective variety that is special in the sense of Campana. We prove that the monodromy group of any complex local system on X is virtually nilpotent of class at most 2. This result sharply refines a theorem of Cadorel, Yamanoi, and the second author.

To establish this result, we develop a deformation theory for certain local systems on quasi-compact Kähler manifolds by constructing universal deformations for such local systems. As a byproduct of our argument, we also show that a general fiber of the quasi-Albanese map of X is special, extending a result of Campana and Claudon from the projective to the quasi-projective setting.

Contents

1. Introduction

1.1. Main theorems. — The main goal of this paper is to prove the following.

Theorem A **(=Theorem 6.2).** — *Let X be a smooth complex quasi-projective variety. Suppose that X is special in the sense of Campana (for example, if its logarithmic Kodaira dimension $\bar{\kappa}(X) = 0$). Then for any linear representation $\varrho\colon \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$, the image $\varrho(\pi_1(X))$ is virtually nilpotent of class at most* 2.

We recall the following definition of nilpotent group.

Definition 1.1 **(Nilpotent group).** — A group G is *nilpotent* if it possesses a central series of finite length. This is defined as a series of normal subgroups

$$\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$$

such that $G_{i+1}/G_i \leq Z(G/G_i)$ for all $0 \leq i < n$. For a nilpotent group G, the minimal integer n for which a central series of length n exists is called the *nilpotency class* of G. Such a group is said to be *nilpotent of class* n. We say that G is n*-step nilpotent* if it is nilpotent of class less than or equal to n. In particular, 1-step nilpotent groups are abelian.

Previously, in [CDY25c], Cadorel, Yamanoi, and the second author proved that for any ϱ as in Theorem A, the image $\varrho(\pi_1(X))$ is virtually nilpotent. Moreover, they constructed examples of special varieties whose fundamental groups are 2-step nilpotent but not virtually abelian. Later, in [DY25], Yamanoi and the second author proved that for any representation $\tau : \pi_1(X) \to \mathrm{GL}_N(K)$ where K is a field of positive characteristic, the image $\tau(\pi_1(X))$ is virtually abelian. With these results in place, Theorem A completes the picture for the monodromies of local systems of arbitrary characteristic over special quasi-projective varieties.

One motivation for this paper is the following fundamental question concerning fundamental groups of complex algebraic varieties.

Question 1.2. — *Let X be a smooth complex quasi-projective variety. Assume that its fundamental group $\pi_1(X)$ is nilpotent. Is $\pi_1(X)$ necessarily 2-step nilpotent?*

Since every finitely generated nilpotent group is linear, Theorem A thus yields a positive answer to Question 1.2 for the class of special varieties.

We remark that Question 1.2 was originally asked by Campana [Cam95, Remarque 1.5] for compact Kähler manifolds, and only recently raised in its present form for quasi-projective varieties in his joint work with Aguilar [AAC25, Question 25]. Very recently, there have been some interesting works regarding Question 1.2 by Shimoji [Shi25], who provided a positive answer when the rank of $\pi_1(X)$ is at most seven. In [Rog25], Rogov proposed a strategy to study Question 1.2 using o-minimality, combined with Hain's higher Albanese maps [Hai87] and the unipotent variations of mixed Hodge structures of Hain–Zucker [HZ87].

In the present paper, our approach to proving Theorem A is essentially self-contained and does not rely on any of these preceding strategies or tools. A key novelty of our work is the establishment of a 2-*step phenomenon in non-abelian Hodge theory*. We begin by clarifying this terminology with the following definition.

***Definition 1.3* (Quasi-compact Kähler).** — A complex manifold X is called *quasi-compact Kähler* if it is a Zariski open subset of a compact Kähler manifold $\overline{X}$, which we refer to as a *compactification* of X. A holomorphic map between two quasi-compact Kähler manifolds is called *compactifiable* if it extends to a meromorphic map between their compactifications.

The following theorem is a simplified version of the so-called *2-step phenomenon*. For a more general statement, see Theorem 5.25.

***Theorem B* (=Theorem 6.1).** — *Let X, Y, and Z be quasi-compact Kähler manifolds. Let $f\colon Y \to X$ and $g\colon Z \to Y$ be compactifiable holomorphic maps. Assume that the images of these maps under the respective quasi-Albanese maps are points; that is,* $\mathrm{alb}_X(f(Y)) = \{\mathrm{pt}\}$ *and* $\mathrm{alb}_Y(g(Z)) = \{\mathrm{pt}\}$, *where* $\mathrm{alb}_X\colon X \to \mathrm{Alb}(X)$ *and* $\mathrm{alb}_Y\colon Y \to \mathrm{Alb}(Y)$ *denote the quasi-Albanese maps of X and Y, respectively.*

Then, for any $N \in \mathbb{Z}_{>0}$ and any representation $\tau\colon \pi_1(X,x) \to \mathrm{GL}_N(\mathbb{C})$ belonging to the irreducible component of the representation variety $\mathrm{Hom}(\pi_1(X,x), \mathrm{GL}_N(\mathbb{C}))$ *that contains the trivial representation, the pullback representation $(f\circ g)^*\tau : \pi_1(Z) \to \mathrm{GL}_N(\mathbb{C})$ is trivial. In particular, for any unipotent representation τ, the pullback $(f\circ g)^*\tau$ is trivial.*

A more conceptual formulation of Theorem B is given in Theorem C. We refer the reader to [Fuj24] for the precise definition and properties of quasi-Albanese maps.

To prove Theorem B, we develop a deformation theory for certain local systems on quasi-compact Kähler manifolds X. More precisely, inspired by the celebrated work of Eyssidieux–Simpson [ES11], we construct a *canonical and universal* connection for extendable unitary local systems on X. Since the precise statements are somewhat technical, we only state a special case here and refer the reader to Theorems 5.21 and 5.23 for the general formulations.

***Theorem C* ($\subset$Theorems 5.21, 5.23 and 5.25).** — *Let $\overline{X}$ be a compact Kähler manifold and let $D = \sum_{i=1}^m D_i$ be a simple normal crossing divisor on $\overline{X}$. Set $X := \overline{X} \setminus D$ and fix a base point $x \in X$. Let (V, ∇) be a trivial flat bundle of rank N on $\overline{X}$ and denote the associated endomorphism bundle by* $(E, \nabla) := (\mathrm{End}(V), \nabla)$.

Let Z be the affine subvariety of the affine space underlying the complex vector space $H^1(\overline{X}, E) \times H^0(D^{(1)}, E)$ defined by the following quasi-homogeneous equations for $(\alpha, \beta) \in H^1(\overline{X}, E) \times$

$H^0(D^{(1)}, E)$ of type $(1,2)$:

$$g(\beta) + \frac{1}{2}[\alpha, \alpha] = 0 \in H^2(\overline{X}, E),$$
$$[\iota^*\alpha, \beta] = 0 \in H^1(D^{(1)}, E),$$
$$\left[\iota_{ij}^*(\beta|_{D_i}), \iota_{ji}^*(\beta|_{D_j})\right] = 0 \in H^0(D_i \cap D_j, E) \quad \text{for any } 1 \le i < j \le m,$$

where $D^{(1)} := \bigsqcup_{i=1}^m D_i$, and $\iota\colon D^{(1)} \to \overline{X}$ is the natural morphism. Here, $g\colon H^0(D^{(1)}, E) \to H^2(X, E)$ denotes the Gysin morphism (cf. Definition 5.3), and $\iota_{ij}\colon D_i \cap D_j \to D_i$ is the inclusion map.

Then there exists an open ball $B \subset H^1(\overline{X}, E) \times H^0(D^{(1)}, E)$ centered at the origin O such that, over the analytic open subset $\mathcal{T} := Z \cap B$, there exists a canonical *and* analytic *family of logarithmic connections $\nabla_{(\alpha,\beta)}$ on V, parametrized by $(\alpha, \beta) \in \mathcal{T}$. These connections are flat over X_0 and satisfy the following property: The natural map defined by the monodromy representations of the flat connections $(V|_X, \nabla_{(\alpha,\beta)})$*

$$\begin{aligned} \mathcal{T} &\to \operatorname{Hom}(\pi_1(X, x), \mathrm{GL}_N(\mathbb{C})) \\ (\alpha, \beta) &\mapsto \operatorname{Mon}(\nabla_{(\alpha,\beta)}) \end{aligned}$$

is a biholomorphism onto an open neighborhood of the trivial representation in the representation variety $\operatorname{Hom}(\pi_1(X, x), \mathrm{GL}_N(\mathbb{C}))$.

Furthermore, we can regard $\nabla_{(\alpha,\beta)}$ as defining a representation

$$\varrho_{\mathcal{T}}\colon \pi_1(X) \to \mathrm{GL}_N(\mathcal{O}_{\mathcal{T},O}),$$

where $\mathcal{O}_{\mathcal{T},O}$ is the local ring of $\mathcal{T}$ at the origin O. Let $\mathcal{V}$ be the local system defined by the composition

$$\pi_1(X) \xrightarrow{\varrho_{\mathcal{T}}} \mathrm{GL}_N(\mathcal{O}_{\mathcal{T},O}) \to \mathrm{GL}_N(\mathcal{O}_{\mathcal{T},O}/\mathfrak{m}^3),$$

where $\mathfrak{m}$ denotes the maximal ideal of $\mathcal{O}_{\mathcal{T},O}$. This local system has 2-step nilpotent monodromy. Moreover, it satisfies the following property: if $f\colon Y \to X$ is a compactifiable holomorphic map from a quasi-Kähler manifold Y such that the pullback $f^*\mathcal{V}$ is trivial, then the pullback representation $f^*\tau\colon \pi_1(Y) \to \mathrm{GL}_N(\mathbb{C})$ is also trivial for any representation $\tau\colon \pi_1(X, x) \to \mathrm{GL}_N(\mathbb{C})$ belonging to the irreducible component of the representation variety $\operatorname{Hom}(\pi_1(X, x), \mathrm{GL}_N(\mathbb{C}))$ *that contains the trivial representation.*

We refer the reader to Theorem 5.21 for the precise formula for $\nabla_{(\alpha,\beta)}$. Notably, when X is compact, the above quasi-homogeneous equations reduce to the homogeneous quadratic equation $[\alpha, \alpha] = 0$. Theorem C provides an analytic viewpoint and a concrete construction of the equivalence of analytic germs between the representation variety $\operatorname{Hom}(\pi_1(X, x), \mathrm{GL}_N)$ and the quadratic cone in $H^1(X, E)$ defined by $[\alpha, \alpha] = 0$, a result originally established by Goldman and Millson [GM88] and Eyssidieux and Simpson [ES11]. The compact case of Theorem C is covered by the more general framework in Theorem 4.4. We refer the reader to Theorem 5.21 for the precise formula for $\nabla_{(\alpha,\beta)}$. Subsequently, we establish the *1-step phenomenon* for local systems on *compact Kähler* manifolds in Theorem 4.5. Together, these phenomena provide a conceptually unifying framework that allows us to reprove several known results in a uniform way:

(1) the theorems of Katzarkov [Kat97] and Leroy [Ler99] establishing the Shafarevich conjecture for compact Kähler manifolds with nilpotent fundamental groups (see Theorem 4.10);
(2) more generally, the theorems of Green–Griffiths–Katzarkov [GGK24] and Aguilar–Campana [AAC25] on the holomorphic convexity of the universal covers of quasi-compact Kähler manifolds with nilpotent fundamental groups and *proper* quasi-Albanese maps (see Theorem 5.26);
(3) Campana's theorem [Cam04] on the virtual abelianity of the monodromy of local systems over special smooth projective varieties (see Theorem 4.11); and
(4) a structure theorem for the linear Shafarevich morphism (see Theorem 4.12), established implicitly in the work of Eyssidieux, Katzarkov, Pantev, and Ramachandran on the linear Shafarevich conjecture [EKPR12], and later applied by Wang and the second author [DW24]

in their proof of the linear Kollár conjecture on the positivity of the holomorphic Euler characteristic for varieties with big fundamental groups.

In the course of proving Theorem C, we derive several ancillary results, including the construction of explicit zigzags of 1-*quasi-isomorphisms* between the differential graded Lie algebras (dglas) associated with E-valued logarithmic complexes and the cohomology complexes. We present this construction here due to its independent interest.

***Theorem D* (=Theorem 5.19). —** *Let $\overline{X}$ be a compact Kähler n-fold and let $D = \sum_{i=1}^m D_i$ be a simple normal crossing divisor on $\overline{X}$. Let (V, ∇) be a unitary flat vector bundle on $\overline{X}$ and denote the associated endomorphism bundle by $(E, \nabla) := (\mathrm{End}(V), \nabla)$. Then for the following maps*

$$\begin{tikzcd}
A^0(\overline{X}, D, E) \arrow[r, "\nabla"] & \cdots \arrow[r, "\nabla"] & A^{2n}(\overline{X}, D, E) \\
A^0(\overline{X}, D, E) \cap \ker \nabla' \arrow[r, "\nabla''"] \arrow[u, "\phi"] \arrow[d, "\psi"] & \cdots \arrow[r, "\nabla''"] \arrow[u, "\phi"] \arrow[d, "\psi"] & A^{2n}(\overline{X}, D, E) \cap \ker \nabla' \arrow[u, "\phi"] \arrow[d, "\psi"] \\
H^0(D^{(0)}, E) \arrow[r, "d"] & \cdots \arrow[r, "d"] & \oplus_{p+q=2n} H^q(D^{(p)}, E)
\end{tikzcd}$$

the vertical morphisms ϕ and ψ are both 1-equivalences (see Definition 3.2) connecting the three horizontal dgla complexes. Here:

- *ϕ denotes the inclusion map;*
- *$D^{(k)}$ denotes the disjoint union of all k-fold intersections*

$$D_I := D_{i_1} \cap \cdots \cap D_{i_k},$$

  where $I = (i_1, \ldots, i_k)$ is a multi-index satisfying $1 \leq i_1 < \cdots < i_k \leq m$, with the convention $D^{(0)} := \overline{X}$;
- *the differential d in the bottom chain complex is given by the Gysin morphism with suitable signs (see (5.15.1)), and the Lie brackets are defined in (5.17.1);*
- *for any $\eta \in A^k(\overline{X}, D, E)$, the map $\psi(\eta)$ is defined by*

$$\psi(\eta) := \Big((-1)^{|I|} \{ \mathcal{H}(\mathrm{Res}_{D_I} \eta) \} \Big)_I \in H^{k-|I|}(D_I, E),$$

  where I ranges over all ordered multi-indices contained in $\{1, \ldots, m\}$, and $\mathcal{H}$ denotes the harmonic projection *of the residue $\mathrm{Res}_{D_I} \eta \in A^{k-|I|}(D_I, \sum_{j \notin I} D_j \cap D_I, E)$.*

In particular, this refines previous results by Morgan [Mor78], Pridham [Pri16], and Lefèvre [Lef19], while avoiding the use of either Sullivan's minimal models or L_∞-algebras. To the best of our knowledge, this provides the first explicit construction of zigzags connecting the dglas of E-valued logarithmic complexes to their cohomology complexes, establishing a 1-quasi-equivalence.

In a forthcoming paper [CDHP26], we will extend Theorem C to a more general setting, with (V, ∇) be replaced by a semisimple flat bundle defined over X.

In the course of the proof of Theorem A, we need to establish a result in birational geometry within Campana's theory of special varieties and orbifolds. We state it here because of its independent interest; its proof is independent of the other results stated above.

***Theorem E* (=Theorem 7.15). —** *Let X be a smooth quasi-projective variety that is special. Then a general fiber of its quasi-Albanese map (which is dominant with connected general fibers) is also special.*

Note that Campana–Claudon [CC16] proved Theorem E in the case where X is projective.

Finally, we state a structure theorem for quasi-projective special varieties admitting a big local system, which is a consequence of the proof of Theorem A.

***Corollary F* (=Corollary 6.3). —** *Let X be a special smooth quasi-projective variety admitting a* big *representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ (cf. Definition 4.8). Then, after replacing X by a suitable finite étale cover, for a very general fiber Y of the quasi-Albanese map $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$, its quasi-Albanese map $\mathrm{alb}_Y : Y \to \mathrm{Alb}(Y)$ is birational.*

The logical dependencies among the main theorems are summarized in Figure 1.

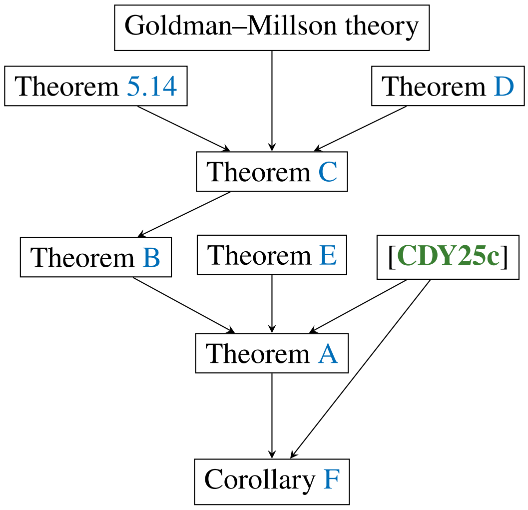

FIGURE 1. Logical dependencies between the main theorems

1.2. Idea of the proof of Theorem A. — In this subsection, we outline the main ideas of the proof of Theorem A, utilizing the deformation theory of local systems on quasi-compact Kähler manifolds presented above.

By [**CDY25c**], after replacing X by a finite étale cover, the Zariski closure of the image $\varrho(\pi_1(X))$ decomposes as a direct product $U \times T$, where U is a unipotent algebraic group and T is an algebraic torus (see Theorem 2.11). Hence, it suffices to prove that the image of any unipotent representation $\sigma : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ is 2-step nilpotent.

According to [**CDY25c**] (see Proposition 2.13), the quasi-Albanese map $a : X \to A$ is dominant with connected general fibers and is π_1-exact; that is, we have a short exact sequence

$$\pi_1(Y) \to \pi_1(X) \to \pi_1(A) \to 0, \tag{1.3.1}$$

where Y is a general fiber of a. By Theorem E, Y is also special, and by Proposition 2.13 again, the quasi-Albanese map $a_Y : Y \to A_Y$ of Y is also π_1-exact. Let F be a general fiber of a_Y.

By Theorem B, we have

$$\sigma\left([\pi_1(F) \to \pi_1(X)]\right) = 0.$$

The π_1-exactness of a_Y yields the short exact sequence

$$\pi_1(F) \to \pi_1(Y) \to \pi_1(A_Y) \to 0.$$

It follows that the representation

$$\iota^*\sigma : \pi_1(Y) \to \mathrm{GL}_N(\mathbb{C})$$

factors through $\pi_1(A_Y)$, thus has an abelian image. Here $\iota : Y \hookrightarrow X$ denotes the inclusion. Finally, by the π_1-exactness of a in (1.3.1), one can see that $\sigma(\pi_1(X))$ is 2-step nilpotent.

Notations and conventions

Throughout this paper, unless otherwise stated, we assume that (X, ω) is a compact Kähler manifold of dimension n and let $D = \sum_{i=1}^m D_i$ be a simple normal crossing divisor on X. We set $X_0 := X \setminus D$.

Throughout this paper, we adopt the convention that any multi-index $I = (i_1, \ldots, i_k)$ consists of *distinct entries* in $\{1, \ldots, m\}$. We say that I is *ordered* if $i_1 < \cdots < i_k$. For any $z = (z_1, \ldots, z_n)$, denote by $z_I := z_{i_1} \cdots z_{i_k}$. For such an index, we define the intersection

$$D_I := D_{i_1} \cap \cdots \cap D_{i_k}.$$

For any integer $k \geq 1$, let $D^{(k)}$ denote the disjoint union of all intersections indexed by ordered multi-indices of length k:

$$D^{(k)} := \bigsqcup_{\substack{|I|=k \\ I \text{ ordered}}} D_I.$$

In particular, $D^{(1)} = \bigsqcup_{i=1}^{m} D_i$.

- Let (V, ∇) be a flat bundle of rank N on X. Unless otherwise mentioned, we denote the associated endomorphism bundle by $(E, \nabla) := (\mathrm{End}(V), \nabla)$, and we write $H^k(X, E)$ for the de Rham cohomology of E with respect to ∇. Throughout this paper, except in § 4, we assume that the flat bundle (V, ∇) is unitary.
- For any unitary flat bundle (V, ∇) on X, we write $\nabla = \nabla' + \nabla''$ for the decomposition of ∇ into its $(1,0)$- and $(0,1)$-parts. We endow $E = \mathrm{End}(V)$ with the holomorphic vector bundle structure induced by the $(0,1)$-component ∇''.
- A local system $\mathcal{L}$ on $X \backslash D$ is called *extendable* if it extends to a local system on X.
- Let $\mathcal{A}^\bullet(X, D, E)$ be the sheaf of E-valued smooth differential forms with at most logarithmic poles along D. We write $A^k(X, E)$ for the space of smooth E-valued k-forms, $A^k(X, D, E)$ for those with at most logarithmic poles along D, $\mathscr{D}^k(X, E)$ for the space of E-valued currents of degree k, $A^k_{\mathrm{cts}}(X, E)$ for the space of continuous E-valued differential forms of degree k, and $\mathcal{H}^k(X, E)$ for the space of E-valued harmonic k-forms.
- For any $\eta \in A^k(X, D, V)$, we denote by $\boldsymbol{\nabla}\eta$ (resp. $\boldsymbol{\nabla}'\eta$ and $\boldsymbol{\nabla}''\eta$, noting the bold font) its covariant derivative in the sense of currents. In general, $\boldsymbol{\nabla}''\eta$ may contain nontrivial residue terms supported on D, whereas $\boldsymbol{\nabla}'\eta = \nabla'\eta$.
- Any element $\eta \in A^k(X, D, E)$ is called $\boldsymbol{\nabla}'$-exact if $\eta = \boldsymbol{\nabla}'\sigma$ for some $\sigma \in \mathscr{D}^{k-1}(X, E)$.
- For any closed form $\eta \in \mathscr{D}^k(X, E)$ with $\boldsymbol{\nabla}\eta = 0$, we let $\mathcal{H}(\eta)$ denote its harmonic projection, which is the unique harmonic representative of the class $\{\eta\} \in H^k(X, E)$. By abuse of notation, if $\alpha \in H^k(X, E)$ is a cohomology class, we also write $\mathcal{H}(\alpha)$ for its unique harmonic representative.
- Let $W_\bullet \mathcal{A}^k(X, D, E) \subset \mathcal{A}^k(X, D, E)$ be the increasing filtration induced by the number of pure logarithmic factors, and let $W_\bullet A^k(X, D, E)$ denote its space of global sections on X.
- Let $\mathcal{A}^\bullet_{X,D}(E)$ be the kernel of the residue morphism

$$\mathrm{Res}_D \colon W_1\mathcal{A}^\bullet(X, D, E) \longrightarrow \mathcal{A}^{\bullet-1}(D^{(1)}, E).$$

- Fix a base point $x \in X_0$. We denote the topological fundamental group of X_0 by $\pi_1(X_0, x)$. By abuse of notation, we usually omit the base point when the context is clear. Furthermore, we let $\mathfrak{g} := E_x$ denote the fiber of E at x, which carries a natural Lie algebra structure.
- Let $R(X_0, N) := \mathrm{Hom}(\pi_1(X_0), \mathrm{GL}_N(\mathbb{C}))$ denote the variety of representations of the fundamental group of X_0 into $\mathrm{GL}_N(\mathbb{C})$.
- A group G is said to be virtually P if it contains a subgroup of finite index that has property P.
- For brevity, we write dgla (resp. dglas) for differential graded Lie algebra(s).
- We denote by Art the category of Artin local $\mathbb{C}$-algebras, and Grp the category of groupoids. For any Artin local $\mathbb{C}$-algebra A, denote by $\mathfrak{m}$ its maximal ideal.
- For any $\varepsilon > 0$, let $\mathbb{B}^{p+q}_\varepsilon$ denote the ball of radius ε centered at the origin in $\mathbb{C}^{p+q}$.
- An *algebraic fiber space* $f \colon X \to Y$ between smooth projective varieties is a surjective morphism with connected fibers. For a divisor E on Y, we let f^*E denote the pullback divisor and $f^{-1}(E) := (f^*E)_{\mathrm{red}}$ denote its reduced support.
- For any quasi-compact Kähler manifold Y, we denote by $\mathrm{alb}_Y : Y \to \mathrm{Alb}(Y)$ its quasi-Albanese map.
- A birational morphism $f : X \to Y$ between smooth quasi-projective varieties is said to be *proper over a big open subset* if there exists an open subset $U \subset Y$ with $\mathrm{codim}_Y(Y \setminus U) \geq 2$ such that the restriction $f : f^{-1}(U) \to U$ is a proper morphism.

2. Technical Preliminaries

2.1. Special varieties and orbifolds. — Special varieties are introduced by Campana [Cam04, Cam11] in his remarkable program of classification of geometric orbifolds. We collect some facts about orbifolds from [Cam04, Cam11].

Recall that a (geometric) orbifold (X, D) is a pair consisting of a variety X and a $\mathbb{Q}$-divisor $D = \sum c_i P_i$, where the P_i are prime divisors and the coefficients $0 \le c_i \le 1$ are of the form $c_i = 1 - \frac{1}{m_i}$ for some $m_i \in \mathbb{Z}_{\ge 1} \cup \{+\infty\}$. We will say that (X, D) is *log smooth* (or just *smooth*) if X is non-singular and the support of D has simple normal crossings. The multiplicities of D along P_i are $m_i = m_{P_i}(D) = \frac{1}{1-c_i}$ so that

$$D = \sum c_i P_i = \sum (1 - \frac{1}{m_i}) P_i.$$

Note that $m_i \in [1, +\infty]$ and $m_i = +\infty$ coincides with the case $c_i = 1$ whilst $m_i = 1$ when $c_i = 0$.

***Definition 2.1* (Multiplicity and Orbifold base). —** Let $f : X \to Y$ be a surjective morphism of normal varieties with connected fibers. Let P be a prime divisor on X. We say P is *f-exceptional* if $\operatorname{codim}_Y f(P) \ge 2$. If P dominates Y, it is called *horizontal*; otherwise, it is *vertical*. If P is an f-vertical divisor that is not f-exceptional, then its image $f(P) = Q$ is a prime divisor on Y. Restricting to the regular loci $X^0 \subset X$ and $Y^0 \subset Y$, we can write the pullback of Q as

$$(f^0)^* Q = m(f, P) P + F$$

where F is an effective divisor whose support does not contain P. We define the integer $m(f, P)$ to be the *multiplicity of f along P*.

Suppose now that Y is smooth, and (X, D) is a smooth orbifold, where the boundary divisor is written as $D = \sum (1 - \frac{1}{m_P(D)}) P$ with multiplicities $m_P(D) \in \mathbb{Z}_{\ge 1} \cup \{\infty\}$. We define the *orbifold base* (or *orbifold divisor of f*) as the $\mathbb{Q}$-divisor on Y given by (cf. [CC16, Définition A.3]):

$$\Delta(f, D) = \sum_Q \left(1 - \frac{1}{m_f(Q)}\right) Q$$

where the sum runs over all prime divisors $Q \subset Y$, and the *multiplicity* of $\Delta(f, D)$ along Q is defined as:

$$m_Q\left(\Delta(f, D)\right) = m_f(Q) := \inf_{f(P)=Q} \left\{ m(f, P) \cdot m_P(D) \right\}.$$

***Definition 2.2* (Orbifold morphism). —** If (X, D_X) and (Y, D_Y) are log smooth orbifolds and $f : X \to Y$ is a morphism such that $m_i m_{P_i}(D_X) \ge m_Q(D_Y)$ for every divisor Q on Y and $f^*(Q) = \sum m_i P_i$ then we say that $f : (X, D_X) \to (Y, D_Y)$ is an *orbifold morphism*. If f is birational and $f_* D_X = D_Y$, then we say that f is *birational morphism* between orbifolds. A birational morphism that is also an orbifold morphism is called an *elementary birational orbifold morphism*.

Note that if $D_Y = \sum Q_i$ is a reduced divisor and $f : (X, D_X) \to (Y, D_Y)$ is an orbifold morphism, then $n_i = \infty$ and so $m_{P_j} = \infty$ for any P_j in the support of $f^*(Q_i)$, i.e. $\lfloor D_X \rfloor \ge \operatorname{Supp}(f^* D_Y)$. Note that it is not always the case that $f : (X, D) \to (Y, \Delta(f, D))$ is an orbifold morphism.

***Definition 2.3* ([Cam11, Définition 4.10]). —** Let $g : (Z, \Delta) \to Y$ be a fibration, with (Z, Δ) and Y smooth. We say that g is *neat* (relative to w) if there exists a diagram:

$$\begin{array}{ccc} (Z, \Delta) & \xrightarrow{w} & (Z', \Delta') \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ Y & \xrightarrow{v} & Y' \end{array}$$

in which:

- w is an orbifold morphism, v and w are bimeromorphic, Z', Y, Y' are smooth, and $w_*(\Delta) = \Delta'$;
- every g-exceptional divisor of Z is w-exceptional.

We say that g is *neat* if it is neat relative to a fibration g' as above.

We say that g is *neat and high* if it is neat, and if $g : (Z, \Delta) \to (Y, \Delta(g, \Delta))$ is an orbifold morphism.

We say that g is *strictly neat* if it is neat, and if, moreover, its orbifold base is smooth.

Definition 2.4 **([Cam11, Définition 5.7]). —** We say that f and f' are *birationally elementarily equivalent* if there exists a commutative diagram:

$$\begin{array}{ccc} (Y', D') & \xrightarrow{v} & (Y, D) \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ X' & \xrightarrow[u]{} & X \end{array}$$

in which u, v are bimeromorphic, u, v are holomorphic, and v is an an elementary birational orbifold morphism.

More generally, f and f' are equivalent (we then denote $f \sim f'$) if they belong to the equivalence relation generated by such diagrams.

Proposition 2.5 **([Cam11, Proposition 5.9]). —** *Consider the following commutative diagram*

$$\begin{array}{ccc} (Y', D') & \xrightarrow{v} & (Y, D) \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ X' & \xrightarrow{u} & X \end{array}$$

in which u and v are birational, and v is an orbifold morphism such that $v_*(D') = D$. Then we have

(i) $u_* (\Delta(f', D')) = \Delta(f, D)$;

(ii) $\kappa (X, \Delta(f, D)) \geqslant \kappa (X', \Delta(f', D'))$. □

Note that the strict inequality in Proposition 2.5.(ii) can indeed occur; thus, the Kodaira dimension of the orbifold base of a fibration is not, in general, a birational invariant. We therefore introduce the following definition (cf. [Cam11, Définitions 5.10 & 5.16]):

Definition 2.6 **(Canonical dimension). —** Let $f : (Y, D) \to X$ be a fibration where both the orbifold pair (Y, D) and the base X are smooth. We define the *canonical dimension* of the orbifold fibration f as:

$$\kappa(f, D) := \inf_{f' \sim f} \{\kappa(X', \Delta(f', D'))\}$$

where $f' : (Y', D') \to X'$ ranges over all models that are birationally equivalent to f, as in Definition 2.4. By definition, $\kappa(f, D)$ is a birational invariant of the fibration. If $\kappa(f, D) = \dim(X)$, we say that f is a *fibration of general type*.

By Proposition 2.5.(ii), we generally have the inequality:

$$\kappa(f, D) \leq \kappa(X, \Delta(f, D)).$$

Equality is achieved when the fibration is neat:

Lemma 2.7 **([Cam11, Corollaire 5.11.(3)]). —** *If the fibration f is neat, then:*

$$\kappa(f, D) = \kappa(X, \Delta(f, D)).$$

Proposition 2.8 **([Cam11, Proposition 4.10]). —** *Let $g' : (Z', \Delta') \to Y'$ be a morphism from a projective orbifold to a projective normal variety. Then there exists a commutative diagram:*

$$\begin{array}{ccc} (Z, \Delta) & \xrightarrow{w} & (Z', \Delta') \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle g'} \\ Y & \xrightarrow{v} & Y' \end{array}$$

such that w and v are both birational, (Z, Δ) and Y are both smooth, and g is strictly neat and high. □

***Definition 2.9* (Special variety).** — Let (X, D) be a smooth projective orbifold, then (X, D) is *special* if, for every elementary birational orbifold morphism $u : (X', D') \longrightarrow (X, D)$ and every neat fibration $f : (X', D') \longrightarrow Y$ with $\dim(Y) > 0$, the following inequality holds:

$$\kappa\,(Y, K_Y + \Delta(f, D')) < \dim(Y).$$

In other words, (X, D) is special if it does not admit any fibration of general type in the sense of Definition 2.6.

We will need the following result of Campana [**Cam11**, Théorème 10.1].

***Theorem 2.10* (Relative core map).** — *Let (X, D) be a smooth projective orbifold, and let $f\colon X \to S$ be an algebraic fiber space over a smooth projective variety S. Then there exists an almost holomorphic fibration $c\colon X \dashrightarrow C$ over S such that:*

(i) *for the orbifold base (C, D_C) induced by $c\colon (X, D) \dashrightarrow C$ (after possibly replacing c by a birational model as in Definition 2.4), the general fibers (C_s, D_{C_s}) of the induced map $(C, D_C) \to S$ are of general type; and*
(ii) *the general fibers (X_c, D_c) of $c\colon (X, D) \dashrightarrow C$ are special.*

The map $c\colon X \dashrightarrow C =: C(X, D/S)$ is called the relative core of (X, D) over S. Such a relative core map is unique up to birational equivalence.

Proof. — This follows easily from [**Cam11**, Théorème 10.1], which establishes the absolute case where $\dim(S) = 0$. We include the details for the reader.

By Proposition 2.8, we may replace $f\colon X \to S$ by a neat model. Let H be a sufficiently ample divisor on S in general position. Then f is clearly a fibration of general type for the pair $(X, D + f^*H)$. It follows that if $c\colon X \dashrightarrow C$ is the absolute core of $(X, D + f^*H)$, then f factors through c (see [**Cam11**, Corollaire 9.13]). Up to replacing c with a birationally equivalent model as in Definition 2.4, we may assume that it is a neat morphism. This yields the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "c"] \arrow[dr, "f"'] & C \arrow[d, "g"] \\ & S \end{tikzcd}$$

The general fibers of $c\colon (X, D) \to C$ agree with the general fibers of $c\colon (X, D + f^*H) \to C$. Since c is the absolute core of $(X, D + f^*H)$, these general fibers are special by [**Cam11**, Théorème 10.1]. This proves Item (ii).

For the orbifold base $\Delta(c, D + f^*H)$ of $c\colon (X, D + f^*H) \to C$, we note that the difference

$$\Delta(c, D + f^*H) - \Delta(c, D)$$

is an effective divisor entirely supported on $g^{-1}(H)$. Consequently, the general fibers of the induced fibrations $(C, \Delta(c, D + f^*H)) \to S$ and $(C, \Delta(c, D)) \to S$ coincide.

Since c is a neat morphism and is the absolute core of $(X, D+f^*H)$, its orbifold base $(C, \Delta(c, D+f^*H))$ is of general type. It is a standard property that the general fibers of an orbifold of general type over any base are again of general type. Hence, the general fibers of $(C, \Delta(c, D + f^*H)) \to S$, and thus the general fibers of $(C, \Delta(c, D)) \to S$, are of general type. This proves Item (i).

Finally, the uniqueness of c up to birational equivalence follows directly from [**Cam11**, Corollaire 9.13]. The theorem is proved. □

In [**CDY25c**], the authors conjectured that smooth special quasi-projective varieties have virtually nilpotent fundamental groups, and prove the following:

***Theorem 2.11* ([CDY25c, Theorem A]).** — *Let X be a smooth quasi-projective variety that is special and let $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ be a representation. Then after replacing X by a finite étale cover, the Zariski closure of the image $\varrho(\pi_1(X))$ decomposes as a direct product $U \times T$, where U is a unipotent algebraic group, and T is an algebraic torus. In particular, $\varrho(\pi_1(X))$ is a nilpotent group.* □

In this paper, we propose a refined version of the above nilpotency conjecture.

***Conjecture 2.12* (2-step nilpotency conjecture).** — *Let X be a smooth special quasi-projective variety. Then $\pi_1(X)$ is virtually nilpotent of class at most 2.*

We will need the following result on the π_1-*exactness* of quasi-Albanese maps of special varieties in the proof of Theorem A.

Proposition 2.13 **([CDY25c, Proposition 4.13]).** — *Let X be a smooth quasi-projective variety that is special. Then its quasi-Albanese map $a : X \to A$ is π_1-exact, i.e., a is a dominant morphism with connected general fibers, such that we have the following short exact sequence*

$$\pi_1(F, x) \to \pi_1(X, x) \to \pi_1(A, a(x)) \to 0,$$

where F is a general smooth fiber, and $x \in F$ is a base point. □

2.2. Hodge decomposition for currents and $\partial\bar{\partial}$-lemma. — Let X be a compact Kähler n-fold, and let (E, ∇) be a semisimple flat bundle on X. By Simpson [Sim92], there exists a harmonic metric h on E. This metric induces a decomposition $\nabla = \nabla' + \nabla''$ such that the following identities hold:

$$\Delta = 2\Delta' = 2\Delta''.$$

Here, the total Laplacian is defined as $\Delta := \nabla^*\nabla + \nabla\nabla^*$, and the partial Laplacians Δ' and Δ'' are defined analogously using ∇' and ∇'' respectively. Note that the splitting $\nabla = \nabla' + \nabla''$ generally differs from the decomposition of ∇ into its standard $(1, 0)$- and $(0, 1)$-parts, unless (E, ∇) is unitary.

Definition 2.14. — We denote by $\mathscr{D}^k(X, E)$ the space of E-valued currents of degree k on X. It is the topological dual of the space of smooth global forms of degree $2n - k$ with values in the dual bundle E^*:

$$\mathscr{D}^k(X, E) := \big(A^{2n-k}(X, E^*)\big)'.$$

The space of test forms $A^{2n-k}(X, E^*)$ is equipped with the natural Fréchet topology, i.e. the topology of uniform convergence of all derivatives.

Note that if D is a simple normal crossing divisor on X, then we have the inclusion

$$A^k(X, D, E) \subset \mathscr{D}^k(X, E).$$

Let G denote the Green operator associated with Δ'. We can extend the operators $\boldsymbol{\nabla}'$, $\boldsymbol{\nabla}''$, Δ', and G to act on the space of E-valued currents of degree k. Since X is compact Kähler, we have the following identities:

$$\begin{aligned} &[\boldsymbol{\nabla}', G] = 0, \quad [\boldsymbol{\nabla}'', G] = 0, \\ &[\boldsymbol{\nabla}'', (\boldsymbol{\nabla}')^*] = 0, \quad [\boldsymbol{\nabla}', (\boldsymbol{\nabla}'')^*] = 0, \\ &\alpha = \mathcal{H}\alpha + \Delta' G\alpha, \end{aligned} \tag{2.14.1}$$

where $\mathcal{H}$ is the harmonic projection.

Theorem 2.15 **(De Rham-Kodaira decomposition for currents).** — *Given a current $T \in \mathscr{D}^k(X, E)$, we have the following Hodge decomposition*

$$T = \mathcal{H}(T) + \Delta' G(T), \tag{2.15.1}$$

where G is the Green operator with respect to the Laplacian $\Delta' := [\boldsymbol{\nabla}', \boldsymbol{\nabla}'^*]$.

Proof. — For the proof, we refer to [dRK50] or [CP23, Section 3]. □

Lemma 2.16 **($\partial\bar{\partial}$-lemma).** — *Let $u \in \mathscr{D}^k(X, E)$ such that $\boldsymbol{\nabla} u = 0$ as currents on X. If $u = \boldsymbol{\nabla}' v$, or $u = \boldsymbol{\nabla}'' v$ on X for some current $v \in \mathscr{D}^{k-1}(X, E)$, then we have*

$$u = \boldsymbol{\nabla}'\boldsymbol{\nabla}''\sigma$$

for some $\sigma \in \mathscr{D}^{k-2}(X, E)$. In particular, if $k = 1$, then $u = 0$.

Proof. — We suppose that $u = \boldsymbol{\nabla}' v$ for some current v. By applying Theorem 2.15 to v, we have

$$v = v_0 + \boldsymbol{\nabla} v_1 + \boldsymbol{\nabla}^* v_2,$$

where v_0 is harmonic and $v_1, v_2 \in \mathscr{D}^{k-2}(X, E)$. Then we have

$$u = \boldsymbol{\nabla}' v = \boldsymbol{\nabla}'\boldsymbol{\nabla} v_1 + \boldsymbol{\nabla}'\boldsymbol{\nabla}^* v_2.$$

Then

$$\boldsymbol{\nabla}^*\boldsymbol{\nabla}' v_2 = u + \boldsymbol{\nabla}\boldsymbol{\nabla}' v_1 \in \ker \boldsymbol{\nabla}.$$

Therefore
$$\Delta(\nabla^*\nabla' v_2) = \nabla^*\nabla(\nabla^*\nabla' v_2) + \nabla\nabla^*(\nabla^*\nabla' v_2) = 0.$$
Then by the elliptic regulairty, $\nabla^*\nabla' v_2$ is smooth and harmonic. Then we have
$$\langle \nabla^*\nabla' v_2, \nabla^*\nabla' v_2\rangle = \langle \nabla' v_2, \nabla\nabla^*\nabla' v_2\rangle = 0.$$
Then $\nabla^*\nabla' v_2 = 0$ and
$$u = \nabla'\nabla v_1.$$
The lemma is thus proved. □

3. A quick tour of Goldman-Millson theory

In this section, we briefly recall some essential ingredients of (Deligne–)Goldman–Millson theory used in the proof of Theorems B and C.

3.1. Transformation groupoids. — Recall that a (transformation) *groupoid* is a small category in which all morphisms are isomorphisms. Let (X, G) be a transformation groupoid and let Y be a set upon which G acts. Then the transformation groupoid $(X \times Y, G)$, where G acts on $X \times Y$ by the diagonal action is a new groupoid which we denote by $(X, G) \bowtie Y$. If $\varphi : (X', G') \to (X, G)$ is a morphism of transformation groupoids and Y is a G-set, then there is a corresponding morphism of transformation groupoids
$$\varphi \bowtie Y : (X', G') \bowtie Y \to (X, G) \bowtie Y$$
where the G'-action on Y is induced from the G-action on Y by the homomorphism $\varphi : G' \to G$.
Lemma 3.1 **([GM88, Lemma 3.8]). —** *If $\varphi : (X', G') \to (X, G)$ is an equivalence of groupoids, then $\varphi \bowtie Y : (X', G') \bowtie Y \to (X, G) \bowtie Y$ is also an equivalence of groupoids.* □

Let us recall the definition of equivalence of categories [GM88, p. 52]. If $\mathscr{A}$ and $\mathscr{B}$ are categories, then a functor $F : \mathscr{A} \to \mathscr{B}$ is an *equivalence* if it satisfies three basic properties:

(a) F is surjective on isomorphism classes, i.e. the induced map $F_* : \operatorname{Iso}\mathscr{A} \to \operatorname{Iso}\mathscr{B}$ is surjective;
(b) F is full, i.e. for any two objects $x, y \in \operatorname{Obj}\mathscr{A}$, the map
$$F(x, y) : \operatorname{Hom}(x, y) \to \operatorname{Hom}(F(x), F(y))$$
is surjective;
(c) F is faithful, i.e. for any two objects $x, y \in \operatorname{Obj}\mathscr{A}$, the map
$$F(x, y) : \operatorname{Hom}(x, y) \to \operatorname{Hom}(F(x), F(y))$$
is injective.

In particular, an equivalence of categories $F : \mathscr{A} \to \mathscr{B}$ induces an bijection of sets
$$F_* : \ \operatorname{Iso}\mathscr{A} \to \ \operatorname{Iso}\mathscr{B}.$$

3.2. DGLA. — Let $\mathfrak{g}$ be a Lie algebra. A $\mathfrak{g}$-augmented differential graded Lie algebra is a triple $(L^\bullet, d, \varepsilon)$ where $(L^\bullet, d)$ is a differential graded Lie algebra (dgla for short) and $\varepsilon : L^0 \to \mathfrak{g}$ is a homomorphism of Lie algebras. A homomorphism of $\mathfrak{g}$-augmented differential graded Lie algebras $(L^\bullet, d, \varepsilon) \to (\overline{L^\bullet}, \bar{d}, \bar{\varepsilon})$ is a differential graded Lie algebra homomorphism $\varphi : (L^\bullet, d) \to (\overline{L^\bullet}, \bar{d})$ such that $\varepsilon \circ \varphi = \bar{\varepsilon}$. If $(L^\bullet, d, \varepsilon)$ is a $\mathfrak{g}$-augmented differential graded Lie algebra, then the augmentation extends trivially to a differential graded Lie algebra homomorphism $\varepsilon : L^\bullet \to \mathfrak{g}$ where $\mathfrak{g}$ is given the dgla structure with no nonzero elements of positive degree.
Definition 3.2 **(1-equivalence). —** A morphism of dglas $f : (L_0^\bullet, d_0) \to (L_1^\bullet, d_1)$ is called *1-equivalent* if the induced map on cohomology is an isomorphism in degrees 0 and 1, and a monomorphism in degree 2.

Definition 3.3 **((1-)quasi-isomorphism). —** Two $\mathfrak{g}$-augmented differential graded Lie algebras $(L^\bullet, d, \varepsilon)$ and $(\overline{L^\bullet}, \bar{d}, \bar{\varepsilon})$ are called *quasi-isomorphic* (resp. *1-quasi-isomorphic*) if there exists a zigzag of $\mathfrak{g}$-augmented dgla homomorphisms

$$L^\bullet = L_0 \longrightarrow L_1 \longleftarrow L_2 \longrightarrow \cdots \longleftarrow L_{m-1} \longrightarrow L_m = \overline{L^\bullet}$$

such that each homomorphism induces an isomorphism in cohomology (resp. is 1-equivalent).

A $\mathfrak{g}$-augmented differential graded Lie algebra is formal if it is quasi-isomorphic to its cohomology. Note that in this paper, the dgla constructed to study the deformation of local systems on quasi-compact Kähler manifolds is, in general, not formal. This makes the present work significantly more involved than [**GM88, ES11, EKPR12**], particularly regarding the analytic aspects of Hodge theory; consequently, we must establish new techniques in Hodge theory to study these dglas.

3.3. DGLA and deformation functor. — Given a dgla $C^\bullet$, there is a functor from Artin local $\mathbb{C}$-algebras to groupoids

$$\mathrm{Def}(C^\bullet) : \mathrm{Art} \to \mathrm{Grp}$$

as follows. Let A be an Artin local $\mathbb{C}$-algebra and $\mathfrak{m}$ be its maximal ideal. Then the objects of $\mathrm{Def}(C^\bullet, A)$ are defined by

$$\mathrm{Obj}\,\mathrm{Def}(C^\bullet, A) = \mathscr{C}(C^\bullet, A) := \{\alpha \in C^1 \otimes \mathfrak{m} \mid d\alpha + \frac{1}{2}[\alpha, \alpha] = 0\},$$

and the morphisms are

$$\mathrm{Hom}(\alpha, \beta) = \exp(C^0 \otimes \mathfrak{m}) := \{\lambda \in C^0 \otimes \mathfrak{m} \mid \exp(\lambda)\alpha = \beta\}$$

where $\exp(\lambda)\alpha$ denotes the gauge transformation $e^\lambda \circ \alpha \circ e^{-\lambda} - d(e^\lambda) \circ e^{-\lambda}$. We often write $[\mathscr{C}(C^\bullet, A)/\exp(C^0 \otimes \mathfrak{m})]$ for this transformation groupoids.

We shall encounter augmented ones. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$, viewed as a dgla in degree 0, and let $\varepsilon : L^\bullet \to \mathfrak{g}$ be an augmentation of the dgla $L^\bullet$. Let $(A, \mathfrak{m})$ be an Artin local $\mathbb{C}$-algebra. Then one defines the small groupoid $\mathrm{Def}(L^\bullet, \varepsilon, A)$ by:

(3.3.1)

$$\mathrm{Obj}\,\mathrm{Def}\,(L^\bullet, \varepsilon, A) = \mathscr{C}(L^\bullet, \varepsilon, A) := \left\{ (\alpha, e^r) \in L^1 \otimes \mathfrak{m} \times \exp(\mathfrak{g} \otimes \mathfrak{m}) \,\middle|\, d\alpha + \frac{1}{2}[\alpha, \alpha] = 0 \right\},$$

$$\mathrm{Hom}\,((\alpha, e^r), (\beta, e^s)) = \left\{ \lambda \in L^0 \otimes \mathfrak{m} \mid \exp(\lambda)\alpha = \beta, \exp(\varepsilon(\lambda)) \cdot e^r = e^s \right\}.$$

In the preceding notation, one has

$$\mathrm{Def}\,(L^\bullet, \varepsilon, A) = \mathrm{Def}\,(L^\bullet, A) \rtimes \exp(\mathfrak{g} \otimes \mathfrak{m})_\varepsilon, \tag{3.3.2}$$

where the ε subscript meaning that the gauge group acts via ε. This gives a covariant functor $\mathrm{Def}\,(L^\bullet, \varepsilon, -)$ on the category Art of $\mathbb{C}$-Artin local rings with values in the category of small groupoids.

A main result by Goldman-Millson in [**GM88**] is the following:

Theorem 3.4 **([GM88, Theorems 2.4 & 3.8]). —** *If $(C_1^\bullet, \varepsilon_1) \to (C_2^\bullet, \varepsilon_2)$ is a 1-quasi-isomorphism of $\mathfrak{g}$-augmented dgla, then it induces an equivalence of groupoids $\mathrm{Def}(C_1^\bullet, \varepsilon_1, A) \to \mathrm{Def}(C_2^\bullet, \varepsilon_2, A)$ for any local Artin $\mathbb{C}$-algebra A.* □

An analytic germ (S, s) defines a (covariant) functor of Artin rings

$$\begin{aligned} F_{S,s} : \mathrm{Art} &\longrightarrow \mathrm{Set} \\ A &\longmapsto \mathrm{Hom}(\widehat{O}_{S,s}, A), \end{aligned} \tag{3.4.1}$$

where the Hom denotes the set of $\mathbb{C}$-algebra homomorphisms. Such a functor is called *pro-representable*.

Theorem 3.5 **([GM88, Theorem 3.1]). —** *Two analytic germs (S_1, s_1) and (S_2, s_2) are analytically isomorphic if and only if the associated pro-representable functors F_{S_1,s_1}, F_{S_2,s_2} are isomorphic.* □

3.4. Application to the variety of representations. — Let X be a differential manifold. For any $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$, we define a functor by

$$\mathscr{R}_\varrho : \mathrm{Art} \longrightarrow \mathrm{Grp},$$

whose objects are

$$\mathrm{Obj}\,\mathscr{R}_\varrho(A) = R_\varrho(A) := \{\tilde{\rho} \in \mathrm{Hom}(\pi_1(X), \mathrm{GL}_N(A)) \mid \tilde{\rho} = \rho \bmod \mathfrak{m}\} \tag{3.5.1}$$

and the morphisms $G_A^0 := \exp(\mathfrak{g} \otimes \mathfrak{m})$, where $\mathfrak{g} := \mathrm{Lie}\,\mathrm{GL}_N(\mathbb{C})$. Here, $\mathfrak{g} \otimes \mathfrak{m}$ is a nilpotent Lie algebra, the Baker-Campbell-Hausdorff series define a nilpotent Lie group structure on the space G_A^0. If we consider the morphism $q : \mathrm{GL}_N(A) \to \mathrm{GL}_N(\mathbb{C})$ induced by $A \to A/\mathfrak{m} \simeq \mathbb{C}$, then its kernel is G_A^0. Note that there exists a natural map

$$q_* : \mathrm{Hom}(\pi_1(X), \mathrm{GL}_N(A)) \to \mathrm{Hom}(\pi_1(X), \mathrm{GL}_N(\mathbb{C}))$$

whose fibers are objects $\mathscr{R}_\varrho(A)$. Since G_A^0 preserves $R_\varrho(A)$, they defines a groupoid

$$[R_\varrho(A)/G_A^0] =: \mathscr{R}_\varrho(A).$$

We now consider a flat bundle (V, ∇) and let $(E, \nabla) := (\mathrm{End}(V), \nabla)$ be the associated endomorphism bundle. We consider a dgla

$$(L^\bullet, d) := (A^\bullet(X, E), \nabla).$$

Fix a base point $x \in X$ and an isomorphism $V_x \simeq \mathbb{C}^N$. Let

$$\varrho : \pi_1(X, x) \to \mathrm{GL}(V_x) \simeq \mathrm{GL}_N(\mathbb{C})$$

be the holonomy representation of ∇. In [GM88, Theorems 6.3 & 6.6], the following result is proved.

***Lemma 3.6.* —** *There is a natural equivalence of groupoids*

$$\begin{aligned} h : [\mathscr{C}(L^\bullet, A)/\exp(L^0 \otimes \mathfrak{m})] &\to [R_\varrho(A)/G_A^0] \\ \alpha &\mapsto \mathrm{hol}_x(\nabla + \alpha). \end{aligned}$$

□

By eliminating the superfluous isotropy, we have the following isomorphism between groupoids

$$\begin{aligned} [R_\varrho(A)/\{\mathrm{Id}\}] &\to [R_\varrho(A)/G_A^0] \rtimes G_A^0 \\ \tilde{\rho} &\mapsto (\tilde{\rho}, 1) \end{aligned}$$

Consider the functor

$$\begin{aligned} \mathscr{R}'_\varrho : \mathrm{Art} &\longrightarrow \mathrm{Grp}, \\ A &\mapsto [R_\varrho(A)/\{\mathrm{Id}\}] \end{aligned}$$

where

$$\mathrm{Obj}\,\mathscr{R}'_\varrho(A) = R_\varrho(A)$$

and the only morphisms in $\mathscr{R}'_\varrho(A)$ are the identity morphisms. Then we have

***Lemma 3.7.* —** *The functor $A \mapsto \mathscr{R}'_\varrho(A)$ it is prorepresented by the analytic germ $(R(X, N), \varrho)$.*

□

Therefore, by Lemmas 3.1 and 3.6, the morphism

$$h \rtimes G_A^0 : [\mathscr{C}(L^\bullet, A)/\exp(L^0 \otimes \mathfrak{m})] \rtimes G_A^0 \to [R_\varrho(A)/\{\mathrm{Id}\}] \tag{3.7.1}$$

is an equivalence of groupoids.

Let $\varepsilon_x : L^\bullet \to \mathrm{End}(V_x) = E_x$ defined by the evaluation $A^0(X, E) \to E_x$. It follows from (3.3.2) that there exists a natural equivalence

$$[\mathscr{C}(L^\bullet, A)/\exp(L^0 \otimes \mathfrak{m})] \rtimes G_A^0 \simeq \mathrm{Def}(L^\bullet, \varepsilon_x, A).$$

Lemma 3.8. — *The map*

$$h : [\mathscr{C}(L^\bullet, \varepsilon_x, A)/\exp(L^0 \otimes \mathfrak{m})] \to [R_\varrho(A)/\mathrm{Id}]$$
$$(\alpha, e^r) \mapsto \exp(-r)\mathrm{hol}_x(\nabla + \alpha)\exp(r)$$

is well-defined, and is an equivalence of groupoids.

Proof. — Recall that

$$\mathrm{Hom}\left((\alpha, e^r), (\beta, e^s)\right) = \left\{\lambda \in L^0 \otimes \mathfrak{m} \mid \exp(\lambda)\alpha = \beta, \exp(\varepsilon_x(\lambda)) \cdot e^r = e^s\right\}.$$

Note that we have

$$\nabla + \exp(\lambda)\alpha = \exp(\lambda) \circ (\nabla + \alpha) \circ \exp(-\lambda).$$

This implies that

$$\mathrm{hol}_x(\nabla + \exp(\lambda)\alpha) = \exp(\varepsilon_x(\lambda))\mathrm{hol}_x(\nabla + \alpha)\exp(\varepsilon_x(-\lambda)).$$

Therefore, we have

$$h(\beta, e^s) = \exp(-s)\exp(\varepsilon_x(\lambda))\mathrm{hol}_x(\nabla + \alpha)\exp(\varepsilon_x(-\lambda))\exp(s) = h(\alpha, e^r).$$

Hence h is well-defined. By the above arguments, it is an equivalence of groupoids. □

3.5. An extension to quasi-compact Kähler manifolds. — Let X be a compact Kähler n-fold and let D be a simple normal crossing divisor on X. Set $X_0 := X \setminus D$. Let (V, ∇) be a semisimple flat bundle on X, and we denote the associated endomorphism bundle by $(E, \nabla) := (\mathrm{End}(V), \nabla)$.

Consider a twisted logarithmic De Rham complex

$$(T^\bullet, d) := (A^\bullet(X, D, E), \nabla) = \left(A^0(X, D, E) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} A^{2n}(X, D, E)\right).$$

which constitutes a dgla on X. Let

$$(L^\bullet, d) := (A^0(X_0, E) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} A^{2\dim X}(X_0, E))$$

be the standard de Rham complex on X_0. Fix a base point $x \in X_0$ and let $\varepsilon_x : A^0(X_0, E) \to E_x$ be the augmentation map. The restriction of $(A^\bullet(X, D, E), \nabla)$ to X_0 induces a natural morphism of augmented dglas

$$a : (T^\bullet, \varepsilon_x, d) \to (L^\bullet, \varepsilon_x, d).$$

Let $\mathcal{V}$ be the local system associated to $(V, \nabla)|_{X_0}$. By [Del70, II, 6.10], there is a natural isomorphism between the logarithmic de Rham cohomology and the singular cohomology with coefficients in $\mathcal{V}$:

$$H^i(T^\bullet, d) \simeq H^i_{\mathrm{sing}}(X_0, \mathcal{V}).$$

This isomorphism factors through the map

$$H^i(T^\bullet, d) \to H^i(L^\bullet, d)$$

induced by the restriction morphism $a : T^\bullet|_{X_0} \to L^\bullet$. This implies that, the dgla morphism of a is a quasi-isomorphism. Hence, by Theorem 3.4 and Lemma 3.8, we have

Lemma 3.9. — *Let ϱ be the monodromy representation of $(V, \nabla)|_{X_0}$. We define*

$$R_\varrho(A) := \{\tilde{\rho} \in \mathrm{Hom}(\pi_1(X_0, x), \mathrm{GL}_N(A)) \mid \tilde{\rho} \equiv \varrho \pmod{\mathfrak{m}}\}$$

The holonomy map h:

$$h : [\mathscr{C}(T^\bullet, \varepsilon_x, A)/\exp(T^0 \otimes \mathfrak{m})] \to [R_\varrho(A)/\mathrm{Id}]$$
$$(\alpha, e^r) \mapsto \exp(-r)\mathrm{hol}_x(\nabla + \alpha)\exp(r)$$

is well-defined and induces an equivalence of groupoids. □

4. New perspectives on deformations of local systems on compact Kähler manifolds

In this section, we offer new perspectives on several foundational results established by Goldman–Millson [GM88], Eyssidieux–Simpson [ES11], and Eyssidieux et al. [EKPR12]. While this section is logically independent of the rest of the paper, it serves as a warm-up for understanding the quasi-compact case. Readers familiar with the standard compact theory may skip this section without loss of continuity.

4.1. Universal connection: analytic construction. — Let X be a compact Kähler manifold and let (V,∇) be a semisimple flat bundle on X, and we denote the associated endomorphism bundle by $(E,\nabla) := (\mathrm{End}(V),\nabla)$. Fix a basis $\eta_1,\ldots,\eta_p \in \mathcal{H}^1(X,E)$. For any multi-index $I=(i_1,\ldots,i_k)$ with $|I|\geq 2$, we define inductively

$$\gamma_I := \sum_{\substack{I=(I_1,I_2);\\ 1\leq |I_1|<|I_2|}} [\eta_{I_1},\eta_{I_2}] + \frac{1}{2}\sum_{\substack{I=(I_1,I_2);\\ |I_1|=|I_2|}} [\eta_{I_1},\eta_{I_2}] \tag{4.0.1}$$

$$\eta_I := -\nabla''^{*}G\gamma_I \in A^1(X,E), \tag{4.0.2}$$

where G is Green operator with respect to the Laplacian Δ'. Note that the decomposition $\nabla = \nabla' + \nabla''$ does not correspond to the standard splitting into $(1,0)$ and $(0,1)$ parts, unless (V,∇) is unitary. Instead, this decomposition arises from the harmonic bundle (see [Sim92]).

Lemma 4.1. — *For any $|I|\geq 2$, we have $\eta_I\in \mathrm{Im}\,\nabla'$, and any $|I|\geq 3$, $\gamma_I\in\mathrm{Im}\,\nabla'$.*

Proof. — We prove by induction on $|I|$, and assume that it holds for any $|I|\in\{2,\ldots,k-1\}$. Fix some I with $|I|=k$, we have

$$\begin{aligned}\nabla'\nabla'^{*}G\eta_I &= (1-\mathcal{H})\eta_I - \nabla'^{*}G\nabla'\eta_I\\ &= \eta_I + \nabla''^{*}G\nabla'\gamma_I = \eta_I.\end{aligned}$$

Together with (4.0.1), it follows that γ_I is also ∇'-exact when $|I|\geq 3$. The lemma is proved. □

Lemma 4.2. — *Set*

$$\sigma_k := -\Big(\sum_{|I|=k} z_I\eta_I\Big). \tag{4.2.1}$$

Let $\mathcal{Q}'$ be the quadratic cone in $\mathbb{C}^p$ defined by

$$\Big\{\Big[\sum_{i=1}^p z_i\eta_i, \sum_{i=1}^p z_i\eta_i\Big]\Big\} = 0 \in H^2(X,E),$$

Then for any $z\in\mathcal{Q}'$, we have

$$\nabla\sigma_k = \nabla''\sigma_k = -\sum_{i=1}^{k-1}\sigma_i\wedge\sigma_{k-i}. \tag{4.2.2}$$

Proof. — Set $\sigma_1(z) := \sum_{i=1}^m z_i\eta_i$. Note that

$$\nabla\sigma_2(z) := -\nabla\nabla''^{*}G\frac{1}{2}[\sigma_1,\sigma_1] = -(1-\mathcal{H})\frac{1}{2}\Big[\sum_{i=1}^p z_i\eta_i, \sum_{i=1}^p z_i\eta_i\Big].$$

By the assumption, for any $z\in\mathcal{Q}$, $[\sigma_1,\sigma_1]$ is cohomologous to zero. Hence we have

$$\nabla\sigma_2(z) + \frac{1}{2}[\sigma_1(z),\sigma_1(z)] = 0$$

for any $z\in\mathcal{Q}$.

We now make the induction for $\ell\geq 3$. Assume that (4.2.2) holds for $k=2,\ldots,\ell-1$. Then we have

$$\nabla''\Big(\sum_{|I|=\ell} z_I\gamma_I\Big) = \nabla''\Big(\sum_{i=1}^{\ell}\sigma_i\wedge\sigma_{\ell-i}\Big) = 0$$

by the induction for (4.2.2) and the Jacobian identity. Thus,

$$\begin{aligned}\nabla\sigma_\ell(z) &= -\sum_{|I|=\ell} z_I \nabla\nabla''^* G\gamma_I \\ &= -(1-\mathcal{H})(\sum_{|I|=\ell} z_I\gamma_I) + \sum_{|I|=\ell} z_I \nabla''^* G\nabla''(\gamma_I) \\ &= -\sum_{|I|=\ell} z_I\gamma_I\end{aligned}$$

where the last equality follows from Lemma 4.1. Hence the induction is proved for ℓ. The lemma is proved. □

For any $z \in \mathbb{C}^p$, we define a connection

$$\nabla_z := \nabla + \sum_{|I|\geq 1} z_I\eta_I. \tag{4.2.3}$$

for V. In general, this connection is neither flat nor smooth. However, by applying elliptic estimates for the Laplacian, there exists an $\varepsilon > 0$ such that on the intersection $\mathcal{Q}'_\varepsilon := \mathcal{Q}' \cap \mathbb{B}^p_\varepsilon$, the operator ∇_z forms an analytic family of flat connections for V. We omit the proof here, as we will treat the more general quasi-projective setting later.

4.2. Universal connection: universal property. —

Consider a dgla defined by

$$(L^\bullet, d) := (A^\bullet(X, E), \nabla).$$

Set

$$(C^\bullet, d) := (\ker\nabla', \nabla''), \quad (D^\bullet, d) := (H^i(L^\bullet, d), 0),$$

both of which are naturally endowed with dgla structures. Then by the formality in [Sim92], we have the quasi-isomorphisms

$$L^\bullet \xleftarrow{\phi} C^\bullet \xrightarrow{\psi} D^\bullet. \tag{4.2.4}$$

wehre ϕ is the inclusion, and $\psi(\eta) := \{\mathcal{H}(\eta)\}$. Fix a base point $x \in X$. Let $\varepsilon : L^0 \to E_x$, $\varepsilon : C^0 \to E_x$, and $\varepsilon : D^0 \to E_x$ be the evaluation maps which define the augmentations. By Theorem 3.4, for any Artin local $\mathbb{C}$-algebra A, the natural maps

(4.2.5)
$$[\mathscr{C}(L^\bullet, \varepsilon, A)/\exp(L^0 \otimes \mathfrak{m})] \leftarrow [\mathscr{C}(C^\bullet, \varepsilon, A)/\exp(C^0 \otimes \mathfrak{m})] \to [\mathscr{C}(D^\bullet, \varepsilon, A)/\exp(D^0 \otimes \mathfrak{m})]$$

are equivalences of groupoids.

Let $\mathcal{Q}_0$ be the quadratic cone in D^1 defined by

$$\mathcal{Q}_0 := \left\{u \in D^1 \otimes \mathfrak{m} \mid [u,u] = 0\right\}. \tag{4.2.6}$$

Denote by

$$\mathcal{Q} := \mathcal{Q}_0 \times \mathfrak{g}/\varepsilon(D^0). \tag{4.2.7}$$

The following lemma was proven in [GM88, Lemma 3.10]. Here, we refine the result by providing an explicit natural transformation between the functors.

Lemma 4.3. — *The analytic germ* $(\mathcal{Q}, 0)$ *(non-canonically) prorepresents the functor*

$$\begin{aligned}\mathrm{Art} &\to \mathrm{Grp}\\ A &\mapsto \mathrm{Def}(D^\bullet, \varepsilon, A)\end{aligned}$$

where $\mathrm{Def}(D^\bullet, \varepsilon, A)$ *is the small groupoids defined in* (3.3.1).

Proof. — Let $\ell := \dim_{\mathbb{C}}(\mathfrak{g}/\varepsilon(C^0))$. We choose a subset $\{v_1, \ldots, v_\ell\}$ of $\mathfrak{g}$ such that the images of $\{v_1, \ldots, v_\ell\}$ form a basis for the quotient space $\mathfrak{g}/\varepsilon(C^0)$. Then the natural linear map $W \to \mathfrak{g}/\varepsilon(D^0)$ is an isomorphism. Since $\varepsilon : D^0 \to E_x$ defined by the evaluation is injective, it follows that $\exp\big(\varepsilon(D^0) \otimes \mathfrak{m}\big)$ acts freely on $G_A^0 := \exp(\mathfrak{g} \otimes \mathfrak{m})$ by left-multiplication. Hence the following (non-canonical) maps

$$\begin{array}{ccccc} \mathfrak{g}/\varepsilon(D_0) \otimes \mathfrak{m} & \longleftarrow & W \otimes \mathfrak{m} & \longrightarrow & \frac{\exp(\mathfrak{g}\otimes\mathfrak{m})}{\exp(\varepsilon(D^0)\otimes\mathfrak{m})} \\ \downarrow & & \downarrow & & \downarrow \\ p(r) & \longleftarrow\!\shortmid & r & \longmapsto & \exp(\varepsilon(D^0) \otimes \mathfrak{m}) \cdot \exp(r) \end{array}$$

are both bijective. Here $\exp(\varepsilon(D^0) \otimes \mathfrak{m}) \cdot \exp(r)$ is the right coset of $\exp(r)$. Hence we can define an equivalence

$$\begin{aligned} \mathscr{C}(D^\bullet, A) \times W \otimes \mathfrak{m} &\to \frac{\mathscr{C}(D^\bullet, \varepsilon, A)}{\exp(D^0 \otimes \mathfrak{m})} \\ (\alpha, r) &\mapsto [(\alpha, \exp(r))]. \end{aligned} \tag{4.3.1}$$

Note that $C^0 = D^0$, the map

$$\begin{aligned} \mathscr{C}(C^\bullet, A) \times W \otimes \mathfrak{m} &\to \frac{\mathscr{C}(C^\bullet, \varepsilon, A)}{\exp(C^0 \otimes \mathfrak{m})} \\ (\alpha, r) &\mapsto [(\alpha, \exp(r))]. \end{aligned} \tag{4.3.2}$$

We consider $D^1 \times W$ as an affine $\mathbb{C}$-space. Let $\mathcal{I}$ be the ideal of $\mathbb{C}[D^1 \times W]$ generated by the quadratic equations

$$[q(u), q(u)] = 0,$$

where $q : D^1 \times W \to D^1$ denotes the projection map. It follows that

$$\mathbb{C}[D^1 \times W]/\mathcal{I} = \mathrm{Spec}\,(\mathcal{Q}).$$

Then for any element $\alpha \in D^1 \otimes \mathfrak{m}$ and $r \in W \otimes \mathfrak{m}$, it can be written as

$$\alpha = \sum_{i=1}^{m} \{\eta_i\} \otimes f_i, \quad r = \sum_{i=1}^{\ell} v_i \otimes g_i, \tag{4.3.3}$$

where $f_i, g_j \in \mathfrak{m}$. This gives rise to a morphism

$$\begin{gathered} \mathrm{Hom}(\mathbb{C}[D^1 \times W], A) \\ (\{\eta_i\}^*, (v_j)^*) \mapsto f_i + g_j \end{gathered}$$

Here we consider $\{\eta_i\}^* \in (D^1)^*$ and $(v_j)^* \in W^*$. Note that the above morphism factors through

$$\mathrm{Hom}(\mathbb{C}[D^1 \times W]/\mathcal{I}, A)$$

if and only if α satisfies

$$\frac{1}{2}[\alpha, \alpha] = d(\alpha) + \frac{1}{2}[\alpha, \alpha] = 0,$$

where $d : D^1 \to D^2$ denotes the differential of the dgla $(D^\bullet, d)$, that is zero.

Let $\mathcal{I}_1$ be the ideal of $\mathbb{C}[\mathcal{Q}]$ generated by the image of $(D^1 \times W)^* \subset \mathbb{C}[D^1 \times W]$ under the natural projection $\mathbb{C}[D^1 \times W] \to \mathbb{C}[D^1 \times W]/\mathcal{I}$. Let $\widehat{\mathbb{C}[\mathcal{Q}]}_{\mathcal{I}_1}$ denote the $\mathcal{I}_1$-adic completion of $\mathbb{C}[\mathcal{Q}]$. Since A is an Artin local $\mathbb{C}$-algebra, it follows that

$$\mathrm{Hom}(\mathbb{C}[D^1 \times W]/\mathcal{I}, A) \simeq \mathrm{Hom}(\widehat{\mathbb{C}[\mathcal{Q}]}_{\mathcal{I}_1}, A).$$

By standard properties of completion of quotient rings, there is a natural isomorphism

$$\widehat{\mathbb{C}[\mathcal{Q}]}_{\mathcal{I}_1} \simeq \frac{\mathbb{C}[[D^1 \times W]]}{\mathcal{I}},$$

where, by abuse of notation, we continue to denote by $\mathcal{I}$ the ideal $\mathcal{I} \cdot \mathbb{C}[[D^1 \times W]]$.

On the other hand, $\mathcal{I}_1$-adic completion commutes with localization at the maximal ideal $\mathcal{I}_1$ of $\mathbb{C}[\mathcal{Q}]$. Therefore we have a natural isomorphism

$$\widehat{\mathbb{C}[\mathcal{Q}]}_{\mathcal{I}_1} \xrightarrow{\simeq} \widehat{\mathcal{O}_{\mathcal{Q},0}}.$$

Therefore, the following map

$$\mathscr{C}(D^\bullet, A) \times W \otimes \mathfrak{m} \to \operatorname{Hom}(\widehat{\mathcal{O}_{\mathcal{Q},0}}, A) \tag{4.3.4}$$
$$(\alpha, r) \mapsto \Big((\{\eta_i\}^*, v_j^*) \mapsto f_i + g_j \Big)$$

is an equivalence, where (α, r) is written as in (4.3.3), and $\{\eta_i\}^*$ and v_j^* are understood as the images of $\{\eta_i\}^*$,v_j^* under the composite morphisms

$$\mathbb{C}[D^1 \times W] \to \mathbb{C}[\mathcal{Q}] \to \mathcal{O}_{\mathcal{Q},0} \to \widehat{\mathcal{O}}_{\mathcal{Q},0}.$$

We remark that $\widehat{\mathcal{O}_{\mathcal{Q},0}}$ is generated by $\{\eta_i\}^*$ and v_j^*. Therefore, for any morphism in $\operatorname{Hom}(\widehat{\mathcal{O}_{\mathcal{Q},0}}, A)$, the images of these generators completely determine the morphism. Combining (4.3.4) with (4.3.1) yields the lemma. □

Fix a positive integer $k \geq 2$. We set

$$\mathcal{Q}'' := \mathcal{Q}' \times \mathbb{C}^\ell.$$

Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{\mathcal{Q}'',0}$ and we consider $A_k := \mathcal{O}_{\mathcal{Q}'',0}/\mathfrak{m}^{k+1}$, that is an Artin local $\mathbb{C}$-algebra. Write $\mathfrak{m}_k := \mathfrak{m}A_k$, that is the maximal ideal for A_k. We now regard z_I and x_j as the image of z_I and x_j under the morphism

$$\mathcal{O}_{\mathbb{C}^{p+\ell},0} \to \mathcal{O}_{\mathcal{Q}'',0} \to \mathcal{O}_{\mathcal{Q}'',0}/\mathfrak{m}^{k+1} = A_k.$$

Then $z_I \in \mathfrak{m}_k$ for each I with $|I| \in \{1, \ldots, k\}$. For any $i \in \{1, \ldots, k\}$, let

$$\sigma_i = \sum_{|I|=i} z_I \eta_I$$

be defined in (4.2.1). By Lemma 4.1, we have $\eta_I \in \operatorname{Im} \nabla'$ for $|I| \geq 2$, and for any $z \in \mathcal{Q}'$, we have

$$\nabla' \sigma_i(z) = 0$$

for any $i \geq 1$. This implies that

$$\psi(\sigma_i) = 0 \in D^1 \otimes \mathfrak{m}_k, \quad \text{if } i \geq 2, \tag{4.3.5}$$

and

$$\delta_k := \sum_{1 \leq |I| \leq k} z_I \eta_I \in C^1 \otimes \mathfrak{m}_k.$$

On the other hand, by Lemma 4.2, we have

$$\delta_k := \sum_{1 \leq |I| \leq k} z_I \eta_I \in \mathscr{C}(C^\bullet, A_k).$$

Together with (4.3.5), this yields

$$\psi(\delta_k) = \sum_{i=1}^{p} z_i \{\eta_i\} \in \mathscr{C}(D^\bullet, A_k).$$

By (4.2.4) and Theorem 3.4, we have the equivalences

$$\Phi : \mathscr{C}(C^\bullet, A) \times W \otimes \mathfrak{m}_k \xrightarrow{\psi \times \mathrm{Id}} \mathscr{C}(D^\bullet, A) \times W \otimes \mathfrak{m}_k \to \operatorname{Hom}(\widehat{\mathcal{O}}_{\mathcal{Q},0}, A_k), \tag{4.3.6}$$

with

$$\Phi(\delta_k, \sum_{i=1}^{\ell} w_i v_i) = \Big((\{\eta_i\}^*, v_j^*) \mapsto z_i + w_j \Big).$$

As remarked above, since $\widehat{\mathcal{O}_{\mathcal{Q},0}}$ is generated by $\{\eta_i\}^*$ and v_j^*, any morphism in $\operatorname{Hom}(\widehat{\mathcal{O}_{\mathcal{Q},0}}, A_k)$ is determined by the images of these generators.

Theorem 4.4. — *Let ∇_z denote the analytic family of flat connections on V, parametrized by $z \in \mathcal{Q}'_\varepsilon$ for some $\varepsilon > 0$, as defined in (4.2.3). Let $\mathrm{ev}_x : H^0(X, E) \to E_x$ be the evaluation map, and let $\{v_1, \dots, v_\ell\}$ be a subset of E_x such that the images of $\{v_1, \dots, v_\ell\}$ form a basis for the quotient space $\mathfrak{g}/\mathrm{ev}_x H^0(X, E)$. Let $\mathrm{Mon}(\nabla_z)$ be the monodromy representation for any $z \in \mathcal{Q}'_\varepsilon$. Then the natural map between analytic germs*

$$\begin{aligned} f : (\mathcal{Q}'_\varepsilon \times \mathbb{C}^\ell, 0) &\to (R(X, N), \varrho) \\ (z, w) &\mapsto \exp(-\sum_{i=1}^{\ell} w_i v_i)\mathrm{Mon}(\nabla_z)\exp(\sum_{i=1}^{\ell} w_i v_i) \end{aligned} \tag{4.4.1}$$

is an analytic isomorphism.

Proof. — We recall that we have morphisms of groupoids

(4.4.2)

$$\begin{tikzcd} & \mathrm{Hom}\left(\widehat{\mathcal{O}}_{\mathcal{Q},0}, A_k\right) & \\ {[R_\varrho(A_k)/\mathrm{Id}]} & \mathscr{C}(C^\bullet, A_k) \times W \otimes \mathfrak{m}_k \arrow[u, "\Phi"] \arrow[r, "\psi\times\mathrm{Id}"] \arrow[d] & \mathscr{C}(D^\bullet, A_k) \times W \otimes \mathfrak{m}_k \arrow[ul] \arrow[d] \\ {[\mathscr{C}(L^\bullet, \varepsilon, A_k)/\exp(L^0 \otimes \mathfrak{m}_k)]} \arrow[u, "h"] & {[\mathscr{C}(C^\bullet, \varepsilon, A_k)/\exp(C^0 \otimes \mathfrak{m}_k)]} \arrow[l, "\phi"'] \arrow[r, "\psi"] & {[\mathscr{C}(D^\bullet, \varepsilon, A_k)/\exp(D^0 \otimes \mathfrak{m}_k)]} \end{tikzcd}$$

in which each of them is an isomorphism by (4.3.1), (4.3.2), (4.2.4) and Theorem 3.4. Here

$$\begin{aligned} h : [\mathscr{C}(L^\bullet, \varepsilon_x, A_k)/\exp(L^0 \otimes \mathfrak{m})] &\to [R_\varrho(A_k)/\mathrm{Id}] \\ (\alpha, e^r) &\mapsto \exp(-r)\mathrm{hol}_x(\nabla + \alpha)\exp(r) \end{aligned}$$

is defined in Lemma 3.8. We denote by

$$\Upsilon : \mathscr{C}(C^\bullet, A_k) \times W \otimes \mathfrak{m} \to [R_\varrho(A_k)/\mathrm{Id}]$$

the induced equivalence of groupoids in (4.4.2). It follows that

$$\Upsilon(\delta_k, \sum_{i=1}^{\ell} w_i v_i) = \exp(-\sum_{i=1}^{\ell} w_i v_i)\mathrm{Mon}(\nabla_z)\exp(\sum_{i=1}^{\ell} w_i v_i) \in R_\varrho(A_k)$$

corresponds to the morphism

$$f_k : \widehat{\mathcal{O}}_{R(X,N),\varrho} \to A_k = \mathcal{O}_{\mathcal{Q}'',0}/\mathfrak{m}^{k+1}$$

induced by f defined in (4.4.1).

Let us define a morphism $g_k \in \mathrm{Hom}(\widehat{\mathcal{O}}_{\mathcal{Q},0}, A_k)$ by setting

$$g_k(\{\eta_i\}^*) = z_i, \quad g_k(v_j^*) = w_j. \tag{4.4.3}$$

for each $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, \ell\}$. By (4.3.6), we have

$$\Phi(\delta_k, \sum_{i=1}^{\ell} w_i v_i) = g_k.$$

One can see the compatibility of g_k by

$$\begin{tikzcd} & \vdots \arrow[d] \\ & A_{k+1} \arrow[d] \\ \widehat{\mathcal{O}}_{\mathcal{Q},0} \arrow[ur, "g_{k+1}"] \arrow[r, "g_k"'] \arrow[ddr, "g_1"'] & A_k \arrow[d] \\ & \vdots \arrow[d] \\ & A_1 \end{tikzcd}$$

In conclusion, there exists natural maps

$$\begin{tikzcd} \mathrm{Hom}\Big(\widehat{\mathcal{O}}_{\mathcal{Q},0}, A_k\Big) \arrow[r] & \mathrm{Hom}\Big(\widehat{\mathcal{O}}_{\mathcal{Q},0}, A_{k-1}\Big) \\ \mathscr{C}(C^\bullet, A_k) \times W \otimes \mathfrak{m}_k \arrow[r] \arrow[u, "\Phi"] \arrow[d, "\Upsilon"'] & \mathscr{C}(C^\bullet, A_{k-1}) \times W \otimes \mathfrak{m}_{k-1} \arrow[u, "\Phi"'] \arrow[d, "\Upsilon"] \\ \mathrm{Hom}(\widehat{\mathcal{O}}_{R(X,N),\varrho}, A_k) \arrow[r] & \mathrm{Hom}(\widehat{\mathcal{O}}_{R(X,N),\varrho}, A_{k-1}) \end{tikzcd}$$

such that all vertical maps are isomorphisms. We have the inverse limit

$$\varprojlim_k g_k \in \mathrm{Hom}\left(\widehat{\mathcal{O}}_{\mathcal{Q},0}, \varprojlim_k A_k\right) = \mathrm{Hom}\Big(\widehat{\mathcal{O}}_{\mathcal{Q},0}, \widehat{\mathcal{O}}_{\mathcal{Q}'',0}\Big)$$

and by (4.4.3), it is an isomorphism. By (4.4.2), it follows that

$$f^* = \varprojlim_k f_k : \widehat{\mathcal{O}}_{R(X,N),\varrho} \to \widehat{\mathcal{O}}_{\mathcal{Q}'',0},$$

is also an isomorphism. By Theorem 3.5, this implies that $f : (\mathcal{Q}'_\varepsilon \times W, 0) \to (R(X,N), \varrho)$ is a *biholomorphism* between analytic germs. The theorem is therefore proved. □

4.3. 1-step phenomenon. — We are now in a position to prove the result on the so-called *1-step phenomenon* for semisimple systems on compact Kähler manifolds.

***Theorem 4.5* (1-step phenomenon). —** *Let $h : Y \to X$ be a holomorphic map between compact Kähler manifolds. Let (V, ∇) be a semisimple flat bundle on X with monodromy representation $\varrho : \pi_1(X,x) \to \mathrm{GL}_N(\mathbb{C})$, and set $y = h(x)$. Denote by*

$$h^* : R(X,N) \longrightarrow R(Y,N)$$

the induced morphism of representation varieties, given by $\tau \mapsto h^*\tau$ for any $\tau \in R(X,N)$.

Assume that:

(a) the pullback representation $h^*\varrho$ has image in the center $Z(\mathrm{GL}_N(\mathbb{C}))$, that is, $h^*\varrho$ factors through a representation $\tau_0 : \pi_1(Y,y) \to Z(\mathrm{GL}_N(\mathbb{C}))$, where

$$Z(\mathrm{GL}_N(\mathbb{C})) = \{\lambda\, \mathrm{Id}_N \mid \lambda \in \mathbb{C}^\times\};$$

(b) for every $\eta \in H^1(X,E)$, one has $h^*\eta = 0$ in $H^1(Y, h^*E)$.

Let R be any (geometrically) irreducible component of $R(X,N)$ containing ϱ. Then the image of R under h^* consists of the single point

$$h^*(R) = \{\tau_0\}.$$

Proof. — We use the notations introduced in this subsection. Let $\eta_1, \ldots, \eta_m \in \mathcal{H}^1(X, E)$ be a basis, and let η_I be defined as in (4.0.2). By assumption, $h^*\eta_i = 0$ for each i. By the construction of η_I in (4.0.2), this yields

$$h^*\eta_I = 0 \qquad \text{for all } I \text{ with } |I| \geq 1.$$

This implies that

$$h^*\nabla_z = h^*\nabla \qquad \text{for every } z \in \mathcal{Q}'_r.$$

On the other hand, by Item (a), for any $g \in \mathrm{GL}_N(\mathbb{C})$ we have $g^{-1}\tau_0 g = \tau_0$; hence the representation τ_0 is fixed under conjugation. Therefore, for the map $f : \mathcal{Q}'_\varepsilon \times W \to R(X, N)$ constructed in Theorem 4.4, the composition

$$h^* \circ f : \mathcal{Q}'_\varepsilon \times W \longrightarrow R(Y, N).$$

is constant. By Theorem 4.4, this implies that h^* is constant on some neighborhood of ϱ in R. Since h^* is an algebraic morphism, it follows that

$$h^*(R) = \{h^*\varrho\} = \{\tau_0\}.$$

The theorem follows. □

4.4. Geometric applications I: Katzarkov's and Campana's theorems. —

***Definition 4.6* (Shafarevich morphism).** — Let X be a quasi-compact Kähler manifold, and let $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ be a linear representation. Assume that there exists a *compactifiable* dominant holomorphic map

$$\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$$

onto a complex normal variety $\mathrm{Sh}_\varrho(X)$ with connected general fibers, such that for every compactifiable holomorphic map $f : Y \to X$, the following properties are equivalent:

(a) $\mathrm{sh}_\varrho \circ f(Y)$ is a point;
(b) $f^*\varrho(\pi_1(Y))$ is finite;

Then such sh_ϱ is called the *Shafarevich morphism* of ϱ.

The following lemma allows us to pass to a finite étale cover when constructing the Shafarevich morphism.

Lemma 4.7. — *Let X be a compact Kähler manifold and let $\varrho\colon \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ be a linear representation. Assume there exists a finite étale Galois cover $\mu\colon \widehat{X} \to X$ such that the Shafarevich morphism $\mathrm{sh}_{\mu^*\varrho}\colon \widehat{X} \to \mathrm{Sh}_{\mu^*\varrho}(\widehat{X})$ associated to the pullback $\mu^*\varrho$ exists. Then the Shafarevich morphism of ϱ exists.*

Proof. — Let $G := \mathrm{Aut}(\widehat{X}/X)$ be the Galois group of the cover. Let F be a fiber of $\mathrm{sh}_{\mu^*\varrho}$. Note that, for any $\gamma \in G$, we have

$$\mu^*\varrho\big(\mathrm{Im}[\pi_1(\gamma \cdot F) \to \pi_1(\widehat{X})]\big) = \mu^*\varrho\big(\mathrm{Im}[\pi_1(F) \to \pi_1(\widehat{X})]\big),$$

which implies that the image of the fundamental group of $\gamma \cdot F$ is also finite. By Definition 4.6, it follows that $\mathrm{sh}_{\mu^*\varrho}$ must contract $\gamma \cdot F$ to a point; in other words, $\gamma \cdot F$ is contained in a fiber of $\mathrm{sh}_{\mu^*\varrho}$.

Consequently, the action of G on $\widehat{X}$ descends to an action on $\mathrm{Sh}_{\mu^*\varrho}(\widehat{X})$, making $\mathrm{sh}_{\mu^*\varrho}$ equivariant with respect to this action. Therefore, the quotient $X \to \mathrm{Sh}_{\mu^*\varrho}(\widehat{X})/G$ is proper, and one can check that it satisfies the properties required to be the Shafarevich morphism of ϱ. □

***Definition 4.8* (Big representation).** — Let X be a smooth quasi-projective variety. A linear representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ is called *big* if, for any closed subvariety $Z \subset X$ passing through a very general point, the group

$$\varrho\big(\mathrm{Im}\big[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)\big]\big)$$

is infinite, where Z^{norm} denotes the normalization of Z.

In [Kat97], Katzarkov proved that for a smooth projective variety with linear nilpotent fundamental groups, its Shafarevich morphism is the Stein factorization of its Albanese map. This was extended to compact Kähler manifolds by Leroy [Ler99] in his PhD thesis (see also the survey paper by Claudon [Cla12]). Based on Theorem 4.5, we can give another proof of Katzarkov's theorem following same ideas in [EKPR12].

We first prove the following lemma.

Lemma 4.9. — *Let X be a compact Kähler manifold, and let $\tau : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ be a linear representation whose image is virtually nilpotent. Let $a : X \to A$ denote the Albanese map, and let $f : Y \to X$ be a holomorphic map from another compact Kähler manifold such that $(a \circ f)(Y)$ is a point. Then $f^*\tau(\pi_1(X))$ has finite image.*

Proof. — Since $\tau(\pi_1(X))$ is virtually nilpotent, after replacing X by a finite étale cover, we may assume that its Zariski closure in $\mathrm{GL}_N(\mathbb{C})$ is a connected nilpotent algebraic group G. Then G decomposes as a direct product of its central torus T and its unipotent radical U. This induces a natural decomposition of the representation

$$\tau = (\tau_1, \tau_2) \colon \pi_1(X) \to T \times U.$$

By fixing a faithful linear representation $U \hookrightarrow \mathrm{GL}_{N_2}(\mathbb{C})$, we can view the projection τ_2 as a unipotent representation.

Let (V, ∇) denote the trivial flat bundle of rank N_2. By the universal property of the Albanese map, one has $f^*\omega = 0$ for every $\omega \in H^0(X, \Omega_X)$. Since (V, ∇) is the trivial flat bundle, it follows that $f^*\eta = 0$ for all $\eta \in H^1(X, \mathrm{End}(V))$. Thus the hypotheses of Theorem 4.5 are satisfied for the trivial representation $\varrho : \pi_1(X) \to \mathrm{GL}_{N_2}(\mathbb{C})$ and the map $f : Y \to X$. Note that τ_2 belongs to the same irreducible component of ϱ. Hence $f^*\tau_2$ is trivial by Theorem 4.5.

On the other hand, since $\tau_1(\pi_1(X))$ is abelian, the representation τ_1 factors through $H_1(X, \mathbb{Z})$. Moreover, the map $a_* : H_1(X, \mathbb{Z})/\mathrm{tor} \to H_1(A, \mathbb{Z})$ is an isomorphism. Thus there exists a factorization

$$\begin{tikzcd} \pi_1(X) \arrow[r] \arrow[dr, "\tau_1"'] & \pi_1(A) \arrow[d] \\ & \mathrm{GL}_{N_1}(\mathbb{C}). \end{tikzcd}$$

It follows that $f^*\tau_1(\pi_1(Y)) = \{1\}$. Consequently, $f^*\tau(\pi_1(Y)) = \{1\}$. The lemma is proved. □

***Theorem 4.10* (Katzarkov, Leroy).** — *Let X be a compact Kähler manifold. Assume that $\pi_1(X)$ is nilpotent. Then the Shafarevich morphism of X is the Stein factorization of its Albanese map $a : X \to A$. Moreover, the universal cover of X is holomorphically convex.*

Proof. — Note that any finitely generated nilpotent group is linear. Hence, there exists a faithful representation $\tau \colon \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ whose image is nilpotent. By Lemma 4.7, it suffices to construct the Shafarevich morphism for this cover.

Let $X \xrightarrow{g} S \to A$ be the Stein factorization of a. Let $f : Y \to X$ be a holomorphic map from another compact Kähler manifold such that $(g \circ f)(Y)$ is a point. By Lemma 4.9, $f^*\tau : \pi_1(Y) \to \mathrm{GL}_N(\mathbb{C})$ has finite image. Since τ is faithful, the image $\mathrm{Im}[\pi_1(Y) \to \pi_1(X)]$ is thus finite.

Finally, by the characterization of the Albanese map, for any holomorphic map $h : Z \to X$ from a compact Kähler manifold, the group $h^*H^1(X, \mathbb{Z})$ is finite if and only if $(a \circ h)(Z)$ is a point. Combining this with the property established above, we conclude that g is the Shafarevich morphism of X.

We omit the proof of holomorphic convexity here, as it is identical to the proof of the more general Theorem 5.26 established later. □

We also provide a new proof of Campana's abelianity conjecture in the linear and compact setting.

***Theorem 4.11* ([Cam04]).** — *Let X be a smooth projective variety. If X is special, then every linear representation $\tau : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ has virtually abelian image.*

Proof. — By [CDY25c], after replacing X by a finite étale cover, we may assume that $\varrho(\pi_1(X))$ is nilpotent and torsion free. By [Cam04], the Albanese map $a : X \to A$ is an algebraic fiber space. Let F be a general smooth fibre of a. By [CDY25c], we have an exact sequence

$$\pi_1(F) \longrightarrow \pi_1(X) \longrightarrow \pi_1(A) \longrightarrow 0. \tag{4.11.1}$$

Let $\iota : F \to X$ denote the inclusion. By Lemma 4.9, the representation $\iota^*\tau : \pi_1(F) \to \mathrm{GL}_N(\mathbb{C})$ has finite image, and hence is trivial since $\tau(\pi_1(X))$ is torsion free. By (4.11.1), the representation τ therefore factors through $\pi_1(A)$, and thus has abelian image. This proves the theorem. □

4.5. Geometric applications II: a structure theorem for the linear Shafarevich morphism. — The following result is implicit in [EKPR12]. It played a crucial role in the work of Wang and the second author on the proof of the linear Kollár conjecture (cf. [DW24, Proof of Theorem 4.6]). Here, we provide an alternative proof using Theorem 4.5.

***Theorem 4.12.* —** *Let X be a smooth projective variety, and let $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ be a big linear representation. Let $\sigma : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ denote its semisimplification. Let $\mathrm{sh}_\sigma : X \to \mathrm{Sh}_\sigma(X)$ be the Shafarevich morphism associated with σ, whose existence is guaranteed by* [Eys04] *(see also* [DYK23]*). If Y is a very general fiber of sh_σ, then the Albanese map of Y is generically finite onto its image.*

Proof. — By Selberg's theorem, we may replace X with a finite étale cover such that the images $\sigma(\pi_1(X))$ and $\varrho(\pi_1(X))$ are torsion-free. Let (V, ∇) be the semisimple flat bundle on X associated with the monodromy representation of σ.

Let $f : Y \to X$ be the natural morphism. Since ϱ is big and Y is a very general fiber of sh_σ, Y is smooth and the pullback $f^*\varrho : \pi_1(Y) \to \mathrm{GL}_N(\mathbb{C})$ remains big. On the other hand, by Definition 4.6, the image of $\pi_1(Y)$ under σ is finite; given our torsion-free assumption, $f^*\sigma$ is trivial. Consequently, the pullback bundle $f^*(V, \nabla)$ is a trivial flat bundle.

Let $a : Y \to A$ be the Albanese map. Let Z be a connected component of a very general smooth fiber of a, and let $g : Z \to Y$ denote the natural inclusion. The pullback $g^*f^*\varrho : \pi_1(Z) \to \mathrm{GL}_N(\mathbb{C})$ is also big. Since Z maps to a point in A, the induced map on cohomology vanishes, i.e., $g^*H^1(Y, \mathbb{C}) = 0$. Since $f^*(V, \nabla)$ is a trivial flat bundle, we have

$$g^*H^1(Y, f^*\operatorname{End}(V)) \simeq g^*H^1(Y, \mathbb{C}) \otimes \mathbb{C}^{N^2} = 0.$$

Define $h := f \circ g$. The conditions in Theorem 4.5 are thus satisfied for $h : Z \to X$.

***Claim 4.13.* —** *The representations ϱ and σ are in the same irreducible component of $R(X, N)$.*

Proof. — Let $V = \mathbb{C}^N$. Since V is a finite-dimensional vector space, there exists a Jordan-Hölder filtration of $\pi_1(X)$-submodules:

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_k = V,$$

such that the successive quotients $W_i := V_i/V_{i-1}$ are irreducible representations. The semisimplification σ is defined as the direct sum representation acting on $\bigoplus_{i=1}^k W_i$.

We choose a basis for V compatible with this filtration. With respect to this basis, for any $\gamma \in \pi_1(X)$, the matrix $\rho(\gamma)$ is block upper-triangular:

$$\rho(\gamma) = \begin{pmatrix} \rho_1(\gamma) & * & \dots & * \\ 0 & \rho_2(\gamma) & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \rho_k(\gamma) \end{pmatrix},$$

where $\rho_i(\gamma)$ denotes the action on the quotient W_i. The semisimplification corresponds to the block-diagonal part:

$$\sigma(\gamma) = \operatorname{diag}(\rho_1(\gamma), \rho_2(\gamma), \dots, \rho_k(\gamma)).$$

To construct the deformation, we define a one-parameter family of diagonal matrices. Let $n_i = \dim(W_i)$. For $t \in \mathbb{C}^*$, define $D(t) \in \mathrm{GL}_N(\mathbb{C})$ as:

$$D(t) = \operatorname{diag}(t^{w_1} I_{n_1}, t^{w_2} I_{n_2}, \dots, t^{w_k} I_{n_k}),$$

where $w_i := k - i$.

We define an algebraic morphism $\Phi : \mathbb{C}^* \to R(X, N)$ by conjugation:

$$\Phi(t)(\gamma) := D(t)\varrho(\gamma)D(t)^{-1}.$$

A direct calculation shows that the limit $\lim_{t\to 0}\Phi(t)(\gamma)$ exists and equals $\sigma(\gamma)$. Thus, Φ extends to a regular morphism on $\mathbb{C}$ such that $\Phi(0)=\sigma$. Since $\Phi(1)=\varrho$, the representations σ and ϱ are connected by an affine line, and therefore lie in the same irreducible component of $R(X,N)(\mathbb{C})$. □

We now apply Theorem 4.5 to conclude that $h^*\varrho$ is trivial. However, recall that $h^*\varrho$ is big when restricted to the very general fiber Z. The only way a representation can be both trivial and big is if the domain is a point. Therefore, $\dim Z=0$. Since Z is the generic fiber of the Albanese map, this implies $\dim a(Y)=\dim Y$. The theorem is thus proved. □

In the remainder of the paper, we pursue a similar strategy to study Conjecture 2.12. However, in general there is no formality as in (4.2.4), and, as we shall see later, the *1-step phenomenon* of Theorem 4.5 does not hold in this setting. We therefore have to develop Hodge-theoretic methods for the deformation of local systems over quasi-compact Kähler manifolds, and subsequently establish a *2-step phenomenon* in this more general context.

5. Deformation theory of local systems over quasi-compact Kähler manifolds

The results in this section form the technical core of the paper. Here, we prove Theorems C and D, which provide the foundation for the deformation theory of local systems over quasi-compact Kähler manifolds.

Throughout this section, let (X,ω) be a compact Kähler manifold and let $D=\sum_{i=1}^m D_i$ be a simple normal crossing divisor on X. Set $X_0=X\setminus D$. Let (V,∇) be a *unitary* flat bundle on X, and let $(E,\nabla)=(\mathrm{End}(V),\nabla)$ be the induced bundle of endomorphisms. Note that the unitary structure on V induces a hermitian flat metric h for (E,∇). We write $\nabla=\nabla'+\nabla''$ where ∇' is the $(1,0)$-part and ∇'' is the $(0,1)$-part. Let $\boldsymbol{\nabla}'^*$ and $\boldsymbol{\nabla}''^*$ denote the formal adjoints of $\boldsymbol{\nabla}'$ and $\boldsymbol{\nabla}''$, defined with respect to the metrics h and ω. Finally, let G be the Green operator associated with the Laplacian $\Delta'=\Delta''$. These operators satisfy the relations given in (2.14.1); we will use these identities freely throughout this section.

5.1. Residue maps. — Let us first recall the definition of residues.

For a multi-index $I=(i_1,\ldots,i_k)$ with entries in $\{1,\ldots,m\}$, we denote its length by $|I|=k$. We say that I is *ordered* if $i_1<\cdots<i_k$.

For any ordered I, we set $D_I:=D_{i_1}\cap\cdots\cap D_{i_j}$. Set $D^{(j)}:=\sqcup_{|I|=j}D_I$ and let

$$D_{(j)}:=\sqcup_{|I|=j}\sum_{p\notin I}D_I\cap D_p$$

that is a simple normal crossing divisor on $D^{(j)}$.

Fix any $\eta\in A^k(X,D,E)$ and any integer $j\le k$. For any point $x\in D_I$ with $|I|=j$, choose a coordinate neighborhood $(U;z_1,\ldots,z_n)$ centered at x such that there exists $j'\ge j$ with

$$D\cap U=(\cup_{\ell=1}^{j'}D_{i_\ell})\cap U;\quad D_{i_\ell}\cap U=\{z_\ell=0\}$$

for each $\ell\in\{1,\ldots,j'\}$. On this neighborhood U, we can decompose η as

$$\eta|_U=\frac{dz_1}{z_1}\wedge\cdots\wedge\frac{dz_j}{z_j}\wedge a(z)+b(z),$$

where $a(z)$ is a section of $\mathscr{A}^{k-j}(X,\sum_{p\notin I}D_p,E)(U)$, and $b(z)$ is a term that does not contain the full wedge product $\frac{dz_1}{z_1}\wedge\cdots\wedge\frac{dz_j}{z_j}$. We then define the residue locally by

$$\mathrm{Res}_{D_I}(\eta|_U):=(2\pi\sqrt{-1})^j a(z)|_{D_I\cap U}\in\mathscr{A}^{k-j}\Big(D_I,\sum_{p\notin I}D_p\cap D_I,E\Big)(U\cap D_I).\tag{5.0.1}$$

One can verify that this definition is well-defined and independent of the choice of local coordinates. Consequently, these local constructions glue together to get

$$\mathrm{Res}_{D_I}\eta\in A^{k-j}(D_I,\sum_{p\notin I}D_I\cap D_p,E).$$

for each strata D_I with $|I| = j$.

We write

$$\operatorname{Res}_D \eta = \left(\operatorname{Res}_{D_i} \eta\right)_{i=1,\dots,m}.$$

We rely on the following fact throughout this section.

Lemma 5.1. — *Let $\eta \in A^k(X, D, E)$. Then we have*

$$\boldsymbol{\nabla}''\eta = \iota_*(\operatorname{Res}_D \eta) + \nabla''\eta. \tag{5.1.1}$$

Proof. — It suffices to verify the statement locally. By (5.0.1), we have

$$\boldsymbol{\nabla}''\eta = \sum_{i=1}^{m} (\iota_i)_* \operatorname{Res}_{D_i} \eta + \nabla''\eta = \iota_*(\operatorname{Res}_D \eta) + \nabla''\eta.$$

The lemma follows immediately. □

5.2. Gysin morphism. — Recall the complex of sheaves of twisted logarithmic forms

$$(\mathcal{A}^\bullet(X, D, E), \nabla) := \mathcal{A}^0(X, D, E) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathcal{A}^{2n}(X, D, E), \tag{5.1.2}$$

where $\mathcal{A}^k(X, D, E)$ denotes the sheaf of smooth differential k-forms on X with logarithmic singularities along D taking values in the flat vector bundle E, and the differential ∇ is the exterior covariant derivative induced by the flat connection on E.

Consider the weight filtration $W_\bullet \mathcal{A}^\bullet(X, D, E)$ defined by the number of logarithmic poles. Locally, $W_k \mathcal{A}^m(X, D, E)$ is spanned by forms containing at most k logarithmic differentials dz_i/z_i. Explicitly, these are linear combinations of terms of the form

$$\alpha \wedge \frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_j}}{z_{i_j}},$$

where α is a smooth form and $j \leq k$. In this paper, we are primarily interested in the case $W_1 \mathcal{A}^\bullet(X, D, E)$, which contains forms with at most one logarithmic factor.

For the general filtration, we have the k-th residue map

$$\operatorname{Res}^{(k)} : W_k \mathcal{A}^\bullet(X, D, E) \to (\iota_k)_* \mathcal{A}^{\bullet - k}_{D^{(k)}}(E), \tag{5.1.3}$$

where $\iota_k : D^{(k)} \hookrightarrow X$ denotes the inclusion. We adopt the convention that $\mathcal{A}^j_{D^{(k)}}(E) = 0$ if $j < 0$. Let $W'_{k-1} \mathcal{A}^\bullet(X, D, E)$ denote the kernel of this residue map, and define its complex of global sections by

$$W'_{k-1} A^\bullet(X, D, E) := \Gamma\left(X, W'_{k-1} \mathcal{A}^\bullet(X, D, E)\right). \tag{5.1.4}$$

Since these are fine sheaves, the functor of taking global sections is exact. Consequently, we obtain the following short exact sequence of complexes:

$$0 \to W'_{k-1} A^\bullet(X, D, E) \to W_k A^\bullet(X, D, E) \xrightarrow{\operatorname{Res}^{(k)}} A^{\bullet - k}_{D^{(k)}}(E) \to 0. \tag{5.1.5}$$

Note that we have the quasi-isomorphism

$$\operatorname{Gr}^W_k \mathcal{A}^\bullet(X, D, E) \to (\iota_k)_* \mathcal{A}^{\bullet - k}_{D^{(k)}}(E) \tag{5.1.6}$$

Since we have

$$0 \to W_{k-1} \mathcal{A}^\bullet(X, D, E) \to W_k \mathcal{A}^\bullet(X, D, E) \to \operatorname{Gr}^W_k \mathcal{A}^\bullet(X, D, E) \to 0,$$

if we take global sections, we have

$$0 \to W_{k-1} A^\bullet(X, D, E) \to W_k A^\bullet(X, D, E) \to \Gamma(X, \operatorname{Gr}^W_k \mathcal{A}^\bullet(X, D, E)) \to 0.$$

This implies that

$$\Gamma(X, \operatorname{Gr}^W_k \mathcal{A}^\bullet(X, D, E)) \simeq \operatorname{Gr}^W_k A^\bullet(X, D, E).$$

Together with (5.1.6), the natural morphism

$$\operatorname{Gr}^W_k A^\bullet(X, D, E) \to A^{\bullet - k}_{D^{(k)}}(E)$$

induced by the residue gives to an isomorphism of their cohomologies

$$H^i(\mathrm{Gr}_k^W A^\bullet(X,D,E)) \to H^i(A^{\bullet-k}_{D^{(k)}}(E)). \tag{5.1.7}$$

It follows that we have the long exact sequence

$$\begin{tikzcd}[column sep=small]
H^i(W_kA^\bullet(X,D,E)) \arrow[r] \arrow[d, "\simeq"] & H^i(\mathrm{Gr}_k^W A^\bullet(X,D,E)) \arrow[r] \arrow[d, "\simeq"] & H^{i+1}(W_{k-1}A^\bullet(X,D,E)) \arrow[r] \arrow[d] & H^{i+1}(W_kA^\bullet(X,D,E)) \arrow[r] \arrow[d, "\simeq"] & H^{i+1}(\mathrm{Gr}_k^W A^\bullet(X,D,E)) \arrow[d, "\simeq"] \\
H^i(W_kA^\bullet(X,D,E)) \arrow[r] & H^i(A^{\bullet-k}_{D^{(k)}}(E)) \arrow[r] & H^{i+1}(W'_{k-1}A^\bullet(X,D,E)) \arrow[r] & H^{i+1}(W_kA^\bullet(X,D,E)) \arrow[r] & H^{i+1}(A^{\bullet-k}_{D^{(k)}}(E))
\end{tikzcd}$$

By the Five Lemma, we obtain:

Lemma 5.2. — *The inclusion*

$$i_k : W_kA^\bullet(X,D,E) \hookrightarrow W'_kA^\bullet(X,D,E)$$

between the two complexes is a quasi-isomorphism.

We write $A^\bullet_{X,D}(E) := W'_0A^\bullet(X,D,E)$ for simplicity. Then the natural morphism $A^\bullet(X,E) \to A^\bullet_{X,D}(E)$ induces an isomorphism

$$H^i(X,E) \to H^i(A^\bullet_{X,D}(E)). \tag{5.2.1}$$

For any $\beta \in H^q(D^{(1)},E)$, we can find an element $\eta \in W_1A^{q+1}(X,D,E)$ such that $\mathrm{Res}_D\,\eta$ is ∇-closed and represents β. Then by the snake lemma, $\nabla\eta \in A^{q+2}_{X,D}(E)$ that is ∇-closed. Hence it represents a cohomology class in $H^{q+2}(A^\bullet_{X,D}(E)) \simeq H^{q+2}(X,E)$

Definition 5.3 **(Gysin morphism).** — The map

$$\begin{aligned} g : H^q(D^{(1)},E) &\to H^{q+2}(X,E) \\ \beta &\mapsto \{\nabla\eta\} \end{aligned}$$

is called the *Gysin morphism*.

We will the following result.

Lemma 5.4. — *For any $\beta \in A^q(D^{(1)},E)$ such that $\nabla\beta = 0$, we have*

$$g(\{\beta\}) = \{-\iota_*\beta\},$$

where $\iota_*\beta \in \mathscr{D}^{q+2}(X,E)$ denotes the E-valued current defined by the pushforward under the natural inclusion $\iota\colon D^{(1)} \to X$.

Proof. — Choose an element $\eta \in W_1A^{q+1}(X,D,E)$ such that $\mathrm{Res}_D\,\eta$ is ∇-closed and represents β. We then have the following formula:

$$\boldsymbol{\nabla}\eta = \boldsymbol{\nabla}'\eta + \boldsymbol{\nabla}''\eta \overset{(5.1.1)}{=} \nabla\eta + \iota_*(\beta).$$

Since $\{\boldsymbol{\nabla}\eta\} = 0$, it follows that

$$g(\{\beta\}) = \{\nabla\eta\} = -\{\iota_*(\beta)\}.$$

The lemma is proved. □

5.3. A canonical lift of residue map. — Let $\pi_i : U_i \to D_i$ be a deformation retraction of some open neighborhood U_i of D_i. Let s_i be a canonical section of $\mathcal{O}_X(D_i)$ and let h_i be a smooth metric for $\mathcal{O}_X(D_i)$. Let ϱ_i be a smooth function on X supported in U_i, that is equal to 1 in a neighborhood of D_i. Consider a flat section $\beta \in H^0(D^{(1)},E)$. Then we have

$$\nabla'\beta = 0, \quad \nabla''\beta = 0. \tag{5.4.1}$$

Since π_i is a deformation retraction, it follows that $\pi_1(D_i) \to \pi_1(U_i)$ is an isomorphism. Therefore, β extends to a flat section of $H^0(U_i,E)$, denoted by β_i. Since $\nabla\beta = 0$, it follows that $\nabla\beta_i = 0$. Therefore, one has

$$\frac{\sqrt{-1}}{2\pi}\nabla'(\sum_i \varrho_i \log|s_i|^2_{h_i} \cdot \beta_i) \in A^1(X,D,E) \tag{5.4.2}$$

and the residue

$$\operatorname{Res}_D \frac{\sqrt{-1}}{2\pi}\nabla'(\sum_i \varrho_i \log |s_i|^2_{h_i} \cdot \beta_i) = -\beta.$$

Hence,

$$h(\beta) := \frac{\sqrt{-1}}{2\pi}\nabla''\nabla'(\sum_i \varrho_i \log |s_i|^2_{h_i} \cdot \beta_i) \in A^2(X, E)$$

is ∇-closed, and lies in the cohomology class of $g(\beta)$.

If we consider the covariant derivative in the sense of currents, then we have

$$\frac{\sqrt{-1}}{2\pi}\boldsymbol{\nabla}''\boldsymbol{\nabla}'(\sum_i \varrho_i \log |s_i|^2_{h_i} \cdot \beta_i) = -\iota_*\beta + h(\beta).$$

Set

$$\gamma := 2\pi\sqrt{-1}(\nabla')^* G(\nabla'')^* G(h(\beta)).$$

Then

$$\frac{\sqrt{-1}}{2\pi}\nabla''\nabla'\gamma = -(1-\mathcal{H})(h(\beta)),$$

where $\mathcal{H}$ is the harmonic projection. Therefore, we have

$$\frac{\sqrt{-1}}{2\pi}\boldsymbol{\nabla}''\boldsymbol{\nabla}'(\sum_i \varrho_i \log |s_i|^2_{h_i} \cdot \beta_i + \gamma) = -\iota_*(\beta) + \mathcal{H}(h(\beta)).$$

Hence, we define a map

$$f : H^0(D^{(1)}, E) \to A^1(X, D, E)$$

by setting

$$f(\beta) := \frac{\sqrt{-1}}{2\pi}\boldsymbol{\nabla}'(\sum_i \varrho_i \log |s_i|^2_{h_i} \cdot \beta_i + \gamma). \tag{5.4.3}$$

Then we have

$$\boldsymbol{\nabla} f(\beta) = \boldsymbol{\nabla}'' f(\beta) = -\iota_*\beta + \mathcal{H}(h(\beta)) = -\iota_*\beta - \mathcal{H}(g(\beta)) \tag{5.4.4}$$

where g is the Gysin morphism defined in Definition 5.3, and

$$\operatorname{Res}_D(f(\beta)) = -\beta.$$

One can check that f is a linear map.

5.4. Regularity. — Note that although the complex $A^\bullet(X, D, E)$ is preserved under the Lie bracket, it is not preserved by Green operator G or $\boldsymbol{\nabla}''^*$. Moreover, since the product of currents is generally ill-defined, extending the analytic constructions in § 4.1 requires establishing additional regularity properties when applying these operators. In this subsection, we establish various regularity theorems used in the construction of Theorem 5.14.

Lemma 5.5. — *Let $\eta \in W_1A^2(X, D, E)$ such that $\eta = \boldsymbol{\nabla}'\sigma$, where $\sigma \in \mathscr{D}^1(X, E)$. Assume that*

–

$$\nabla\eta = 0$$

– *$\operatorname{Res}_D \eta \in A^1(D^{(1)}, E)$ is ∇-closed, such that its cohomology class in $H^1(D^{(1)}, E)$ is trivial.*

Then $\operatorname{Res}_D \eta = 0$. In particular, $\boldsymbol{\nabla}''\eta = \boldsymbol{\nabla}\eta = 0$.

Proof. — Let $\delta := \operatorname{Res}_D \eta \in A^1(D^{(1)}, E)$. Then we have

$$\boldsymbol{\nabla}\eta = \iota_*(\delta).$$

Since $\boldsymbol{\nabla}'\boldsymbol{\nabla}\eta = 0$, it follows that

$$\boldsymbol{\nabla}'\boldsymbol{\nabla}\eta = \iota_*\nabla'\delta = 0.$$

Since $\iota_* : A^k(D^{(1)}, E) \to \mathscr{D}^{k+2}(X, E)$ is injective, it follows that

$$\nabla'\delta = 0.$$

By the assumption, $\delta = \nabla h$ for some $h \in A^0(D^{(1)}, E)$. Therefore, we have

$$\nabla''\nabla' h = 0.$$

This implies that

$$\Lambda\nabla''\nabla' h = \Delta' h = 0.$$

Hence, $\delta = \nabla h = 0$. Therefore, we have

$$\boldsymbol{\nabla}\eta = 0, \quad \boldsymbol{\nabla}''\eta = 0.$$

□

Lemma 5.6. — *Let $\mathbb{B}^n$ be the unit ball in $\mathbb{C}^n$. Let σ be a distribution on $\mathbb{B}^n$ such that*

$$\partial\sigma = \sum_{i=1}^{m} a_i(z)\frac{dz_i}{z_i} + \sum_{j=m+1}^{n} a_j(z)dz_j.$$

Assume that $a_i(z)$ is smooth for each $i \in \{1, \dots, n\}$, and $a_i(z)|_{D_i} = 0$ when restricted on the divisor $D_i := (z_i = 0)$ for each $i \in \{1, \dots, m\}$. Then σ is a continuous function on $\mathbb{B}^n$. Moreover, for each $i \in \{1, \dots, m\}$, we have

$$\partial(\sigma|_{D_i}) = \Big(\sum_{j=1,\dots,m; j\neq i} a_j(z)\frac{dz_j}{z_j} + \sum_{j=m+1}^{n} a_j(z)dz_j\Big)|_{D_i}. \tag{5.6.1}$$

Proof. — Write $\hat{z}_i := (z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n)$. Since a_i is smooth and vanishes on $(z_i = 0)$, the functions defined by

$$f_i(\hat{z}_i) := \frac{\partial a_i(z_1, \dots, z_{i-1}, 0, z_{i+1}, \dots, z_n)}{\partial z_i}, \quad g_i(\hat{z}_i) := \frac{\overline{\partial} a_i(z_1, \dots, z_{i-1}, 0, z_{i+1}, \dots, z_n)}{\partial \bar{z}_i},$$

are smooth functions. Then we have

$$b_i(z) := a_i(z) - z_i f_i(\hat{z}_i) - \bar{z}_i g_i(\hat{z}_i) \in O(|z_i|^2). \tag{5.6.2}$$

Therefore,

$$\frac{1}{z_i}\frac{\partial b_i(z)}{\partial \bar{z}_i} \in L^\infty(\mathbb{B}^n_\varepsilon) \tag{5.6.3}$$

for any $\varepsilon \in (0, 1)$. Consider

$$f := \sigma - \sum_{i=1}^{m} \bar{z}_i \log|z_i|^2 g_i(\hat{z}_i) - z_i f_i(\hat{z}_i),$$

then we have

$$\begin{aligned}
\partial f &= \partial\sigma - \sum_{i=1}^{m}(\bar{z}_i g_i(\hat{z}_i)\frac{dz_i}{z_i} + \bar{z}_i \log|z_i|^2 \partial g_i + f_i(\hat{z}_i)dz_i + z_i\partial f_i) \\
&= \sum_{i=1}^{m}(a_i(z) - g_i(\hat{z}_i)\bar{z}_i - z_i f_i(\hat{z}_i))\frac{dz_i}{z_i} + \sum_{i=1}^{m}(\bar{z}_i \log|z_i|^2\partial g_i + z_i\partial f_i) + \sum_{j=m+1}^{n} a_j(z)dz_j \\
&= \sum_{i=1}^{m} b_i(z)\frac{dz_i}{z_i} + \sum_{i=1}^{m}(\bar{z}_i \log|z_i|^2\partial g_i + z_i\partial f_i) + \sum_{j=m+1}^{n} a_j(z)dz_j.
\end{aligned}$$

Hence

$$\begin{aligned}
\Delta' f = \Lambda\bar{\partial}\partial f &= \sum_{i=1}^{m} \frac{\partial b_i(z)}{\partial \bar{z}_i}\frac{dz_i}{z_i} + \Lambda\sum_{i=1}^{m}\bar{\partial}(\bar{z}_i \log|z_i|^2\partial g_i + z_i\partial f_i) + \Lambda\bar{\partial}(\sum_{j=m+1}^{n} a_j(z)dz_j) \\
&= \sum_{i=1}^{m} \frac{\partial b_i(z)}{\partial \bar{z}_i}\frac{dz_i}{z_i} + \text{ smooth terms.}
\end{aligned}$$

It follows from (5.6.3) that $\Delta' f \in L^\infty(\mathbb{B}^n_\varepsilon)$. Therefore, $f \in C^{1,\alpha}(\mathbb{B}^n)$ for any $\alpha \in (0,1)$. This implies that

$$\sigma - \sum_{i=1}^m \bar{z}_i \log|z_i|^2 g_i(\hat{z}_i) \in C^{1,\alpha}(\mathbb{B}^n).$$

In particular, σ is a continuous function.

Write

$$h_i(z) := \sigma - \bar{z}_i \log|z_i|^2 g_i(\hat{z}_i) - z_i f_i(\hat{z}_i),$$

which is a continuous function. Note that

$$\sigma|_{D_i} = h_i|_{D_i}.$$

Since we have

$$\begin{aligned}
\partial h_i = & \sum_{j=1,\ldots,m; j\neq i} a_j(z)\frac{dz_j}{z_j} + \sum_{j=m+1}^n a_j(z)dz_j \\
& + (a_i - g_i(\hat{z}_i)\overline{z_i} - z_i f_i(\hat{z}_i))\frac{dz_i}{z_i} + (\bar{z}_i \log|z_i|^2 \partial g_i + z_i \partial f_i) \\
= & \sum_{j=1,\ldots,m; j\neq i} a_j(z)\frac{dz_j}{z_j} + \sum_{j=m+1}^n a_j(z)dz_j \\
& + \frac{b_i(z)}{z_i}dz_i + (\bar{z}_i \log|z_i|^2 \partial g_i + z_i \partial f_i)
\end{aligned}$$

By (5.6.2), $\frac{b_i(z)}{z_i}$ is continuous and $\frac{b_i(z)}{z_i}|_{D_i} = 0$. It follows that

$$\partial(\sigma|_{D_i}) = \partial(h_i|_{D_i}) = \Big(\sum_{j=1,\ldots,m; j\neq i} a_j(z)\frac{dz_j}{z_j} + \sum_{j=m+1}^n a_j(z)dz_j\Big)|_{D_i}.$$

The lemma is proved. □

Lemma 5.7. — *Let $\eta \in A^1(X, D, E)$ be such that $\operatorname{Res}_D \eta = 0$. If there exists some $f \in \mathscr{D}^0(X, E)$ such that $\nabla' f = \eta$, then f is smooth on X_0 and $f \in A^0_{\mathrm{cts}}(X, E)$. Moreover, for any D_i, we have*

$$\nabla'(f|_{D_i}) \in A^1(D_i, \sum_{j\neq i} D_j \cap D_i, E).$$

The following property is satisfied: For any coordinate system $(U; z_1, \ldots, z_n)$ such that $D \cap U = (z_1 \cdots z_k = 0)$ with $D_i \cap U = (z_1 = 0)$, if we write

$$\eta|_U = \sum_{j=1,\ldots,k} a_j(z)\frac{dz_j}{z_j} + \sum_{j=k+1}^n a_j(z)dz_j,$$

then we have

$$\nabla'(f|_{D_i})|_{U\cap D_i} = \Big(\sum_{j=2,\ldots,k} a_j(z)\frac{dz_j}{z_j} + \sum_{j=k+1}^n a_j(z)dz_j\Big)|_{(z_1=0)}. \tag{5.7.1}$$

In the above setting, we shall abusively write

$$\eta|_{D_i} := \nabla'(f|_{D_i}) \in A^1(D_i, \sum_{j\neq i} D_i \cap D_j, E). \tag{5.7.2}$$

Then for any $j \neq i$, by (5.7.1), we have

$$\operatorname{Res}_{D_i\cap D_j}(\eta|_{D_i}) = (\operatorname{Res}_{D_j}\eta)|_{D_i\cap D_j} = 0. \tag{5.7.3}$$

Proof of Lemma 5.7. — Fix any point $p \in D$. We take admissible coordinates $(U; z_1, \ldots, z_n)$ centered at p such that $U \cap D = (z_1 \cdots z_m = 0)$. Let $e_1, \ldots, e_r$ be a flat frame for $E|_U$. Then

$$\nabla'' e_i = \nabla' e_i = 0.$$

Let us write $f = \sum_i f_i e_i$ and $\eta = \sum_i \eta_i e_i$ where $f_i \in \mathscr{D}^0(U)$ and $\eta_i \in \mathscr{A}^1(X, D)(U)$. Then we have

$$\partial f_i = \eta_i.$$

We write

$$\eta_i = \sum_{j=1}^{m} a_j(z) \frac{dz_j}{z_j} + \sum_{j=m+1}^{n} a_j(z) dz_j$$

such that $a_j(z)$ is a smooth function for each j. Since $\mathrm{Res}_D\, \eta = 0$, it follows that $a_j(z)|_{(z_j=0)} \equiv 0$ for each $j \in \{1, \ldots, m\}$. By Lemma 5.6, f_i is continuous, and hence f is continuous. The second assertion follows from (5.6.1). The lemma is proved. □

The following criterion determines when the Lie bracket of two log forms is ∇'-exact.

Lemma 5.8. — *Let $\eta_1, \eta_2 \in A^1(X, D, E)$. Assume that η_1 is ∇'-closed and η_2 is ∇'-exact. Then $[\eta_1, \eta_2]$ is ∇'-exact.*

Proof. — As η_2 is ∇'-exact, we have $\eta_2 = \nabla' \sigma$ for some current σ. As σ is of degree 0, we have

$$(\nabla' + (\nabla')^*)\sigma = \eta_2.$$

As $\eta_2 \in L^{2-\varepsilon}$ for any $\varepsilon > 0$, by the elliptic regularity of $\nabla' + (\nabla')^*$, we know that $\sigma \in L^{2+c_0}$ for some $c_0 > 0$ and σ is smooth on $X \backslash D$. Since η_1 has logarithmic poles, the bracket product

$$[\eta_1, \sigma] \in L^1,$$

is a well-defined current in $\mathscr{D}^1(X, E)$.

Now we show that

$$\nabla'[\eta_1, \sigma] = -[\eta_1, \eta_2] \qquad \text{on } X$$

in the sense of currents. Let μ_ε be the standard cut-off functions with respect to $X \setminus D$, namely

$$\mu_\varepsilon = \rho_\varepsilon(\log\log \frac{1}{|s_D|_h^2}),$$

where s_D is the canonical section of $O_X(D)$, h is a smooth metric, ρ_ε is a function which is equal to 1 on $[1, \varepsilon^{-1}]$ and is equal to 0 on $[1 + \varepsilon^{-1}, +\infty[$ and $|\rho'_\varepsilon| \leq 2$.

Let $\beta \in A^{2n-2}(X, E^*)$. Then

$$\int_X [\eta_1, \sigma] \wedge (\nabla' \beta) = \lim_{\varepsilon \to 0} \int_X \mu_\varepsilon \cdot [\eta_1, \sigma] \wedge (\nabla' \beta)$$

$$= -\lim_{\varepsilon \to 0} \int_X \partial \mu_\varepsilon \wedge [\eta_1, \sigma] \wedge \beta - \lim_{\varepsilon \to 0} \int_X \mu_\varepsilon \cdot \nabla'[\eta_1, \sigma] \wedge \beta.$$

We now compute the two terms on the right-hand side. Note that on the support of μ_ε, η_1 and σ are smooth. We have thus $\mu_\varepsilon \cdot \nabla'[\eta_1, \sigma] = -\mu_\varepsilon[\eta_1, \eta_2]$. Therefore we have

$$-\lim_{\varepsilon \to 0} \int_X \mu_\varepsilon \cdot \nabla'[\eta_1, \sigma] \wedge \beta = \lim_{\varepsilon \to 0} \int_X \mu_\varepsilon [\eta_1, \eta_2] \wedge \beta = \int_X [\eta_1, \eta_2] \wedge \beta.$$

For the other term $\lim_{\varepsilon \to 0} \int_X \partial \rho_\varepsilon \wedge [\eta_1, \sigma] \wedge \beta$, note that

$$\partial \mu_\varepsilon \wedge \eta_1 = \rho'_\varepsilon \cdot \frac{1}{\log \frac{1}{|s_D|_h^2}} (\partial(\log |s_D|_h^2) \wedge \eta_1).$$

As $\partial(\log |s_D|_h^2) \wedge \eta_1 \in A^1(X, D, E)$ and $\sigma \in L^{2+c_0}$ for some $c_0 > 0$, we know that

$$\partial(\log |s_D|_h^2) \wedge \eta_1 \cdot \sigma \in L^1.$$

Then by Lebesgue's dominated convergence theorem, we have

$$\lim_{\varepsilon \to 0} \int_X \partial \mu_\varepsilon \wedge [\eta_1, \sigma] \wedge \beta = 0.$$

As a consequence, we obtain

$$\int_X [\eta_1, \sigma] \wedge (\nabla' \beta) = -\int_X [\eta_1, \eta_2] \wedge \beta.$$

Thus, $[\eta_1, \eta_2] = \nabla' (-[\eta_1, \sigma])$ in the sense of currents. The lemma is thus proved. □

Corollary 5.9. — *Let $\eta_1, \eta_2 \in A^1(X, D, E)$. We assume that η_1 is ∇'-closed and η_2 is ∇'-exact, and $\operatorname{Res}_D \eta_2 = 0$. Then $\operatorname{Res}_D(\eta_1 \wedge \eta_2)$ is ∇'-exact.*

Proof. — By Lemma 5.7 and (5.7.1), we can check that, for each D_i, we have

$$\operatorname{Res}_{D_i}(\eta_1 \wedge \eta_2) = \operatorname{Res}_{D_i}(\eta_1) \wedge (\eta_2|_{D_i}) - (\eta_1|_{D_i}) \wedge \operatorname{Res}_{D_i}(\eta_2) = \operatorname{Res}_{D_i}(\eta_1) \wedge (\eta_2|_{D_i}), \tag{5.9.1}$$

where $\eta_1|_{D_i}$ and $\eta_2|_{D_i}$ are defined in (5.7.2). By Lemma 5.7 again, $\eta_2|_{D_i}$ is ∇'-exact. It follows that

$$\operatorname{Res}_{D_i}(\eta_1) \wedge (\eta_2|_{D_i})$$

is also ∇'-exact. The corollary is thus proved. □

Proposition 5.10. — *Let $\eta \in A^2(X, D, E)$ such that η is ∇-closed and ∇'-exact in the sense of currents. If we define*

$$\omega := \nabla' \nabla'^* G \nabla''^* G \eta = \nabla''^* G \eta,$$

then $\omega \in A^1(X, D, E)$, and ω is the unique element in $\mathscr{D}^1(X, E)$ that is ∇'-exact, and

$$\nabla \omega = \nabla'' \omega = \eta.$$

Proof. — It is straightforward to see that

$$\omega = \nabla' \nabla'^* G \nabla''^* G \eta = (1 - \mathcal{H}) \nabla''^* G \eta - \nabla'^* \nabla' G \nabla''^* G \eta = \nabla''^* G \eta.$$

If $\nabla \nabla' f = 0$ for some $f \in \mathscr{D}^0(X, E)$, then by elliptic regularity, f is smooth. So f is pluri-harmonic. By the compactness of X, we know that $\nabla' f = 0$. This in particular shows the uniqueness of ω.

Now we would like to show that $\omega \in A^1(X, D, E)$. Write $f := \nabla''^* G \eta$. Fix any point $p \in D$. We take an admissible coordinate $(U; z_1, \ldots, z_n)$ centered at p such that $U \cap D = (z_1 \cdots z_m = 0)$. Let $e_1, \ldots, e_r$ be a flat frame for $E|_U$. Then

$$\nabla'' e_i = \nabla' e_i = 0$$

Let us write $f = \sum_i f_i e_i$ and $\eta = \sum_i \eta_i e_i$ where $f_i \in \mathscr{D}^0(U)$ and $\eta_i \in \mathscr{A}^2(X, D)(U)$. Then we have

$$\bar{\boldsymbol{\partial}} \boldsymbol{\partial} f_i = \eta_i.$$

Therefore, $\eta_i \in \mathscr{A}^{1,1}(X, D)(U)$.

Note that $(\mathscr{A}^{p,\bullet}(X, D), \bar{\partial})$ is a fine resolution for $\Omega_X^p(\log D)$. It follows that there exists some $h_i \in \mathscr{A}^{1,0}(X, D)(U)$ such that

$$\bar{\partial} h_i = \eta_i.$$

Since $\operatorname{Res}_D \eta_i = 0$, it follows that $\operatorname{Res}_{D_j} \bar{\partial} h_i = 0$, which implies that

$$\bar{\partial} \operatorname{Res}_{D_j} h_i = 0.$$

Hence $\operatorname{Res}_{D_j} h_i$ is a holomorphic function on $D_j \cap U$. Define a holomorphic function $g_{i,j} \in \mathcal{O}(U)$ such that $g_{i,j}|_{D_j} = \operatorname{Res}_{D_j} h_i$. We replace h_i by

$$h_i - \sum_{j=1}^m g_{i,j} \frac{dz_j}{z_j}.$$

Then $\operatorname{Res}_D h_i = 0$. It follows from Lemma 5.1 that

$$\bar{\boldsymbol{\partial}} h_i = \bar{\partial} h_i = \eta_i.$$

Therefore, we have

$$\bar{\boldsymbol{\partial}}(h_i - \boldsymbol{\partial} f_i) = 0.$$

By the regularity of $\bar{\partial}$-operator, it follows that $h_i - \boldsymbol{\partial} f_i$ is a holomorphic 1-form. Hence $\boldsymbol{\partial} f_i \in \mathscr{A}^1(X, D)(U)$. Therefore, we obtain $\omega = \nabla' f \in A^1(X, D, E)$. □

Following [CDHP25, LRW19], we prove an effective estimate of the solution in Proposition 5.10. To begin with, we consider the vector bundle

$$E_p := \Omega_X^p(\log D) \otimes E$$

on X. We have an identification

$$\phi : A^i(X, D, E) \simeq \oplus_{p+q=i} C^\infty_{(0,q)}(X, E_p)$$

We equip the vector bundle E_p with a fixed smooth hermitian metric h_p, and we fix a Kähler metric ω_X on X. For any constant $k \in \mathbb{N}$, and any $\alpha \in A^i(X, D, E)$, we define

$$\|\alpha\|_{W^k} := \|\phi(\alpha)\|_{W^k}, \tag{5.10.1}$$

where $\|\phi(\alpha)\|_{W^k}$ is the norm of the Sobolev space whose derivatives up to order k are L^2 with respect to h_p and ω_X. We have the following effective version of Proposition 5.10.

Corollary 5.11. — *Let $\eta \in A^2(X, D, E)$ such that η is $\boldsymbol{\nabla}$-closed and $\boldsymbol{\nabla}'$-exact in the sense of currents. Fix $k \in \mathbb{N}^*$. Let $\omega \in A^1(X, D, E)$ be the unique $\boldsymbol{\nabla}'$-exact solution of*

$$\boldsymbol{\nabla}\omega = \eta \qquad \text{on } X$$

as in Proposition 5.10. Then there exists a constant C (depending only on k) such that

$$\|\omega\|_{W^k} \le C\|\eta\|_{W^k}.$$

Proof. — We follow ideas from the proof of [CDHP25, Thm 8.7]. Suppose by contradiction that there is no such uniform constant C. Then there exists a sequence of $\eta_i \in A^2(X, D, E)$ and $\omega_i \in A^1(X, D, E)$ such that $\omega_i \in \operatorname{Im} \boldsymbol{\nabla}'$,

$$\boldsymbol{\nabla}\omega_i = \eta_i \qquad \text{on } X,$$

$\|\omega_i\|_{W^k} = 1$ for every i, and $\lim_{i\to+\infty} \|\eta_i\|_{W^k} \to 0$.

After passing to some subsequence, we may assume that ω_i converges weakly to some ω in W^k. Then $\boldsymbol{\nabla}\omega = 0$, and by the closeness of $\operatorname{Im} \nabla'$ in the Hodge decomposition, it follows that $\omega \in \operatorname{Im} \nabla'$. Therefore $\omega = 0$.

For every open set $U \Subset X \setminus D$, by elliptic regularity, we know that $\omega_i|_U$ converges strongly to $\omega|_U = 0$ in $W^k(U)$. Near the boundary D, we suppose that D is defined locall by $\prod_{s\in I} z_s = 0$. Then ω_i is locally of type

$$\omega_i = \sum_{s\in I} \frac{dz_s}{z_s} f_{i,s} + g_{i,s},$$

where $g_{i,s}$ is $(0,1)$-type or $(1,0)$-type with dz_s for $s \notin I$. By using the fact that $\boldsymbol{\nabla}'\omega_i = 0$ and $\boldsymbol{\nabla}\omega_i \to 0$, we know that

$$\bar\partial f_{i,s} \to 0$$

in the W^k sense. Together with the fact that $f_{i,s} \to 0$ on $U \Subset X \setminus D$ in W^k, we know that $f_{i,s} \to 0$ near D in the W^k sense. For the term $g_{i,s}$, by the same argument, we know also that $g_{i,s} \to 0$ near D in the W^k sense. As a consequence, we know that $\omega_i \to 0$ in W^k. This contradicts the fact that $\|\omega_i\|_{W^k} = 1$ for every i. □

Lemma 5.12. — *Consider $\eta_1, \eta_2 \in A^1(X, D, E)$. Assume that*

- *$\boldsymbol{\nabla}\eta_i = \nabla\eta_i$ for $i = 1, 2$.*
- *η_1, η_2 are both $\boldsymbol{\nabla}'$-exact.*

Then

(i) $\boldsymbol{\nabla}[\eta_1, \eta_2] = \nabla[\eta_1, \eta_2]$, *or equivalently,* $\operatorname{Res}_D[\eta_1, \eta_2] = 0$.
(ii) $[\eta_1, \eta_2]$ *is $\boldsymbol{\nabla}'$-exact.*

Proof. — Fix any point $p \in D$. We take admissible coordinates $(U; z_1, \ldots, z_n)$ centered at p such that $U \cap D = (z_1 \cdots z_m = 0)$. With these coordinates, we can write

$$\eta_1 = \sum_{i=1}^m \frac{dz_i}{z_i} \otimes a_i + \sum_{j=m+1}^n dz_j \otimes e_j$$

$$\eta_2 = \sum_{i=1}^{m} \frac{dz_i}{z_i} \otimes b_i + \sum_{j=m+1}^{n} dz_j \otimes f_j$$

where $a_i, b_i, e_j, f_j \in A^0(U, E)$. Since $\boldsymbol{\nabla}\eta_i = \nabla\eta_i$, by Lemma 5.1, we have $\iota_i^* a_i = 0$ and $\iota_i^* b_i = 0$ for any $i \in \{1, \ldots, m\}$. Here $\iota_i : D_i \to X$ is the inclusion map. Note that

$$\begin{aligned}[\eta_1, \eta_2] = &\sum_{1\le i<j\le m} ([a_i, b_j] - [a_j, b_i]) \frac{dz_i}{z_i} \wedge \frac{dz_j}{z_j} \\ &+ \sum_{1\le i\le m<j\le n} ([a_i, f_j] - [e_j, b_i]) \frac{dz_i}{z_i} \wedge dz_j + \sum_{m\le i<j\le n} ([e_i, f_j] - [e_j, f_i]) dz_i \wedge dz_j\end{aligned}$$

It follows that $\mathrm{Res}_D[\eta_1, \eta_2] = 0$. This implies that $\boldsymbol{\nabla}[\eta_1, \eta_2] = \nabla[\eta_1, \eta_2]$ by Lemma 5.1. The first item is proved.

By Lemma 5.7 and our assumption, there exists $\sigma_i \in A^0_{\mathrm{cts}}(X, E)$ such that $\boldsymbol{\nabla}'\sigma_i = \eta_i$. Therefore, $[\sigma_1, \eta_2] \in \mathscr{D}^1(X, E)$, and by the proof of Lemma 5.8, we have

$$[\eta_1, \eta_2] = \boldsymbol{\nabla}'[\sigma_1, \eta_2].$$

Hence $[\eta_1, \eta_2]$ is $\boldsymbol{\nabla}'$-exact. The lemma is thus proved. □

Lemma 5.13. — *Let $\eta = f(\beta) \in A^1(X, D, E)$ for some $\beta \in H^0(D^{(1)}, E)$, where f is defined in* (5.4.3)*. Assume also that for any $i < j$, we have*

$$[\iota_{ij}^*(\beta|_{D_i}), \iota_{ji}^*(\beta|_{D_j})] = 0, \tag{5.13.1}$$

where we denote by $\iota_{ij} : D_{ij} \to D_i$ the natural map. Then there exists some $h \in \mathscr{D}^0(D^{(1)}, E)$ such that

$$\mathrm{Res}_D[\eta, \eta] = \boldsymbol{\nabla}' h \tag{5.13.2}$$

$$\boldsymbol{\nabla}\, \mathrm{Res}_D[\eta, \eta] = 2[\beta, \iota^*\nabla\eta]. \tag{5.13.3}$$

In particular, if $g(\beta) = 0$, where g is the Gysin morphism, then we have

$$\begin{aligned}\mathrm{Res}_D[\eta, \eta] &= 0\\ \boldsymbol{\nabla}[\eta, \eta] &= 0.\end{aligned}$$

Proof. — We keep the notations from § 5.3 and do not recall their meaning. By (5.4.4), there exists some $\gamma \in A^0(X, E)$ such that we have

$$\begin{aligned}f(\beta) &:= \frac{\sqrt{-1}}{2\pi} \boldsymbol{\nabla}'(\sum_{i=1}^{m} \varrho_i \log |s_i|^2_{h_i} \cdot \beta_i + \gamma) \\ &= \frac{\sqrt{-1}}{2\pi} (\sum_{i=1}^{m} \partial(\varrho_i \log |s_i|^2_{h_i}) \beta_i + \nabla'\gamma)\end{aligned}$$

We then have

$$-4\pi^2[\eta, \eta] = \sum_{i\ne j} \partial(\varrho_i \log |s_i|^2_{h_i}) \wedge \partial(\varrho_j \log |s_j|^2_{h_j}) [\beta_i, \beta_j] + 2[\sum_{i=1}^{m} \partial(\varrho_i \log |s_i|^2_{h_i}) \beta_i, \nabla'\gamma] + [\nabla'\gamma, \nabla'\gamma].$$

It follows that

$$\begin{aligned}
\mathrm{Res}_{D_i}[\eta,\eta] &= \frac{-\sqrt{-1}}{\pi}\left([\beta|_{D_i}, \iota_i^*(\sum_{j\neq i}\partial(\varrho_j\log|s_j|^2_{h_j})\beta_j+\nabla'\gamma)]\right)\\
&= \frac{-\sqrt{-1}}{\pi}\left([\beta|_{D_i}, \iota_i^*(\sum_{j=1}^{m}\partial(\varrho_j\log|s_j|^2_{h_j})\beta_j+\nabla'\gamma)]\right)\\
&= \frac{-\sqrt{-1}}{\pi}\left([\beta|_{D_i}, \iota_i^*\eta]\right) \qquad (5.13.4)\\
&= \frac{-\sqrt{-1}}{\pi}\boldsymbol{\nabla}'\left([\beta|_{D_i}, \iota_i^*(\sum_{j=1}^{m}\varrho_j\log|s_j|^2_{h_j}\beta_j+\gamma)]\right) \qquad (5.13.5)
\end{aligned}$$

This proves (5.13.2).

Let us prove (5.13.3). Recall that we have assumed that $\mathrm{Res}_{D_{ij}}[\eta,\eta]=0$ for any $1\leq i<j\leq m$. By (5.13.4) and (5.13.1), this implies that for any fixed $j\neq i$, we have

$$\mathrm{Res}_{D_j\cap D_i}\mathrm{Res}_{D_i}[\eta,\eta]=2[\iota_{ij}^*(\beta|_{D_i}),\iota_{ji}^*(\beta|_{D_j})]=0$$

for each $j>i$. Lemma 5.1 together with (5.13.5) imply that

$$\boldsymbol{\nabla}\,\mathrm{Res}_{D_i}[\eta,\eta]=\boldsymbol{\nabla}''\,\mathrm{Res}_{D_i}[\eta,\eta]=\nabla''\,\mathrm{Res}_{D_i}[\eta,\eta]=2[\beta,\iota^*\nabla\eta].$$

Equation (5.13.3) is thus proved.

If $g(\beta)=0$, then by the construction of f, we have $\nabla\eta=0$ and $\boldsymbol{\nabla}\eta=\iota_*(\beta)$. By (5.13.3),

$$\boldsymbol{\nabla}\,\mathrm{Res}_D[\eta,\eta]=0.$$

By (5.13.2),

$$\Delta' h=\sqrt{-1}\Lambda\boldsymbol{\nabla}''\boldsymbol{\nabla}'h=\sqrt{-1}\Lambda\boldsymbol{\nabla}\boldsymbol{\nabla}'h=0.$$

Hence

$$\mathrm{Res}_D[\eta,\eta]=\boldsymbol{\nabla}'h=0.$$

The last claim is proved. The lemma is proved. □

5.5. Explicit construction of the deformation. —

Theorem 5.14. — *Let X be a compact Kähler manifold and let D be a simple normal crossing divisor on X. Let (V,∇) be a unitary bundle on X. Denote by $(E,\nabla):=(\mathrm{End}(V),\nabla)$. Assume that there exist $\eta_1\in\mathcal{H}^1(X,E)$ and $\beta\in H^0(D^{(1)},E)$ (it might happen that $\eta_1=0$ or $\beta=0$) such that*

(a) $g(\beta)+\frac{1}{2}\{[\eta_1,\eta_1]\}=0$, where $g: H^0(D^{(1)},E)\to H^2(X,E)$ is the Gysin morphism.
(b) $\{[\iota^*\eta_1,\beta]\}=0\in H^1(D^{(1)},E)$, where $\iota: D^{(1)}\to X$ is the natural morphism.
(c) For any $i\neq j$, $[\iota_{ij}^*(\beta|_{D_i}),\iota_{ji}^*(\beta|_{D_j})]=0\in H^0(D_i\cap D_j,E)$, where $\iota_{ij}: D_i\cap D_j\to D_i$ is the natural morphism.

Then

(i) *for every $i\geq 2$, there exists $\eta_i\in A^1(X,D,E)\cap\mathrm{Im}\,\boldsymbol{\nabla}'$ such that they solve the following equations:*

–$\mathrm{Res}_D\,\eta_2=-\beta$ *and*

$$\boldsymbol{\nabla}\eta_2=-\iota_*(\beta)-\eta_1\wedge\eta_1 \tag{5.14.1}$$

–*For every $k\geq 3$, we have*

$$\eta_k:=-\boldsymbol{\nabla}''^*G(\sum_{i=1}^{k-1}\eta_i\wedge\eta_{k-i}),$$

such that

$$\nabla\eta_k = \sum_{i=1}^{k-1} \eta_i \wedge \eta_{k-i}. \tag{5.14.2}$$

In particular, we have

$$\mathrm{Res}_D\, \eta_k = 0 \quad \textit{for } k \geq 3. \tag{5.14.3}$$

(ii) *There exists some $\varepsilon > 0$ such that for any $t \in \mathbb{D}_\varepsilon$, we have*

$$\sum_{i=1}^{\infty} t^i \eta_i \in A^1(X, D, E).$$

As a consequence, we have an analytic *famlly of* logarithmic connections ∇_t *for* V *parametrized by $t \in \mathbb{D}_\varepsilon$ in the following explicit way:*

$$\nabla_t := \nabla + \sum_{i=1}^{\infty} t^i \eta_i,$$

with its $(0,1)$-part given by

$$\nabla_t'' = \nabla'' + t\eta_1^{0,1}.$$

(iii) *Let $f : Y \to X$ be a holomorphic map from a compact Kähler manifold Y such that $D_Y := f^{-1}(D)$ is a simple normal crossing divisor. Let f_0 denote the restriction $f|_{Y_0}$. If $f^*\eta_1 = 0$ and $f^*\eta_2 = 0$, then $f_0^*\nabla_t = f_0^*\nabla$ for all $t \in \mathbb{D}_\varepsilon$.*

Proof. — **Step 1.** *Construction η_2.*

Consider the map

$$f : H^0(D^{(1)}, E) \to A^1(X, D, E) \tag{5.14.4}$$

defined in (5.4.3). Then by (5.4.4), we have $\mathrm{Res}_D\, f(\beta) = -\beta$, $f(\beta)$ is ∇'-exact and

$$\nabla f(\beta) = -\mathcal{H}(g(\beta)) - \iota_*(\beta).$$

We define

$$\eta_2 := f(\beta) - \frac{1}{2}\nabla''^* G[\eta_1, \eta_1].$$

Then we have

$$\nabla\eta_2 = -\mathcal{H}(g(\beta)) - \iota_*(\beta) + (\mathcal{H} - 1)\frac{1}{2}[\eta_1, \eta_1].$$

Item (a) implies that,

$$\nabla\eta_2 = -\iota_*(\beta) - \frac{1}{2}[\eta_1, \eta_1]. \tag{5.14.5}$$

It remains to check that $\eta_2 \in A^1(X, D, E)$. Indeed,

$$\nabla(\eta_2 - f(\beta)) = -\frac{1}{2}[\eta_1, \eta_1] + \mathcal{H}(g(\beta)) \in A^2(X, E).$$

As $\eta_2 + f(\beta) \in \mathrm{Im}\,\nabla'$, by Lemma 2.16, we know that $\eta_2 + f(\beta) \in A^1(X, E)$. Therefore $\eta_2 \in A^1(X, D, E)$. Since $f(\beta)$ is ∇'-exact, and

$$\nabla''^* G[\eta_1, \eta_1] = \nabla'\nabla'^* G\nabla''^* G[\eta_1, \eta_1],$$

it follows that η_2 is ∇'-exact.

Step 2. *Constructing η_3.* Since $\eta_2 \in \mathrm{Im}\,\nabla'$, and η_1 is harmonic, we know that

$$[\eta_1, \eta_2] \in \mathrm{Im}\,\nabla' \cap W_1 A^2(X, D, E).$$

Note that

$$\nabla[\eta_1, \eta_2] = \frac{1}{2}[\eta_1, [\eta_1, \eta_1]] = 0 \tag{5.14.6}$$

by the Jacobian identity. We also have

$$\mathrm{Res}_D[\eta_1,\eta_2] = [\iota^*\eta_1,\beta] = 0 \in H^1(D^{(1)},E),$$

where the last equality comes from Item (b). Then by Lemma 5.1, $[\eta_1,\eta_2] \in \mathrm{Im}\,\nabla' \cap \ker\nabla \cap A^2(X,D,E)$.

Let us define

$$\eta_3 := -\nabla''^* G[\eta_1,\eta_2].$$

Applying Lemma 2.16 and Proposition 5.10 to $[\eta_1,\eta_2]$, it follows that $\eta_3 \in A^1(X,D,E)\cap \mathrm{Im}\,\nabla'$, and that

$$\nabla\eta_3 = -[\eta_1,\eta_2]. \tag{5.14.7}$$

In particular, this implies that

$$\mathrm{Res}_D[\eta_1,\eta_2] = 0. \tag{5.14.8}$$

Step 3. *Constructing η_4.*

We consider $\sum_{i=1}^3 \eta_i\wedge\eta_{4-i}$. Thanks to Lemma 5.8, $\sum_{i=1}^3 \eta_i\wedge\eta_{4-i}$ is ∇'-exact.

Now we show that $\sum_{i=1}^3 \eta_i\wedge\eta_{4-i}$ is ∇-closed.

$$\nabla(\frac{1}{2}[\eta_2,\eta_2]+[\eta_1,\eta_3]) = [\eta_2,\eta_2]+[\eta_1,\eta_3] = -\frac{1}{2}[\eta_2,[\eta_1,\eta_1]]+[\eta_1,-[\eta_1,\eta_2]] = 0 \tag{5.14.9}$$

By the Jacobian identity, we have

$$[\eta_2,[\eta_1,\eta_1]] - [\eta_1,[\eta_2,\eta_1]] + [\eta_1,[\eta_1,\eta_2]] = 0 \tag{5.14.10}$$

By Lemma 5.1, $\mathrm{Res}_D\,\eta_3 = 0$. Hence

$$\mathrm{Res}_D[\eta_1,\eta_3] = 0.$$

Hence we have

$$\nabla(\frac{1}{2}[\eta_2,\eta_2]+[\eta_1,\eta_3]) = \frac{1}{2}\iota_*(\mathrm{Res}_D[\eta_2,\eta_2]). \tag{5.14.11}$$

Using Item (c) and by Lemma 5.13, we know that $\mathrm{Res}_{D_i}[\eta_2,\eta_2] \in \mathrm{Im}\,\nabla'$ and

$$\begin{aligned}\nabla\,\mathrm{Res}_{D_i}[\eta_2,\eta_2] &= 2[\beta_i,\iota^*\nabla\eta_2]\\ &= -[\beta_i,\iota^*[\eta_1,\eta_1]]\\ &= 2[\iota^*\eta_1,[\beta_i,\iota^*\eta_1]]\\ &\overset{(5.14.8)}{=} 0.\end{aligned}$$

Hence we have $\mathrm{Res}_{D_i}[\eta_2,\eta_2] \in \mathrm{Im}\,\nabla'\cap\ker\nabla$. By Lemma 2.16 and the fact that $\mathrm{Res}_{D_i}[\eta_2,\eta_2]$ is a 1-form, we know that $\mathrm{Res}_{D_i}[\eta_2,\eta_2]=0$. Therefore, we have

$$\nabla(\sum_{i=1}^3 \eta_i\wedge\eta_{4-i}) = 0. \tag{5.14.12}$$

Since $\sum_{i=1}^3 \eta_i\wedge\eta_{4-i}$ is ∇'-exact, we have

$$\sum_{i=1}^3 \eta_i\wedge\eta_{4-i} \in \mathrm{Im}\,\nabla'\cap\ker\nabla\cap A^1(X,D,E).$$

Let us define

$$\eta_4 := -\nabla''^* G(\sum_{i=1}^3 \eta_i\wedge\eta_{4-i}).$$

By Proposition 5.10, $\eta_4 \in A^1(X,D,E)\cap\mathrm{Im}\,\nabla'$ and

$$\nabla\eta_4 = -(\sum_{i=1}^3 \eta_i\wedge\eta_{4-i}). \tag{5.14.13}$$

In particular, $\mathrm{Res}_D\,\eta_4 = 0$.

Step 4. *Constructing general η_k for $k \geq 5$.* The construction of higher order η_k is similar and we continue this process by induction. Assume that for some $k \geq 5$, we obtain a sequence of $\eta_3, \ldots, \eta_{k-1} \in A^1(X, D, E)$ such that for any $i \in \{3, \ldots, k-1\}$, we have

(a) $\eta_i \in \mathrm{Im}\,\boldsymbol{\nabla}'$
(b) $\boldsymbol{\nabla}\eta_i = -\sum_{j=1}^{i-1} \eta_j \wedge \eta_{i-j}$
(c) $\mathrm{Res}_D\,\eta_i = 0$.

Let us define

$$\gamma_k := -\sum_{j=1}^{k-1} \eta_j \wedge \eta_{k-j}.$$

It follows from Items (a) and (b) together with Lemma 5.8 that γ_k is $\boldsymbol{\nabla}'$-exact. Now we prove that γ_k is $\boldsymbol{\nabla}$-closed. First, by indcution ((b)), we have

$$\begin{aligned}
-\nabla\gamma_k &= \sum_{j=1}^{k-1} \nabla\eta_j \wedge \eta_{k-j} - \sum_{j=1}^{k-1} \eta_j \wedge \nabla\eta_{k-j} \\
&= -\sum_{j=1}^{k-1}\sum_{i=1}^{j-1} \eta_i \wedge \eta_{j-i-1} \wedge \eta_{k-j} + \sum_{j=1}^{k-1}\sum_{i=1}^{k-j-1} \eta_j \wedge \eta_i \wedge \eta_{k-j-i-1} \\
&= -\sum_{i+j+s=k-1} \eta_i \wedge \eta_j \wedge \eta_s + \sum_{i+j+s=k-1} \eta_i \wedge \eta_j \wedge \eta_s = 0,
\end{aligned}$$

where the index $i, j, s \geqq 1$.

By (5.9.1), for each D_i, we have

$$\begin{aligned}
\mathrm{Res}_{D_i}\,\gamma_k &= -\mathrm{Res}_{D_i}\,\eta_2 \wedge \eta_{k-2} + \eta_{k-2} \wedge \mathrm{Res}_{D_i}\,\eta_2 \\
&= -\beta_i \wedge (\eta_{k-2}|_{D_i}) + (\eta_{k-2}|_{D_i}) \wedge \beta_i.
\end{aligned}$$

Here $(\eta_{k-2}|_{D_i})$ is defined in (5.7.2). By Item (c) together with (5.7.3), we have

$$\mathrm{Res}_{D_i \cap D_j}(\eta_{k-2}|_{D_i}) = 0$$

for each $j \neq i$. This implies that

$$\mathrm{Res}_{D_j \cap D_i}\,\mathrm{Res}_{D_i}\,\gamma_k = 0. \tag{5.14.14}$$

for each $j \neq i$.

As $(\eta_{k-2}|_{D_i})$ is $\boldsymbol{\nabla}'$-exact by (5.7.2), $\beta_i \wedge (\eta_{k-2}|_{D_i})$ is $\boldsymbol{\nabla}'$-exact. It follows that $\mathrm{Res}_{D_i}\,\gamma_k$ is also $\boldsymbol{\nabla}'$-exact. Note that

$$\nabla\,\mathrm{Res}_{D_i}\,\gamma_k = -\mathrm{Res}_{D_i}(\nabla\gamma_k) = 0.$$

By Lemma 5.1 together with (5.14.14),

$$\boldsymbol{\nabla}\,\mathrm{Res}_{D_i}\,\gamma_k = 0$$

for each i. Then

$$\mathrm{Res}_{D_i}\,\gamma_k \in \mathrm{Im}\,\boldsymbol{\nabla}' \cap \ker\boldsymbol{\nabla}.$$

Since $\mathrm{Res}_{D_i}\,\gamma_k \in A^1(D_i, \sum_{j\neq i} D_j \cap D_i, E)$, by Lemma 2.16, $\mathrm{Res}_{D_i}\,\gamma_k = 0$. It follows from Lemma 5.1 that γ_k is $\boldsymbol{\nabla}$-closed. Hence we have

$$\gamma_k \in \mathrm{Im}\,\boldsymbol{\nabla}' \cap \ker\boldsymbol{\nabla}.$$

Let us define

$$\eta_k := \boldsymbol{\nabla}''^{*}G\gamma_k.$$

Then by Proposition 5.10, $\eta_k \in A^1(X, D, E) \cap \mathrm{Im}\,\boldsymbol{\nabla}'$ and we have

$$\boldsymbol{\nabla}\eta_k = -\sum_{j=1}^{k-1} \eta_j \wedge \eta_{k-j}.$$

In particular, $\mathrm{Res}_D\,\eta_k = 0$. Therefore, Items (a) to (c) all hold for η_k. So the induction is proved. We conclude the existence of each η_k that satisfies the above conditions. This proves Item (i).

Step 5. We would like to show the convergence of $\sum_i t^i\eta_i$ for $|t| \ll 1$.

By our construction, for $k \geq 3$, η_k is the unique solution such that η_k is $\boldsymbol{\nabla}'$-exact and satisfying

$$\boldsymbol{\nabla}\eta_k = \sum_{i=1}^{k-1} \eta_i \wedge \eta_{k-i}.$$

By using Corollary 5.11, there exists some *uniform* constant $C > 1$ such that

$$\|\eta_k\|_{W^1} \leq C(\sum_{i=1}^{k-1} \|\eta_i\|_{W^1} \cdot \|\eta_{k-i}\|_{W^1}),$$

where the W^1-norm is defined in (5.10.1). Therefore by induction and the recurrence formula for Catalan numbers, we have

$$\|\eta_k\|_{W^1} \leq \sum_{n=\lceil k/2\rceil}^{k} \frac{1}{n}\binom{2n-2}{n-1} C^{n-1} \binom{n}{k-n} \|\eta_1\|_{W^1}^{2n-k} \|\eta_2\|_{W^1}^{k-n}.$$

This yields the following estimate valid for all $k \geq 3$:

$$\|\eta_k\|_{W^1} \leq \frac{1}{2C}\Lambda^k. \tag{5.14.15}$$

where

$$\Lambda = 2C\|\eta_1\|_{W^1} + \sqrt{4C^2\|\eta_1\|_{W^1}^2 + 4C\|\eta_2\|_{W^1}}.$$

Then $\sum_i t^i\eta_i$ converges in the W^1-normed space for $|t| \leq \frac{1}{2\Lambda}$.

Set $\eta(t) := \sum_{i=1}^{\infty} t^i\eta_i$. It remains to show that $\sum_{i=1}^{\infty} t^i\eta_i \in A^1(X, D, E)$ for $|t| \ll 1$. By the above paragraph, we know that $\|\eta(t)\|_{W^1} < +\infty$ for $|t| \ll 1$. Moreover, by construction, we have

$$\nabla\eta(t) + \eta(t) \wedge \eta(t) = 0.$$

Moreover, as $\boldsymbol{\nabla}'\eta_i = 0$ for every i, we know that and $\nabla'\eta(t) = 0$.

Then by the standard elliptic bootstrapping argument, we know that $\eta(t) \in A^1(X, D, E)$. To see this, we suppose by induction that $\eta(t)$ is in W^k for some $k \in \mathbb{N}$ in the sense of (5.10.1). We suppose that D is locally defined by $\prod_{i\in I} z_i = 0$. Then $\eta(t)$ can be locally written as

$$\eta(t) = \sum_{i\in I} f_i \frac{dz_i}{z_i} + g$$

where f_i is E-valued function in W^k and $g \in W^k$ is of $(0,1)$-type or $(1,0)$ not containing dz_i for $i \in I$. As $\nabla(\eta(t))$ and $\nabla'\eta(t)$ are in W^k, we know that $\nabla''\eta(t)$ is also in W^k. This implies that $\bar\partial f_i \in W^k$. Then $f_i \in W^{k+1}$ by elliptic regularity. The term g is also in W^{k+1} by the classical elliptic estimates. Therefore $\eta(t)$ is in W^{k+1}. It follows that $\eta(t) \in W^k$ for every $k \in \mathbb{N}$. Then $\eta(t) \in A^1(X, D, E)$. Furthermore, since $\eta_i \in A^1(X, D, E) \cap \mathrm{Im}\,\boldsymbol{\nabla}'$ for $i \geq 2$, we have

$$\nabla''_t = \nabla'' + t\eta_1^{0,1}.$$

This proves Item (ii).

Step 6. We now prove Item (iii). It suffices to show that $f^*\eta_k = 0$ for all $k \geq 2$. We proceed by induction on k. The base cases $k = 1, 2$ hold by assumption. Assume that $f^*\eta_i = 0$ for all $1 \leq i \leq k-1$. Then by (5.14.2), we have

$$\boldsymbol{\nabla} f^*\eta_k = -\sum_{j=1}^{k-1} f^*\eta_j \wedge f^*\eta_{k-j} = 0. \tag{5.14.16}$$

Since η_k is $\boldsymbol{\nabla}'$-exact, its pullback $f^*\eta_k$ lies in $\mathrm{Im}\,\boldsymbol{\nabla}' \cap A^1(Y, D_Y, f^*E)$. By Lemma 2.16, we conclude that $f^*\eta_k = 0$. This completes the inductive step and the proof of Item (iii).

The theorem is proved. □

5.6. Precise Zigzag between dglas. — Consider the global dgla $(T^\bullet, d)$ defined by the logarithmic Dolbeault complex:

$$(T^\bullet, d) := A^0(X, D, E) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} A^{2\dim X}(X, D, E).$$

Although this dgla is not formal in general, we can nevertheless construct a quasi-isomorphic dgla $(C^\bullet, d)$ where each C^i is a finite-dimensional vector space. This construction enables us to compute the analytic germ that pro-represents the deformation functor associated with $(T^\bullet, d)$. Roughly speaking, the terms of $(C^\bullet, d)$ are formed by the direct sums of the cohomology groups of the divisor strata:

$$C^k := \bigoplus_{p+q=k} H^q(D^{(p)}, E).$$

The differential $d : C^k \to C^{k+1}$ is defined using Gysin morphisms with suitable signs, and the graded Lie bracket is naturally induced by the intersection structure. Precise definitions will be given later in this subsection.

***Lemma 5.15.* —** *Let $L^\bullet := (\ker \nabla', \nabla'')$. Then $\phi : L^\bullet \to (A^\bullet(X, D, E), \nabla)$ is 1-equivalent.*

Proof. — It is obvious that $H^0(\phi)$ is an isomorphism. Note that by the degeneration of E_1-level of Deligne (see also [LRW19]), we have

$$H^1(A^\bullet(X, D, E), \nabla) = H^0(X, \Omega_X(\log D) \otimes E) \oplus H^{0,1}(X, E).$$

Here we endow E with the holomorphic vector bundle structure induced by ∇''. Assume that $\alpha \in A^1(X, D, E)$ such that $\nabla'\alpha = 0$ and $\nabla''\alpha = 0$. We decompose $\alpha = \alpha' + \alpha''$, where $\alpha' \in A^{1,0}(X, D, E)$ and $\alpha'' \in A^{0,1}(X, E)$. Then we have

$$\nabla''\alpha' = 0 \quad \nabla''\alpha'' = 0$$
$$\nabla'\alpha' = 0 \quad \nabla'\alpha'' = 0.$$

This implies that $\alpha' \in H^0(X, \Omega_X(\log D) \otimes E)$ and $\alpha'' \in \mathcal{H}^{0,1}(X, E)$. Hence, ϕ induces an isomorphism for H^1.

Let us prove that $H^2(\phi)$ is injective. Again by the degeneration at the E_1-level, we have

$$H^2(A^\bullet(X, D, E), \nabla) = H^0(X, \Omega_X^2(\log D) \otimes E) \oplus H^{0,2}(X, E) \oplus H^1(X, \Omega_X^1(\log D) \otimes E).$$

Assume that $\alpha \in A^2(X, D, E)$ such that $\nabla'\alpha = 0$ and $\nabla''\alpha = 0$. We decompose $\alpha = \alpha^{2,0} + \alpha^{1,1} + \alpha^{0,2}$, where $\alpha^{i,2-i} \in A^{i,2-i}(X, D, E)$. Then we have

$$\nabla''\alpha^{i,2-i} = 0, \quad \nabla'\alpha^{i,2-i} = 0.$$

This implies that $\alpha^{2,0} \in H^0(X, \Omega_X^2(\log D) \otimes E)$ and $\alpha^{0,2} \in \mathcal{H}^{0,2}(X, E)$.

Assume that $\phi(\alpha) = 0$. Then there exists some $\beta \in A^1(X, D, E)$ such that $\alpha = \nabla\beta$. It follows that

$$\alpha^{2,0} = \nabla'\beta^{1,0}, \quad \alpha^{0,2} = \nabla''\beta^{0,1}, \quad \alpha^{1,1} = \nabla''\beta^{1,0} + \nabla'\beta^{0,1},$$

where $\beta^{1,0} \in A^{1,0}(X, D, E)$ and $\beta^{0,1} \in A^{0,1}(X, E)$. Since

$$\nabla'\nabla''\beta^{0,1} = \nabla'\alpha^{0,2} = 0,$$

by Lemma 2.16, we have

$$\nabla''\beta^{0,1} = 0, \quad \nabla'\beta^{0,1} = \nabla'\nabla''h$$

for some $h \in A^0(X, E)$. Therefore, if we replace $\beta^{0,1}$ by $\beta^{0,1} - \nabla''h$, and $\beta^{1,0} = \beta^{1,0} + \nabla'h$, then we have

$$\alpha^{0,2} = \nabla''\beta^{0,1}, \quad \nabla'\beta^{0,1} = 0.$$

Thus, by replacing α with $\alpha - \nabla''\beta^{0,1}$, we may assume without loss of generality that $\alpha^{0,2} = 0$ and that β is of type $(1, 0)$. It follows that

$$\alpha^{2,0} + \alpha^{1,1} = \nabla\beta^{1,0}.$$

Note that we have

$$\nabla'\nabla''\beta^{1,0} = \nabla'\alpha^{1,1} = 0.$$

This implies that
$$\nabla'\nabla'' \operatorname{Res}_D \beta^{1,0} = \operatorname{Res}_D(\nabla'\nabla''\beta^{1,0}) = 0.$$
Hence $\operatorname{Res}_D \beta^{1,0} \in H^0(D^{(1)}, E)$. By Lemma 5.1, we have
$$\boldsymbol{\nabla}''\beta^{1,0} = \nabla''\beta^{1,0} + \iota_* \operatorname{Res}_D \beta^{1,0}.$$
This implies that
$$\boldsymbol{\nabla}'\boldsymbol{\nabla}''\beta^{1,0} = \boldsymbol{\nabla}'\nabla''\beta^{1,0} + \iota_*\boldsymbol{\nabla}' \operatorname{Res}_D \beta^{1,0} = \boldsymbol{\nabla}'\nabla''\beta^{1,0} = \nabla'\nabla''\beta^{1,0} = 0.$$
By Lemma 2.16, we have
$$\boldsymbol{\nabla}'\beta^{1,0} = \boldsymbol{\nabla}'\boldsymbol{\nabla}'' h$$
for some $h \in \mathscr{D}^1(X, E)$. By the type comparison, it implies that $\boldsymbol{\nabla}'\beta^{1,0} = 0$. Hence $\alpha^{2,0} = 0$, and we have
$$\alpha^{1,1} = \nabla''\beta^{1,0}$$
for some $\beta^{1,0} \in \ker \nabla'$. This proves that the class $\{\alpha\}$ vanishes in $H^2(L^\bullet, d)$, thereby establishing the injectivity of $H^2(\phi)$. This completes the proof of the lemma. □

Let us now introduce a new dgla. Let $D = \sum_{i=1}^m D_i$. Dentoe by $D^{(k)} := \sqcup_{|I|=k} D_I$ and let
$$D_{(k)} := \sqcup_{|I|=k} \sum_{j \notin I} D_I \cap D_j$$
that is a simple normal crossing divisor on $D^{(k)}$.

We first introduce some terminology. For an arbitrary multi-index I, let $\sigma \in S_{|I|}$ be the unique permutation such that $(i_{\sigma(1)}, \dots, i_{\sigma(j)})$ is ordered. We denote by $\sigma(I) := (i_{\sigma(1)}, \dots, i_{\sigma(j)})$ this ordered multi-index. The sign of this permutation by $\varepsilon(I) := \operatorname{sgn}(\sigma)$. We set $C^{p,q} := H^q(D^{(p)}, E)$ and the *differential* $d : C^{p,q} \to C^{p-1,q+2}$ is defined to be the *signed* Gysin morphism as follows: Let I be an ordered multi-index with $|I| = p$. For any $i \in I$, let I_i denote the ordered multi-index corresponding to the set $I \setminus \{i\}$. Then for any element $\alpha \in H^q(D_I, E)$, let
$$g_{I,i} : H^q(D_I, E) \to H^{q+2}(D_{I_i}, E)$$
be the Gysin morphism defined in (5.3). Then we define a differential

$$d\alpha := \sum_{i \in I} (-1)^{\varepsilon(I,i)} g_{I,i}(\alpha), \tag{5.15.1}$$

where $\varepsilon(I, i)$ denotes the number of elements in I that are strictly smaller than i.

For any element $\alpha \in C^k$, we denote by $|\alpha| = k$ its degree.

Lemma 5.16. — *$(C^\bullet, d)$ is a chain complex.*

Proof. — It suffices to verify that $d^2 = 0$. For any ordered multi-index I and any distinct i and j in I, we denote by K the ordered multi-index $I \setminus \{i, j\}$. Then the summmand of $d^2(\alpha)$ in $H^{q+4}(D_K, E)$ is given by
$$(-1)^{\varepsilon(I_i,j)} g_{I_i,j}\left((-1)^{\varepsilon(I,i)} g_{I,i}(\alpha)\right) + (-1)^{\varepsilon(I_j,i)} g_{I_j,i}\left((-1)^{\varepsilon(I,j)} g_{I,j}(\alpha)\right)$$
By Lemma 5.4, we have
$$g_{I_i,j} \circ g_{I,i}(\alpha) = g_{I_j,i} \circ g_{I,j}(\alpha) = \{(\iota_{K,I})_* \eta\}$$
where $\eta \in A^q(D_I, E)$ such that $\nabla\eta = 0$, and is a representative of α, and $\iota_{I,K} : D_I \to D_K$ is the inclusion map. We also note that
$$(-1)^{\varepsilon(I_i,j)}(-1)^{\varepsilon(I,i)} + (-1)^{\varepsilon(I_j,i)}(-1)^{\varepsilon(I,j)} = 0.$$
This implies that $d^2(\alpha) = 0$. Hence $(C^\bullet, d)$ is a chain complex. □

Definition-Lemma 5.17. — We define a bracket

$$C^{p,q} \times C^{k,\ell} \to C^{p+k,q+\ell}$$

as follows. Pick any $\alpha \in A^q(D_I, E)$ and $\beta \in A^\ell(D_J, E)$ with $|I| = p$ and $|J| = k$, that are both ∇-closed. We abusively denote by α and β their cohomolgy classes. We define

$$[\alpha, \beta] := \begin{cases} 0 & \text{if } I \cap J \neq \varnothing \\ (-1)^{\varepsilon(IJ)+q|J|}\{[\iota_1^*\alpha, \iota_2^*\beta]\} & \text{if } I \cap J = \varnothing \end{cases} \tag{5.17.1}$$

Here,

- IJ denotes the multi-index obtained by appending J to I
- K denotes the ordered multi-index corresponding to the set $I \cup J$.
- $\iota_1 : D_K \to D_I$ and $\iota_2 : D_K \to D_J$ denote the inclusion maps.

Then the chain complex $(C^\bullet, d)$ endowed with the above bracket is a dgla.

Proof. — It is straightforward to see that the above bracket is a (graded) Lie algebra. Let us prove the graded Leibniz rule.

Pick any $i \in I$. Let L be the ordered multi-index corresponding to the set $K \backslash \{i\}$. Let $\eta \in A^{q+1}(D_{I_i}, D_I, E)$ be such that $\mathrm{Res}_{D_I} \eta = \alpha$. Let $\iota_3 : D_L \to D_{I_i}$ and $\iota_4 : D_L \to D_J$. Then

$$\iota_3^*\eta \in A^{q+1}(D_L, D_K, E).$$

Hence

$$[\iota_3^*\eta, \iota_4^*\beta] \in A^{q+\ell+1}(D_L, D_K, E)$$

such that

$$\mathrm{Res}_{D_K}[\iota_3^*\eta, \iota_4^*\beta] = [\iota_1^*\alpha, \iota_2^*\beta].$$

It follows that $\nabla[\iota_3^*\eta, \iota_4^*\beta]$ represents

$$g_{K,i}(\{[\iota_1^*\alpha, \iota_2^*\beta]\}) \in H^{q+\ell+2}(D_L, E),$$

and thus

$$g_{K,i}([\alpha, \beta]) = (-1)^{\varepsilon(IJ)+qk}\{\nabla[\iota_3^*\eta, \iota_4^*\beta]\}.$$

On the other hand, note that

$$\nabla[\iota_3^*\eta, \iota_4^*\beta] = [\iota_3^*\nabla\eta, \iota_4^*\beta].$$

Since we have

$$g_{I,i}(\alpha) = \{\nabla\eta\} \in H^{q+2}(D_{I_i}, E),$$

it follows that

$$\{\nabla[\iota_3^*\eta, \iota_4^*\beta]\} = (-1)^{\varepsilon(I_iJ)+(q+2)k}[g_{I,i}(\alpha), \beta].$$

This implies that

$$\begin{aligned} (d[\alpha, \beta])_L &= (-1)^{\varepsilon(K,i)} g_{K,i}([\alpha, \beta]) = (-1)^{\varepsilon(K,i)+\varepsilon(IJ)+qk}\{\nabla[\iota_3^*\eta, \iota_4^*\beta]\}. \\ &= (-1)^{\varepsilon(K,i)+\varepsilon(IJ)+qk+\varepsilon(I_iJ)+(q+2)k}[g_{I,i}(\alpha), \{\beta\}] \\ &= (-1)^{\varepsilon(K,i)+\varepsilon(IJ)+qk+\varepsilon(I_iJ)+(q+2)k+\varepsilon(I,i)}[(d\alpha)_{I_i}, \{\beta\}] \\ &= [(d\alpha)_{I_i}, \{\beta\}]. \end{aligned} \tag{5.17.2}$$

On the other hand, pick any $i \in I$. Let L be the ordered multi-index corresponding to the set $K \backslash \{i\}$. Then

$$\begin{aligned} (d[\alpha, \beta])_L &= (-1)^{|\alpha||\beta|+1}(d[\beta, \alpha])_L \\ &\overset{(5.17.2)}{=} (-1)^{|\alpha||\beta|+1}([(d\beta)_{J_i}, \alpha]) \\ &= (-1)^{|\alpha||\beta|+1}(-1)^{|\alpha|(|\beta|+1)+1}([\alpha, (d\beta)_{J_i}]) \\ &= (-1)^{|\alpha|}([\alpha, (d\beta)_{J_i}]). \end{aligned}$$

Hence, we have

$$d[\alpha, \beta] = [d\alpha, \beta] + (-1)^{|\alpha|}[\alpha, d\beta].$$

This shows the graded Leibniz rule. The lemma is proved. ◻

For any *ordered* multi-index I, we set $\eta_I := \mathrm{Res}_{D_I}\,\eta$. If I not not ordered, then we set

$$\eta_I := (-1)^{\varepsilon(I)}\eta_{\sigma(I)}. \tag{5.17.3}$$

By Lemma 5.1 and (5.17.3), we obtain the identity

$$\boldsymbol{\nabla}\eta_I = \nabla\eta_I + \sum_{i\notin I}(\iota_{I,i})_*\eta_{Ii}, \tag{5.17.4}$$

where $\iota_{I,i}\colon D_I\cap D_i \hookrightarrow D_I$ is the natural inclusion, and the subscript Ii denotes the multi-index formed by appending i to I.

Note that

$$\boldsymbol{\nabla}'\eta_I = \nabla'\eta_I = (-1)^{|I|}\,\mathrm{Res}_{D_I}(\nabla'\eta) = 0. \tag{5.17.5}$$

We are ready to construct the morphism of $\psi : L^\bullet \to C^\bullet$. Consider any $\eta \in L^k$. Define

$$\psi(\eta) = \Big((-1)^{|I|}\{\mathcal{H}(\eta_I)\}\Big)_I \in C^k \tag{5.17.6}$$

where I ranges over all ordered multi-index contained in $\{1,\dots,m\}$, and $\mathcal{H}(I) \in A^{k-|I|}(D_I,E)$ is the harmonic projection of η_I.

Proposition 5.18. — *The map $\psi : L^\bullet \to C^\bullet$ is a morphism of dgla, and is moreover a 1-quasi-isomorphism.*

Proof. — **Step 1.** We first show that ψ is a morphism of chain complexes.

Pick any $\eta \in L^k$. Note that

$$(\nabla''\eta)_I = (-1)^{|I|}(\nabla''\eta_I) \overset{(5.17.4)}{=} (-1)^{|I|}(\boldsymbol{\nabla}\eta_I + \sum_{i\notin I}(\iota_{I,i})_*\eta_{Ii}). \tag{5.18.1}$$

It follows that

$$\mathcal{H}((\nabla''\eta)_I) = (-1)^{|I|}\mathcal{H}(\sum_{i\notin I}(\iota_{I,i})_*\eta_{Ii}).$$

Since

$$\boldsymbol{\nabla}'\eta_I = (\boldsymbol{\nabla}'\eta)_I = 0,$$

it follows that

$$0 = \boldsymbol{\nabla}'\eta_I = \boldsymbol{\nabla}'\boldsymbol{\nabla}'^*\boldsymbol{\nabla}'G\eta_I = \Delta'\boldsymbol{\nabla}'G\eta_I - \boldsymbol{\nabla}'^*\boldsymbol{\nabla}'\boldsymbol{\nabla}'G\eta_I = \Delta'\boldsymbol{\nabla}'G\eta_I$$

Therefore, $\boldsymbol{\nabla}'G\eta_I = 0$, and we have

$$\eta_I = \mathcal{H}(\eta_I) + \boldsymbol{\nabla}'\boldsymbol{\nabla}'^*G\eta_I. \tag{5.18.2}$$

Hence for any $i \notin I$, we have

$$(\iota_{I,i})_*\eta_{Ii} = (\iota_{I,i})_*\mathcal{H}(\eta_{Ii}) + \boldsymbol{\nabla}'(\iota_{I,i})_*\left(\boldsymbol{\nabla}'^*G\eta_{Ii}\right),$$

which yields

$$\mathcal{H}\left((\iota_{I,i})_*\eta_{Ii}\right) = \mathcal{H}\left((\iota_{I,i})_*\mathcal{H}(\eta_{Ii})\right).$$

Therefore, we have

$$\mathcal{H}((\nabla''\eta)_I) = (-1)^{|I|}\sum_{i\notin I}\mathcal{H}((\iota_{I,i})_*\mathcal{H}(\eta_{Ii})).$$

By Lemma 5.4, we have

$$\{\mathcal{H}\left((\iota_{I,i})_*\mathcal{H}(\eta_{Ii})\right)\} = -g_{K(i)}(\{\mathcal{H}(\eta_{Ii})\}) \tag{5.18.3}$$

where we denote by $K(i)$ the ordered multi-index corresponding to the set $I\cup\{i\}$, and

$$g_{K(i)} : H^q(D_{K(i)},E) \to H^{q+2}(D_I,E)$$

be the Gysin morphism defined in (5.3). This implies that

$$\{\mathcal{H}((\nabla''\eta)_I)\} = (-1)^{|I|+1}\sum_{i\notin I}g_{K(i)}(\{\mathcal{H}(\eta_{Ii})\})$$

Hence

$$\psi(\nabla''\eta)_I = -\sum_{i\notin I}g_{K(i)}(\{\mathcal{H}(\eta_{Ii})\})$$

On the other hand, by (5.15.1), we note that

$$\begin{aligned}(d\psi(\eta))_I &= \sum_{i\notin I}(-1)^{\varepsilon(K(i),i)} g_{K(i)}\left((-1)^{|I|+1}\{\mathcal{H}(\eta_{K(i)})\}\right)\\ &\overset{(5.17.3)}{=} \sum_{i\notin I}(-1)^{\varepsilon(K(i),i)+\varepsilon(Ii)+|I|+1} g_{K(i)}\left(\{\mathcal{H}(\eta_{Ii})\}\right)\\ &= -\sum_{i\notin I} g_{K(i)}(\{\mathcal{H}(\eta_{Ii})\})\end{aligned}$$

This implies that

$$d\psi(\eta)=\psi(\nabla''\eta).$$

Hence ψ is a morphism of complexes.

Step 2. We show ψ preserves the Lie brackets.

For any $\eta\in L^k$ and $\sigma\in L^\ell$, we have

$$\mathrm{Res}_{D_K}[\eta,\sigma]=\sum_{I,J}(-1)^{\varepsilon(IJ)+|J|(k-|I|)}[\iota_1^*\eta_I,\iota_2^*\sigma_J]$$

where the sum varies among all ordered multi-indicies I and J such that $I\cap J=\emptyset$, and $K=I\cup J$. Here we denote by $\iota_1:D_K\to D_I$ and $\iota_2:D_K\to D_J$ the inclusion maps.

By (5.18.2), we have

$$\mathcal{H}(\mathrm{Res}_{D_K}[\eta,\sigma])=\sum_{I,J}(-1)^{\varepsilon(IJ)+|J|(k-|I|)}\mathcal{H}\left([\iota_1^*\mathcal{H}(\eta_I),\iota_2^*\mathcal{H}(\sigma_J)]\right).$$

On the other hand, by (5.17.1), we have

$$[\{H(\eta_I)\},\{\mathcal{H}(\sigma_J)\}]=\sum_{I,J}(-1)^{\varepsilon(IJ)+|J|(k-|I|)}\{\left([\iota_1^*\mathcal{H}(\eta_I),\iota_2^*\mathcal{H}(\sigma_J)]\right)\}.$$

which yields

$$[\psi(\eta),\psi(\sigma)]_K=\sum_{I,J}(-1)^{\varepsilon(IJ)+(k-|I|)|J|}\{\left([(-1)^{|I|}\iota_1^*\mathcal{H}(\eta_I),(-1)^{|J|}\iota_2^*\mathcal{H}(\sigma_J)]\right)\},$$

where the sum is also taken as above. It follows that

$$[\psi(\eta),\psi(\sigma)]_K=(-1)^{|I|+|J|}\{\mathcal{H}(\mathrm{Res}_{D_K}[\eta,\sigma])\}=(-1)^{|K|}\{\mathcal{H}(\mathrm{Res}_{D_K}[\eta,\sigma])\}=\psi([\eta,\sigma])_K.$$

Hence ψ preserves the Lie bracket. In conclusion, ψ is a morphism of dglas.

Step 3. We show that $H^i(\psi)$ is an isomorphim for $i=0,1$.

Note that $d:C^0\to C^1$ is the zero map. Therefore, $H^0(\psi):H^0(L^\bullet)\to H^0(C^\bullet)$ is an identity map. Consider $H^1(\psi):H^1(L^\bullet)\to H^1(C^\bullet)$. By the proof of Lemma 5.15, any element $\alpha\in H^1(L^\bullet)$ has a unique representative

$$(\eta^{1,0},\eta^{0,1})\in H^0(X,\Omega_X(\log D)\otimes E)\oplus\mathcal{H}^{0,1}(X,E).$$

Write $\eta:=\eta^{1,0}+\eta^{0,1}$, and $\eta_{(1)}:=\mathrm{Res}_D\,\eta\in H^0(D^{(1)},E)$. Then

$$\nabla\eta^{1,0}=\iota_*\eta_{(1)},$$

and $f(\eta_{(1)})\in A^{1,0}(X,D,E)$ where f defined in (5.4.3) satisfies

$$\nabla f(\eta_{(1)})=\iota_*\eta_{(1)},$$

and $f(\eta_{(1)})$ is ∇'-exact. It follows that

$$\Delta''(\eta^{1,0}-f(\eta_{(1)})=0$$

which yields

$$\eta^{1,0}-f(\eta_{(1)})\in H^0(X,\Omega_X\otimes E).$$

In particular,

$$\eta^{1,0}-f(\eta_{(1)})=\mathcal{H}(\eta^{1,0}-f(\eta_{(1)}))=\mathcal{H}(\eta^{1,0}).$$

Hence we have
$$\psi(\eta) = (\{\eta^{1,0} - f(\eta_{(1)}) + \eta^{0,1}\}, -\{\eta_{(1)}\}).$$
Thus if $\{\psi(\eta)\} = 0$ in $H^1(C^\bullet)$, then
$$\eta_{(1)} = 0, \quad \eta^{1,0} - f(\eta_{(1)}) = 0, \quad \eta^{0,1} = 0.$$
This implies that $\eta = 0$. We obtain the injectivity of $H^1(\psi)$.

Let $(a^{0,1}, a^{1,0}) \in C^{0,1} \oplus C^{1,0}$ such that $d(a^{0,1}, a^{1,0}) = 0$. It follows that
$$g(a^{1,0}) = 0$$
where $g : H^0(D^{(1)}, E) \to H^2(X, E)$ is the Gysin morphism. By the construction of f in (5.4.3), this implies that
$$\nabla f(a^{1,0}) = \iota_* a^{1,0}.$$
In particular,
$$f(a^{1,0}) \in H^0(X, \Omega_X \otimes E).$$
Let $\eta \in \mathcal{H}^1(X, E)$ be a representative of $a^{0,1} \in H^1(X, E)$. Then $\eta + f(a^{1,0}) \in L^1$, such that
$$\psi(\eta - f(a^{1,0})) = (\{\eta - f(a^{1,0})\}, -\{\mathcal{H}(\mathrm{Res}_{\eta - f(a^{1,0})})\} = (a^{0,1}, a^{1,0}).$$
Here we use the fact that $f(a^{1,0})$ is ∇'-exact. This shows that $H^1(\psi)$ is surjective.

Step 4. We now prove that $H^2(\psi)$ is injective. Pick any $\eta \in L^2$ such that $\nabla''\eta = 0$. We decompose $\eta = \eta^{2,0} + \eta^{1,1} + \eta^{0,2}$, where $\eta^{i,2-i} \in A^{i,2-i}(X, D, E)$. Then we have
$$\nabla''\eta^{i,2-i} = 0, \quad \nabla'\eta^{i,2-i} = 0.$$
This implies that $\eta^{2,0} \in H^0(X, \Omega_X^2(\log D) \otimes E)$ and $\eta^{0,2} \in \mathcal{H}^{0,2}(X, E)$.

If $\psi(\eta) = d(a^{0,1}, a^{1,0})$ for some $(a^{0,1}, a^{0,1}) \in C^{0,1} \oplus C^{1,0}$, then we have $\psi(\eta) \in C^{0,2}$, which yields that
$$\{\mathcal{H}(\eta)\} = g(a^{1,0}), \quad \{\mathcal{H}(\eta_{(1)})\} = 0, \quad \{\mathcal{H}(\eta_I)\} = 0 \tag{5.18.4}$$
for each multi-index I with $|I| = 2$ by (5.15.1). By the definition of f in (5.4.3), we have
$$\nabla f(a^{1,0}) = \mathcal{H}(g(a^{1,0})) + \iota_*(a^{1,0}) \overset{(5.18.4)}{=} \mathcal{H}(\eta) + \iota_*(a^{1,0}).$$
Note that $\nabla'' f(a^{1,0}) = \mathcal{H}(\eta) \in A^{1,1}(X, E)$, that is ∇-closed. It follows (5.18.4) that
$$\psi(\eta - \nabla'' f(a^{1,0})) = 0.$$
Therefore, we can replace η by $\eta - \nabla'' f(a^{1,0})$ and assume that
$$\{\mathcal{H}(\eta)\} = 0, \quad \{\mathcal{H}(\eta_{(1)})\} = 0, \quad \{\mathcal{H}(\eta_I)\} = 0. \tag{5.18.5}$$
for each multi-index I with $|I| = 2$. Fix any such I. Since $\nabla'\eta = 0$, it follows that
$$\nabla'\eta_I = 0.$$
(5.18.5) then implies that $\eta_I = \eta_I^{2,0} = 0$. Note that
$$\eta_{(1)} = \eta_{(1)}^{2,0} + \eta_{(1)}^{1,1}$$
with
$$\eta_{(1)}^{2,0} \in H^0(D^{(1)}, \Omega_{D^{(1)}}(\log D_{(1)}) \otimes E)$$
and
$$\eta_{(1)}^{1,1} \in A^{0,1}(D^{(1)}, E).$$
By (5.18.1), we have
$$\nabla''\eta_{(1)} = -(\nabla''\eta)_{(1)} = 0, \quad \nabla'\eta_{(1)} = -(\nabla'\eta)_{(1)} = 0.$$
This implies that
$$\eta_{(1)}^{1,1} \in \mathcal{H}^{0,1}(D^{(1)}, E).$$
Since
$$\{\mathcal{H}(\eta_{(1)})\} = 0, \quad \{\mathcal{H}(\eta_{ij})\} = \{\mathcal{H}(\mathrm{Res}_{D_i \cap D_j} \eta_i)\} = 0 \quad \text{for any } i < j,$$

the arguments in Step 3 in proving the injectivity of $H^1(\psi)$ show that $\eta_{(1)} = 0$. Hence, we have

$$\eta^{2,0} \in H^0(X, \Omega_X^2 \otimes E), \quad \mathrm{Res}_D \eta^{1,1} = 0.$$

Hence

$$\boldsymbol{\nabla}\eta^{1,1} = \boldsymbol{\nabla}''\eta^{1,1} = \nabla''\eta^{1,1} = 0.$$

Note that

$$\mathcal{H}(\eta^{1,1}) \in \mathcal{H}^{1,1}(X, E).$$

By (5.18.5) together with the Hodge composition of harmonic forms

$$\mathcal{H}^2(X, E) = \mathcal{H}^{2,0}(X, E) \oplus \mathcal{H}^{1,1}(X, E) \oplus \mathcal{H}^{0,2}(X, E),$$

we deduce that

$$\eta^{2,0} = 0, \quad \eta^{0,2} = 0, \quad \{\eta^{1,1}\} = 0 \in H^2(X, E).$$

By Lemma 2.16, we have

$$\eta^{1,1} = \boldsymbol{\nabla}''\boldsymbol{\nabla}'\sigma$$

for some $\sigma \in \mathscr{D}^0(X, E)$. By Proposition 5.10, there exists $\omega \in A^1(X, D, E)$ such that

$$\boldsymbol{\nabla}''\omega = \eta^{1,1}.$$

Hence by Lemma 5.1, $\nabla''\omega = \eta^{1,1}$. This implies that $\{\eta\} = 0 \in H^2(L^\bullet)$. The injectivity of $H^2(\psi)$ is proved.

The proof of the proposition is accomplished. □

In summary, we obtain the following result:

Theorem 5.19. — *Let $T^\bullet := (A^\bullet(X, D, E), \nabla)$ be the dgla of logarithmic forms. Let $L^\bullet := (\ker \nabla', \nabla'')$ be the dgla defined previously, and let $(C^\bullet, d)$ be the dgla introduced in Definition-Lemma 5.17. Let $\varepsilon : T^0 \to E_x$, $\varepsilon : L^0 \to E_x$, and $\varepsilon : C^0 \to E_x$ denote the evaluation maps at x, which define the corresponding augmentations.*

Then, the inclusion map $\phi : L^\bullet \to T^\bullet$ and the map $\psi : (L^\bullet, \nabla'') \to (C^\bullet, d)$ defined in (5.17.6) are morphisms of dglas, and both are 1-quasi-isomorphisms. In particular, the diagram

$$T^\bullet \xleftarrow{\phi} L^\bullet \xrightarrow{\psi} C^\bullet$$

establishes a zigzag of 1-quasi-equivalences connecting $(T^\bullet, d)$ and $(C^\bullet, d)$. □

5.7. Constructing the universal connection (I). — In this subsection, we construct the *universal* connection on the representation variety, in the sense specified in Theorem 5.23. This subsection is dedicated to the analytic construction; we defer the proof of its "universal property" to Theorem 5.23, using the Goldman–Millson theory in § 3 and the zigzags of 1-quasi-equivalences established in § 5.6.

Fix a basis $\alpha_1, \ldots, \alpha_p \in \mathcal{H}^1(X, E)$, and a basis $\beta_1, \ldots, \beta_q \in H^0(D^{(1)}, E)$. Consider the polynomial ring $\mathbb{C}[z_1, \ldots, z_p, w_1, \ldots, w_q]$. Let $\mathscr{Q}' \subset \mathbb{C}^{p+q}$ be the quasi-homogeneous cone defined by

(5.19.1) $$d\Big(\sum_{\gamma=1}^{q} w_\gamma \beta_\gamma\Big) + \frac{1}{2}\Big[\sum_{i=1}^{p} z_i \alpha_i, \sum_{i=1}^{p} z_i \alpha_i\Big] = 0 \in C^2$$

where $d\colon C^1 \to C^2$ is the differential of the dgla $(C^\bullet, d)$ defined in (5.15.1), and $[-,-]$ is the Lie bracket defined in (5.17.1). Let $f(\beta_\gamma) \in A^1(X, D, E)$ be defined in (5.4.3). Then $\mathrm{Res}_D f(\beta_\gamma) = -\beta_\gamma$ and we have $\nabla f(\beta_\gamma) \in \mathcal{H}^2(X, E)$ by our construction. Let us denote by $\eta_{i,0} := \alpha_i$ and $\eta_{0,\gamma} := f(\beta_\gamma)$.

For any $(z, w) := (z_1, \ldots, z_p; w_1, \ldots, w_q) \in \mathcal{Q}'$, we define

$$\eta_1(z, w) := \sum_{i=1}^{p} z_i \eta_{i,0} \tag{5.19.2}$$

$$\begin{aligned} \eta_2(z, w) &:= \sum_{\gamma=1}^{q} w_\gamma \eta_{0,\gamma} - \frac{1}{2}\nabla''^{*} G[\eta_1(z, w), \eta_1(z, w)] \\ &= \sum_{\gamma=1}^{q} w_\gamma \eta_{0,\gamma} - \frac{1}{2} \sum_{1 \le i,j \le p} z_i z_j \nabla''^{*} G[\eta_{i,0}, \eta_{j,0}] \end{aligned} \tag{5.19.3}$$

We proceed by induction on k. Let $k \ge 3$ and assume that $\eta_j(z, w) \in A^1(X, D, E) \cap \operatorname{Im}\boldsymbol{\nabla}'$ has been defined for all $1 \le j \le k-1$. Then the 2-form $\gamma_k(z, w)$ given by

$$\gamma_k(z, w) := -\sum_{j=1}^{k-1} \eta_j(z, w) \wedge \eta_{k-j}(z, w), \tag{5.19.4}$$

lies in $A^2(X, D, E)$. Next, we define $\eta_k(z, w)$ using the Green operator:

$$\eta_k(z, w) := \boldsymbol{\nabla}''^{*} G \gamma_k(z, w).$$

Note that (5.19.1) is exactly the three equations in Theorem 5.14. By Theorem 5.14, when $(z, w) \in \mathcal{Q}'$, this $\eta_k(z, w)$ satisfies the regularity condition

$$\eta_k(z, w) \in A^1(X, D, E) \cap \operatorname{Im}\boldsymbol{\nabla}',$$

and solves the equation

$$\boldsymbol{\nabla}\eta_k(z, w) = \gamma_k(z, w).$$

Moreover, there exists some sufficiently small $\varepsilon > 0$ such that, for $(z, w) \in \mathcal{Q}'_\varepsilon := \mathcal{Q}' \cap \mathbb{B}^{p+q}_\varepsilon$, where $\mathbb{B}^{p+q}_\varepsilon$ is the ball in $\mathbb{C}^{p+q}$ of radius ε, the connection

$$\nabla_{z,w} := \nabla + \sum_{i=1}^{\infty} \eta_i(z, w).$$

is flat over X_0 and has logarithmic poles.

Let us write explicitly the formula of $\eta_i(z, w)$. Inductively, for any multi-indices $I = (i_1, \ldots, i_k)$ and $\Gamma = (\gamma_1, \ldots, \gamma_\ell)$ with entries satisfying $1 \le i_j \le p$ and $1 \le \gamma_m \le q$, we define 1-forms on X_0 by

$$\Phi_{I,\Gamma} := -\Big(\sum_{\substack{I=(I_1,I_2);\Gamma=(\Gamma_1,\Gamma_2) \\ 1 \le |I_1|+2|\Gamma_1| < |I_2|+2|\Gamma_2|}} [\eta_{I_1,\Gamma_1}, \eta_{I_2,\Gamma_2}] + \frac{1}{2} \sum_{\substack{I=(I_1,I_2);\Gamma=(\Gamma_1,\Gamma_2) \\ |I_1|+2|\Gamma_1| = |I_2|+2|\Gamma_2|}} [\eta_{I_1,\Gamma_1}, \eta_{I_2,\Gamma_2}] \Big) \tag{5.19.5}$$

$$\eta_{I,\Gamma} := \boldsymbol{\nabla}''^{*} G \Phi_{I,\Gamma} \tag{5.19.6}$$

Note that the $\eta_{I,\Gamma}$ might not extend to log 1-forms on $A^1(X, D, E)$!

Then for any $k \ge 1$, we have

$$\eta_k(z, w) = \sum_{|I|+2|\Gamma|=k} z_I w_\Gamma \eta_{I,\Gamma},$$

where we denote by $z_I := z_{i_1} \cdots z_{i_{|I|}}$, and $w_\Gamma := w_{\gamma_1} \cdots w_{\gamma_{|\Gamma|}}$.

We list one fact that will be used later.

Lemma 5.20. — *For any $k \ge 2$ and any $(z, w) \in \mathcal{Q}'_\varepsilon$, $\eta_k(z, w)$ is $\boldsymbol{\nabla}'$-exact.*

Proof. — It follows from Theorem 5.14. □

We extend Theorem 5.14 to the universal setting as follows.

Theorem 5.21. — *Let X be a compact Kähler manifold and let D be a simple normal crossing divisor on X. Let (V, ∇) be a unitary bundle on X. Denote by $(E, \nabla) := (\operatorname{End}(V), \nabla)$.*

Fix a basis $\eta_{1,0}, \ldots, \eta_{p,0} \in \mathcal{H}^1(X, E)$, and a basis $\beta_1, \ldots, \beta_q \in H^0(D^{(1)}, E)$. Set $\eta_{0,\gamma} := f(\beta_\gamma)$ where $f : H^0(D^{(1)}, E) \to A^1(X, D, E)$ is defined in (5.4.3). For any multi-indices I with *entries in* $\{1,\dots,p\}$ *and* Γ *with entries in* $\{1,\dots,q\}$*, we define inductively* E*-valued smooth 1-forms on* X_0*:*

(5.21.1)

$$\eta_{I,\Gamma} := -\nabla''^{*}G\Big(\sum_{\substack{I=(I_1,I_2);\Gamma=(\Gamma_1,\Gamma_2)\\ 1\le|I_1|+2|\Gamma_1|<|I_2|+2|\Gamma_2|}}[\eta_{I_1,\Gamma_1},\eta_{I_2,\Gamma_2}]+\frac{1}{2}\sum_{\substack{I=(I_1,I_2);\Gamma=(\Gamma_1,\Gamma_2)\\ |I_1|+2|\Gamma_1|=|I_2|+2|\Gamma_2|}}[\eta_{I_1,\Gamma_1},\eta_{I_2,\Gamma_2}]\Big)$$

For any $k\ge 1$*, we define*

$$\eta_k(z,w)=\sum_{|I|+2|\Gamma|=k} z_I w_\Gamma \eta_{I,\Gamma}, \tag{5.21.2}$$

where we write $z_I := z_{i_1}\cdots z_{i_{|I|}}$*, and* $w_\Gamma := w_{\gamma_1}\cdots w_{\gamma_{|\Gamma|}}$*, and*

$$\nabla_{z,w} := \nabla+\sum_{i=1}^{\infty}\eta_k(z,w). \tag{5.21.3}$$

Then for every $(z,w)\in \mathcal{Q}'\cap\mathbb{B}_\varepsilon^{p+q}$*,* $\eta_k(z,w)$ *extend to logarithmic 1-forms in* $\in A^1(X,D,E)$ *such that* $\sum_{i=1}^{\infty}\eta_k(z,w)$ *converges and lies in* $A^1(X,D,E)$*.*

Let $\mathcal{Q}'\subset\mathbb{C}^{p+q}$ *be the quasi-homogeneous cone defined by the following three equations with values in* $H^2(X,E)$*,* $H^1(D^{(1)},E)$ *and* $H^0(D^{(2)},E)$ *respectively:*

(i)
$$g\Big(\sum_{i=1}^{q}w_i\beta_i\Big)+\frac{1}{2}\Big\{\Big[\sum_{i=1}^{p}z_i\eta_{i,0},\sum_{i=1}^{p}z_i\eta_{i,0}\Big]\Big\}=0,$$

where $g: H^0(D^{(1)},E)\to H^2(X,E)$ *is the Gysin morphism defined in Definition 5.3.*

(ii)
$$\Big\{\Big[\iota^*\sum_{i=1}^{p}z_i\eta_{i,0},\sum_{j=1}^{q}w_j\beta_j\Big]\Big\}=0\in H^1(D^{(1)},E),$$

where $\iota: D^{(1)}\to X$ *denotes the natural morphism.*

(iii) *For any* $i\neq j$*,*
$$\Big[\iota_{ij}^*\Big(\sum_{\ell=1}^{q}w_k\beta_k|_{D_i}\Big),\iota_{ji}^*\Big(\sum_{k=1}^{q}w_k\beta_k|_{D_i}\Big)\Big]=0\in H^0(D_i\cap D_j,E),$$

where $\iota_{ij}: D_i\cap D_j\to D_i$ *is the natural morphism.*

Then there exists some $\varepsilon>0$ *such that,* $\nabla_{z,w}$ *is an analytic family of flat connections for* $V|_{X_0}$ *with logarithmic poles around* D *parametrized by* $(z,w)\in\mathcal{Q}'_\varepsilon := \mathcal{Q}'\cap\mathbb{B}_\varepsilon^{p+q}$*.*

Proof. — Suppose that $(z,w)\in\mathcal{Q}'$. Then $\eta_1(z,w)$ and

$$\mathrm{Res}_D\,\eta_2(z,w)=-\sum_{j=1}^{q}w_j\beta_j$$

satisfy the three conditions in Theorem 5.21. Moreover, $\eta_k(z,w)$ defined in (5.21.2) satisfies the inductive equations in the proof of Theorem 5.14 for each $k\ge 1$. Therefore, we have

$$\nabla\eta_k(z,w)+\sum_{i=1}^{k}\eta_i(z,w)\wedge\eta_{k-i}(z,w)=0$$

for $k\ge 3$. By (5.14.15), we have

$$\|\eta_k(z,w)\|_{W^1}\le\frac{1}{2C}\Lambda(z,w)^k, \tag{5.21.4}$$

where

$$\Lambda(z,w) := 2C\|\eta_1(z,w)\|_{W^1}+\sqrt{4C^2\|\eta_1(z,w)\|_{W^1}^2+4C\|\eta_2(z,w)\|_{W^1}}.$$

By the definitions of η_1 and η_2, there exists some $\varepsilon \in (0, \frac{1}{2})$ such that $\Lambda(z, w) \leq 1$ for any $(z, w) \in \mathcal{Q}'_\varepsilon$. Consequently, $\|\eta(z, w)\|_{W^1} < \infty$ for all $(z, w) \in \mathcal{Q}'_\varepsilon$. The bootstrapping arguments used in the proof of Theorem 5.14 imply that $\eta(z, w) \in A^1(X, D, E)$ and satisfies

$$\nabla\eta(z, w) + \eta(z, w) \wedge \eta(z, w) = 0.$$

Thus, we conclude that $\nabla_{z,w} := \nabla + \eta(z, w)$ defines an analytic family of flat logarithmic connections on V, parametrized by $(z, w) \in \mathcal{Q}'_\varepsilon$. This completes the proof. □

5.8. Constructing the universal connection (II). — In this subsection, we prove that the connection constructed in Theorem 5.21 is universal. The argument closely follows that of Theorem 4.4, and we include all details for the sake of completeness.

Consider the dgla $T^\bullet := (A^\bullet(X, D, E), \nabla)$. Define $L^\bullet := (\ker\nabla', \nabla'')$. Let $\varepsilon : T^0 \to E_x$, $\varepsilon : L^0 \to E_x$, and $\varepsilon : C^0 \to E_x$ be the evaluation maps at a fixed base point $x \in X_0$, which define the corresponding augmentations. Then by Theorem 5.19, we have the 1-quasi-equivalences

$$(T^\bullet, d) \xleftarrow{\phi} (L^\bullet, d) \xrightarrow{\psi} (C^\bullet, d).$$

By Theorem 3.4, for any Artin local $\mathbb{C}$-algebra, the natural maps

(5.21.5)
$$[\mathscr{C}(T^\bullet, \varepsilon, A)/\exp(T^0 \otimes \mathfrak{m})] \leftarrow [\mathscr{C}(L^\bullet, \varepsilon, A)/\exp(L^0 \otimes \mathfrak{m})] \to [\mathscr{C}(C^\bullet, \varepsilon, A)/\exp(C^0 \otimes \mathfrak{m})]$$

are equivalences of groupoids.

We note that augmentation

$$\varepsilon : C^0 \to E_x$$

defined by the evaluation is injective. In this case, we have the following result. Let $\mathcal{Q}$ be the quasi-homogeneous cone in $C^1 = C^{1,0} \oplus C^{0,1}$ defined by

$$\mathcal{Q}_0 := \left\{ u \in C^1 \mid du + \frac{1}{2}[u, u] = 0 \right\}. \tag{5.21.6}$$

Denote by

$$\mathcal{Q} := \mathcal{Q}_0 \times \mathfrak{g}/\varepsilon(C^0). \tag{5.21.7}$$

Lemma 5.22. — *The analytic germ $(\mathcal{Q}, 0)$ (non-canonically) prorepresents the functor*

$$\begin{aligned} \mathrm{Art} &\to \mathrm{Grp} \\ A &\mapsto \mathrm{Def}(C^\bullet, \varepsilon, A) \end{aligned}$$

where $\mathrm{Def}(C^\bullet, \varepsilon, A)$ is the small groupoid defined in (3.3.1).

Proof. — Let $\ell := \dim_{\mathbb{C}}(\mathfrak{g}/\varepsilon(C^0))$. We choose a subset $\{v_1, \ldots, v_\ell\}$ of $\mathfrak{g}$ such that the images of $\{v_1, \ldots, v_\ell\}$ form a basis for the quotient space $\mathfrak{g}/\varepsilon(C^0)$. Let W be the subspace of $\mathfrak{g}$ spanned by $\{v_1, \ldots, v_\ell\}$. Then the natural map $p : W \to \mathfrak{g}/\varepsilon(C^0)$ is an isomorphism. Since $\varepsilon : C^0 \to E_x$ and $\varepsilon : D^0 \to E_x$ defined by the evaluation is injective, it follows that $\exp\left(C^0 \otimes \mathrm{m}\right)$ acts freely on $G^0_A := \exp(\mathfrak{g} \otimes \mathfrak{m})$ by left-multiplication. Hence the following (non-canonical) maps

$$\begin{array}{ccccc} \mathfrak{g}/\varepsilon(C^0) \otimes \mathfrak{m} & \leftarrow & W \otimes \mathfrak{m} & \to & \dfrac{\exp(\mathfrak{g} \otimes \mathfrak{m})}{\exp(\varepsilon(C^0) \otimes \mathfrak{m})} \\ p(r) & \leftarrow\!\shortmid & r & \mapsto & \exp(\varepsilon(C^0) \otimes \mathfrak{m}) \cdot \exp(r). \end{array}$$

are both bijective. Here $\exp(\varepsilon(C^0) \otimes \mathfrak{m}) \cdot \exp(r)$ is the right coset of $\exp(r)$. Hence the following maps

$$\begin{aligned} \mathscr{C}(C^\bullet, A) \times W \otimes \mathfrak{m} &\to \frac{\mathscr{C}(C^\bullet, \varepsilon, A)}{\exp(C^0 \otimes \mathfrak{m})} \\ (\alpha, r) &\mapsto [(\alpha, \exp(r))]. \end{aligned} \tag{5.22.1}$$

and

$$\begin{aligned} (5.22.2) \qquad \mathscr{C}(L^\bullet, A) \times W \otimes \mathfrak{m} &\to \frac{\mathscr{C}(L^\bullet, \varepsilon, A)}{\exp(L^0 \otimes \mathfrak{m})} \\ (\alpha, r) &\mapsto [(\alpha, \exp(r))]. \end{aligned}$$

are both equivalences of groupoids.

Since C^1 is a finite-dimensional complex vector space, we view $C^1 \times W$ as an affine space over $\mathbb{C}$. Let $\mathcal{I}$ be the ideal of $\mathbb{C}[C^1 \times W]$ generated by the quasi-homogeneous equations

$$dq(u) + \frac{1}{2}[q(u), q(u)] = 0$$

for $u \in C^1 \times W$, where $q : C^1 \times W \to C^1$ denotes the projection map, $d\colon C^1 \to C^2$ is the differential of the dgla $(C^\bullet, d)$ defined in (5.15.1), and $[-,-]$ is the Lie bracket defined in (5.17.1). It follows that

$$\operatorname{Spec} \mathbb{C}[C^1 \times W]/\mathcal{I} = \mathcal{Q}.$$

Let $\{\alpha_1, \dots, \alpha_p\}$ be a basis of $\mathcal{H}^1(X, E)$, and we denote abusively $\{\alpha_1, \dots, \alpha_p\}$ their cohomology classes in $H^1(X, E) = C^{0,1}$. Let $\{\beta_1, \dots, \beta_q\} \in C^{1,0}$ be a basis. Then, any elements $\alpha \in C^1 \otimes \mathfrak{m}$ and $r \in W \otimes \mathfrak{m}$, can be written as

$$(5.22.3) \qquad \alpha = \Big(\sum_{i=1}^{p} \alpha_i \otimes f_i, \sum_{\gamma=1}^{q} \beta_\gamma \otimes g_\gamma\Big), \quad r = \sum_{j=1}^{\ell} v_j \otimes h_j,$$

where $f_i, g_\gamma, h_j \in \mathfrak{m}$. Hence, (α, r) gives rise to a morphism

$$\begin{aligned} &\operatorname{Hom}(\mathbb{C}[C^1 \times W], A) \\ ((\alpha_i)^*, (\beta_\gamma)^*, (v_j)^*) &\mapsto f_i + g_\gamma + h_j \end{aligned}$$

Here we consider $(\alpha_i)^* \in (C^{0,1})^*$, $(\beta_\gamma)^* \in (C^{1,0})^*$ and $(v_j)^* \in W^*$. Note that the above morphism factors through

$$\operatorname{Hom}(\mathbb{C}[C^1 \times W]/\mathcal{I}, A)$$

if and only if

$$d(\alpha) + \frac{1}{2}[\alpha, \alpha] = 0.$$

Equivalently, this holds if and only if

$$\alpha \in \mathscr{C}^1(C^\bullet, A).$$

Let $\mathcal{I}_1$ be the ideal of $\mathbb{C}[\mathcal{Q}]$ generated by the image of $(C^1 \times W)^* \subset \mathbb{C}[C^1 \times W]$ under the natural projection $\mathbb{C}[C^1 \times W] \to \mathbb{C}[C^1 \times W]/\mathcal{I}$. Let $\widehat{\mathbb{C}[\mathcal{Q}]}_{\mathcal{I}_1}$ denote the $\mathcal{I}_1$-adic completion of $\mathbb{C}[\mathcal{Q}]$. Since A is an Artin local $\mathbb{C}$-algebra, it follows that

$$\operatorname{Hom}(\mathbb{C}[C^1 \times W]/\mathcal{I}, A) \simeq \operatorname{Hom}(\widehat{\mathbb{C}[\mathcal{Q}]}_{\mathcal{I}_1}, A).$$

By standard properties of completion of quotient rings, there is a natural isomorphism

$$\widehat{\mathbb{C}[\mathcal{Q}]}_{\mathcal{I}_1} \simeq \frac{\mathbb{C}[[C^1 \times W]]}{\mathcal{I}},$$

where, by abuse of notation, we continue to denote by $\mathcal{I}$ the ideal $\mathcal{I} \cdot \mathbb{C}[[C^1 \times W]]$.

On the other hand, $\mathcal{I}_1$-adic completion commutes with localization at the maximal ideal $\mathcal{I}_1$ of $\mathbb{C}[\mathcal{Q}]$. Therefore we have a natural isomorphism

$$\widehat{\mathbb{C}[\mathcal{Q}]}_{\mathcal{I}_1} \xrightarrow{\simeq} \widehat{\mathcal{O}_{\mathcal{Q},0}}.$$

Therefore, the following map

$$\begin{aligned} (5.22.4) \qquad \mathscr{C}(C^\bullet, A) \times W \otimes \mathfrak{m} &\to \operatorname{Hom}(\widehat{\mathcal{O}_{\mathcal{Q},0}}, A) \\ (\alpha, r) &\mapsto \big(((\alpha_i)^*, (\beta_\gamma)^*, (v_j)^*) \mapsto f_i + g_\gamma + h_j\big) \end{aligned}$$

is an equivalence, where (α, r) is written as in (5.22.3), and α_i^*, β_γ^*, and ν_j^* are understood as the images of α_i^*, β_γ^*, and ν_j^* under the composite morphisms

$$\mathbb{C}[C^1 \times W] \to \mathbb{C}[\mathcal{Q}] \to \mathcal{O}_{\mathcal{Q},0} \to \widehat{\mathcal{O}}_{\mathcal{Q},0}.$$

We remark that $\widehat{\mathcal{O}_{\mathcal{Q},0}}$ is generated by α_i^*, β_γ^*, and ν_j^*. Therefore, for any morphism in $\mathrm{Hom}(\widehat{\mathcal{O}_{\mathcal{Q},0}}, A)$, the images of these generators completely determine the morphism. Combining (5.22.4) with (5.22.1) yields the lemma. □

Consider the polynomial ring $\mathbb{C}[z_1, \ldots, z_p, w_1, \ldots, w_q]$. Let $\mathcal{Q}' \subset \mathbb{C}^{p+q}$ denote the quasi-homogeneous cone defined in (5.19.1). We set

$$\mathcal{Q}'' := \mathcal{Q}' \times \mathbb{C}^\ell.$$

Write $\eta_{i,0} := \alpha_i$ and let $\eta_{0,\gamma} := f(\beta_\gamma) \in A^1(X, D, E)$ be defined in (5.4.3). Then $\mathrm{Res}_D\, \eta_{0,\gamma} = -\beta_\gamma$ and we have

$$\nabla f(\beta_\gamma) \in \mathcal{H}^2(X, E)$$

by our construction. Let

$$\nabla_{z,w} := \nabla + \sum_{i=1}^{\infty} \eta_k(z, w) \tag{5.22.5}$$

be the connection parametrized by $(z, w) \in \mathcal{Q}'$, where

$$\eta_k(z, w) := \sum_{|I|+2|\Gamma|=k} z_I w_\Gamma \eta_{I,\Gamma},$$

with the notation $z_I := z_{i_1} \cdots z_{i_{|I|}}$, and $w_\Gamma := w_{\gamma_1} \cdots w_{\gamma_{|\Gamma|}}$, where the forms $\eta_{I,\Gamma} \in A^1(X, D, E)$ are defined in (5.21.1).

By Lemma 5.20, we know for each $(z, w) \in \mathcal{Q}'$, each $\eta_k(z, w) \in A^1(X, D, E)$ is $\boldsymbol{\nabla}'$-exact for $k \geq 2$. Hence we have

$$\nabla' \eta_k(z, w) = \boldsymbol{\nabla}' \eta_k(z, w) = 0 \quad \text{ for } k \geq 1. \tag{5.22.6}$$

Fix a $k \in \mathbb{Z}_{>0}$. Set $A_k := \mathcal{O}_{\mathcal{Q}'',0}/\mathfrak{m}^{k+1}$, that is an Artin local $\mathbb{C}$-algebra. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{\mathcal{Q}'',0}$. Write $\mathfrak{m}_k := \mathfrak{m}A_k$, that is the maximal ideal for A_k. We now regard z_I, w_Γ, and x_j as the image of z_I, w_Γ and x_j under the morphism

$$\mathcal{O}_{\mathbb{C}^{p+q+\ell},0} \to \mathcal{O}_{\mathcal{Q}'',0} \to \mathcal{O}_{\mathcal{Q}'',0}/\mathfrak{m}^{k+1} = A_k.$$

Then we have $z_I \in \mathfrak{m}_k$ and $w_\Gamma \in \mathfrak{m}_k$ for $|I| \geq 1$ and $|\Gamma| \geq 1$, and $x_i \in \mathfrak{m}_k$. Thus by (5.22.6), we can consider

$$\eta_i(z, w) \in L^{1,0} \otimes \mathfrak{m}_k$$

for any $i \in \mathbb{N}$. It follows that

$$\eta_j(z, w) = 0 \in L^{1,0} \otimes \mathfrak{m}_k$$

for any $j \geq 2k + 1$. By Theorem 5.21, we have

$$\nabla\Big(\sum_{j=1}^{2k} \eta_j(z, w)\Big) + \frac{1}{2}\Big[\sum_{j=1}^{2k} \eta_j(z, w), \sum_{j=1}^{2k} \eta_j(z, w)\Big] = 0 \in L^1 \otimes \mathfrak{m}_k.$$

This implies that,

$$\sum_{j=1}^{2k} \eta_j(z, w) \in \mathrm{Obj}\,\mathrm{Def}(L^\bullet, A_k). \tag{5.22.7}$$

Consider the morphism $\psi : L^\bullet \to C^\bullet$ in (5.17.6). By Lemma 5.20, each $\eta_j(z, w)$ is $\boldsymbol{\nabla}'$-exact for any $j \geq 2$. By (5.14.3) and (5.17.6), for any $k \geq 2$, we have

$$\psi\Big(\sum_{j=1}^{2k} \eta_j(z, w)\Big) = \sum_{i=1}^{p} z_i \eta_{i,0} + \sum_{\gamma=1}^{q} w_\gamma \beta_\gamma \in \mathscr{C}(C^\bullet, A_k).$$

Here we denote abusively by $\{\eta_{1,0},\dots,\eta_{p,0}\}$ and $\{\beta_1,\dots,\beta_q\}$ their cohomology class in $H^1(X,E)=C^{0,1}$ and $H^0(D^{(1)},E)=C^{1,0}$ respectively. Therefore, using the isomorphisms of groupoids defined in (5.22.1) and (5.22.4), we can see that we have the isomorphism

$$\Phi:\mathscr{C}(L^\bullet,A)\times W\otimes\mathfrak{m}_k\to\mathscr{C}(C^\bullet,A)\times W\otimes\mathfrak{m}_k\to\operatorname{Hom}(\widehat{\mathcal{O}}_{\mathcal{Q},0},A_k)\tag{5.22.8}$$

such that

$$\Phi\Big(\sum_{j=1}^{2k}\eta_j(z,w),\sum_{j=1}^{\ell}x_jv_j\Big)=\Big((\eta^*_{i,0},\beta^*_\gamma,v^*_j)\mapsto z_i+w_\gamma+x_j\Big).$$

As remarked above, since $\widehat{\mathcal{O}_{\mathcal{Q},0}}$ is generated by $\eta^*_{i,0}$, β^*_γ, and v^*_j, any morphism in $\operatorname{Hom}(\widehat{\mathcal{O}_{\mathcal{Q},0}},A_k)$ is determined by images of these generators.

We state and prove the universal property of the connection constructed in Theorem 5.21.

Theorem 5.23. — *Let X be a compact Kähler manifold and let D be a simple normal crossing divisor on X. Assume that (V,∇) be a unitary flat bundle of rank N on X and we denote by $(E,\nabla):=(\operatorname{End}(V),\nabla)$. Let $\nabla_{z,w}$ denote the analytic family of flat connections on $V|_{X_0}$, parametrized by $(z,w)\in\mathcal{Q}'_r$ for some $r>0$, as defined in (5.22.5). Let $\operatorname{Mon}(\nabla_{z,w})$ be the monodromy representation for any $(z,w)\in\mathcal{Q}'_r$. Let $R(X_0,N)$ be the representation variety for $\pi_1(X_0,x)$ into GL_N. Consider the evaluation map $\mathrm{ev}_x:H^0(X,E)\to E_x$. Let $\{v_1,\dots,v_\ell\}\subset E_x$ be a subset such that its image forms a basis of the quotient space $E_x/\operatorname{Im}(\mathrm{ev}_x)$. Then the natural map between analytic germs*

$$f:\Big(\mathcal{Q}'_r\times\mathbb{C}^\ell,0\Big)\to R(X_0,N)$$
$$(z,w;x)\mapsto\exp\Big(-\sum_{i=1}^{\ell}x_iv_i\Big)\operatorname{Mon}(\nabla_{z,w})\exp\Big(\sum_{i=1}^{\ell}x_iv_i\Big)$$

is an isomorphism.

Proof. — We recall that we have morphisms of groupoids

(5.23.1)

$$\begin{tikzcd}
 & \operatorname{Hom}\Big(\widehat{\mathcal{O}}_{\mathcal{Q},0},A_k\Big) & \\
{[R_\varrho(A_k)/\mathrm{Id}]} & \mathscr{C}(L^\bullet,A_k)\times W\otimes\mathfrak{m}_k \arrow[u,"\Phi"] \arrow[r,"\psi\times\mathrm{Id}"] \arrow[d] & \mathscr{C}(C^\bullet,A_k)\times W\otimes\mathfrak{m}_k \arrow[ul] \arrow[d] \\
{[\mathscr{C}(T^\bullet,\varepsilon,A_k)/\exp(T^0\otimes\mathfrak{m}_k)]} \arrow[u,"h"] & {[\mathscr{C}(L^\bullet,\varepsilon,A_k)/\exp(L^0\otimes\mathfrak{m}_k)]} \arrow[l,"\phi"'] \arrow[r,"\psi"] & {[\mathscr{C}(C^\bullet,\varepsilon,A_k)/\exp(C^0\otimes\mathfrak{m}_k)]}
\end{tikzcd}$$

in which each of them is an isomorphism by (5.22.1), (5.22.2), and Theorems 3.4 and 5.19 and Lemma 3.9. Here

$$h:[\mathscr{C}(T^\bullet,\varepsilon_x,A_k)/\exp(T^0\otimes\mathfrak{m})]\to[R_\varrho(A_k)/\mathrm{Id}]$$
$$(\alpha,e^r)\mapsto\exp(-r)\mathrm{hol}_x(\nabla+\alpha)\exp(r)$$

is defined in Lemma 3.9. We denote by

$$\Upsilon:\mathscr{C}(L^\bullet,A_k)\times W\otimes\mathfrak{m}\to[R_\varrho(A_k)/\mathrm{Id}]$$

the induced equivalence of groupoids in (5.23.1). Then, for

$$\sum_{j=1}^{2k}\eta_j(z,w)\in\mathscr{C}(L^\bullet,A_k)$$

defined in (5.22.7),

$$\Upsilon\Big(\sum_{j=1}^{2k}\eta_j(z,w),\sum_{j=1}^{\ell}x_jv_j\Big)=\exp\Big(-\sum_{i=1}^{\ell}x_iv_i\Big)\operatorname{Mon}(\nabla_{z,w})\exp\Big(\sum_{i=1}^{\ell}x_iv_i\Big),$$

corresponds to the morphism

$$f_k : \widehat{\mathcal{O}}_{R(X_0,N),\varrho} \to A_k = \mathcal{O}_{\mathcal{Q}'',0}/\mathfrak{m}^{k+1}$$

induced by f.

Let us define a morphism $g_k \in \mathrm{Hom}(\widehat{\mathcal{O}}_{\mathcal{Q},0}, A_k)$ by setting

$$g_k(\alpha_i^*) = z_i, \quad g_k(\beta_\gamma^*) = w_\gamma, \quad g_k(v_j^*) = x_j. \tag{5.23.2}$$

By (5.22.8), we have

$$\Phi(\sum_{j=1}^{2k} \eta_j(z,w), \sum_{i=1}^{\ell} x_i v_i) = g_k.$$

One can see the compatibility of g_k by

$$\begin{tikzcd} & \vdots \arrow[d] \\ & A_{k+1} \arrow[d] \\ \widehat{\mathcal{O}}_{\mathcal{Q},0} \arrow[ur, "g_{k+1}"] \arrow[r, "g_k"'] \arrow[ddr, "g_1"'] & A_k \arrow[d] \\ & \vdots \arrow[d] \\ & A_1 \end{tikzcd}$$

In conclusion, there exists natural maps

$$\begin{tikzcd} \mathrm{Hom}\Big(\widehat{\mathcal{O}}_{\mathcal{Q},0}, A_k\Big) \arrow[r] & \mathrm{Hom}\Big(\widehat{\mathcal{O}}_{\mathcal{Q},0}, A_{k-1}\Big) \\ \mathscr{C}(C^\bullet, A_k) \times W \otimes \mathfrak{m}_k \arrow[r] \arrow[u, "\Phi"] \arrow[d, "\Upsilon"'] & \mathscr{C}(C^\bullet, A_{k-1}) \times W \otimes \mathfrak{m}_{k-1} \arrow[u, "\Phi"'] \arrow[d, "\Upsilon"] \\ \mathrm{Hom}(\widehat{\mathcal{O}}_{R(X_0,N),\varrho}, A_k) \arrow[r] & \mathrm{Hom}(\widehat{\mathcal{O}}_{R(X_0,N),\varrho}, A_{k-1}) \end{tikzcd}$$

such that all vertical maps are isomorphisms. We have the inverse limit

$$\varprojlim_k g_k \in \mathrm{Hom}\Bigg(\widehat{\mathcal{O}}_{\mathcal{Q},0}, \varprojlim_k A_k\Bigg) = \mathrm{Hom}\Big(\widehat{\mathcal{O}}_{\mathcal{Q},0}, \widehat{\mathcal{O}}_{\mathcal{Q}'',0}\Big)$$

and by (5.23.2), it is an isomorphism. By (5.23.1), it follows that

$$f^* = \varprojlim_k f_k : \widehat{\mathcal{O}}_{R(X_0,N),\varrho} \to \widehat{\mathcal{O}}_{\mathcal{Q}'',0},$$

is also an isomorphism. By Theorem 3.5, this implies that $f : (\mathcal{Q}'_r \times \mathbb{C}^\ell, 0) \to (R(X_0,N), \varrho)$ is an *isomorphism* between analytic germs. The theorem is therefore proved. □

Remark 5.24. — The above theorem shows that, for extendable unitary representation $\varrho : \pi_1(X_0) \to \mathrm{GL}_N(\mathbb{C})$, the analytic germ of the representation variety $R(X_0,N)$ at ϱ is isomorphic to a quasi-homogeneous singularity defined by generators of weights 1 and 2, subject to relations of weights 2, 3, and 4. Similar result for representations with finite image were obtained by Kapovich-Millson [**KM98**] based on Morgan's theorems [**Mor78**]. See also [**Lef19**].

5.9. 2-step phenomenon. — We are now in a position to establish the main theorem concerning the so-called *2-step phenomenon* for extendable unitary systems on quasi-compact Kähler manifolds.

Let ϱ be a unitary representation of the fundamental group $\pi_1(X)$, and let (V,∇) be the associated flat bundle. We denote by $(E,\nabla) := (\mathrm{End}(V),\nabla)$ the induced bundle of endomorphisms. Let $\mathscr{Q}' \subset \mathbb{C}^{p+q}$ be the quasi-homogeneous cone defined in (5.19.1). Fix some $r>0$ sufficiently small as in Theorem 5.23, let $\mathcal{T} := \mathscr{Q}' \cap \mathbb{B}_r^{p+q}$. Then the universal connection $\nabla_{z,w}$ defined in (5.22.5) induces a tautological flat connection on $V \otimes \mathcal{O}(\mathcal{T})$. It corresponds to a representation

$$\pi_1(X_0) \xrightarrow{\varrho_{\mathcal{T}}} \mathrm{GL}_N(\mathcal{O}_{\mathcal{T},0}).$$

Let $\mathfrak{m}$ be the maximal ideal for the local ring $\mathcal{O}_{\mathcal{T},0}$. For each i, we have

$$\varrho_{\mathcal{T},i} : \pi_1(X_0) \xrightarrow{\varrho_{\mathcal{T}}} \mathrm{GL}_N(\mathcal{O}_{\mathcal{T},0}) \to \mathrm{GL}_N(\mathcal{O}_{\mathcal{T},0}/\mathfrak{m}^i). \tag{5.24.1}$$

We are interested in the case $i=3$. Note that as a complex vector bundle, we have a decomposition

$$\mathbf{V}_3 \overset{C^\infty}{\simeq} \oplus_{i=0}^2 E \otimes \mathfrak{m}^i/\mathfrak{m}^{i+1}. \tag{5.24.2}$$

With respect to this decomposition, we write the connection explicitly.

Recall that we consider $z_1,\dots,z_p,w_1,\dots,w_q$ as linear functions on the vector space $H^1(X,E)\oplus H^0(D^{(1)},E)$ given by

$$\begin{aligned} z_k\Big(\sum_{i=1}^p a_i\{\eta_{i,0}\}, \sum_{\gamma=1}^q b_\gamma\eta_\gamma\Big) &\mapsto a_k \\ w_\alpha\Big(\sum_{i=1}^p a_i\{\eta_{i,0}\}, \sum_{\gamma=1}^q b_\gamma\beta_\gamma\Big) &\mapsto b_\alpha. \end{aligned} \tag{5.24.3}$$

Now we rearrange the basis $\beta_1,\dots,\beta_q$ of $H^0(D^{(1)},E)$ so that

$$\beta_1,\dots,\beta_\ell$$

is a basis for $\ker g$, where $g : H^0(D^{(1)},E) \to H^2(X,E)$ is the Gysin morphism defined in Definition 5.3. We have

$$\ell := \dim H^0(D^{(1)},E) - \mathrm{rank}\, g.$$

Let $\mathcal{E}$ be the local system on X_0 corresponding to (E,∇). By [Del70, II, 6.10] together with Theorem 5.19, we have

$$H^1(C^\bullet,d) \simeq H^1(A^\bullet(X,D,E),\nabla) \simeq H^1(X_0,\mathcal{E}),$$

where $(C^\bullet,d)$ is the dgla defined in Definition-Lemma 5.17. Then we have

$$\mathbb{E}_{-1} : = \mathfrak{m}/\mathfrak{m}^2 = \frac{(H^1(X,E)\oplus H^0(D^{(1)},E))^*}{(w_{\ell+1},\dots,w_q)} \cong \Big(H^1(C^\bullet,d)\Big)^* \cong \Big(H^1(X_0,\mathcal{E})\Big)^* \tag{5.24.4}$$

where the isomorphism follows from Theorem 5.19, and

(5.24.5)

$$\mathbb{E}_{-2} : = \mathfrak{m}^2/\mathfrak{m}^3 = \frac{\mathrm{Sym}^2\,(H^1(X,E)\oplus H^0(D^{(1)},E))^*}{Q} = \frac{\mathrm{Sym}^2\,(z_1,\dots,z_p,w_1,\dots,w_\ell)}{\overline{Q}}$$

where $\overline{Q}$ is the linear space generated by the quadratic equations in Theorem 5.21.(ii) and Theorem 5.21.(iii) restricted to the subspace $w_{\ell+1}=\dots=w_q=0$. In other words, we set

$$w_{\ell+1}=\dots=w_q=0$$

in the equations in Theorems 5.21.(ii) and 5.21.(iii). These subsequently become linear equations in the quadratic monomials

$$\{z_iz_j,\ z_iw_\gamma,\ w_\alpha w_\beta\}_{i,j\in\{1,\dots,p\};\,\alpha,\beta,\gamma\in\{1,\dots,\ell\}}.$$

The above quadratic monomials generate $\mathfrak{m}^2/\mathfrak{m}^3$, and the linear equations describe all the relations among them. Hence, a basis of $\mathfrak{m}^2/\mathfrak{m}^3$ is obtained by taking these quadratic monomials *modulo these linear relations*.

Since $g(\beta_{\ell+1}),\dots,g(\beta_q) \in H^2(X,E)$ are linearly independent, the equations in Theorem 5.21.(i) simplify to the following form:

$$w_\alpha = Q_\alpha(z_1,\dots,z_p), \quad \text{for } \alpha = \ell+1,\dots,q, \tag{5.24.6}$$

where each $Q_\alpha(z_1,\dots,z_p)$ is a quadratic polynomial. Consequently, we identify the class of w_α with $Q_\alpha(z_1,\dots,z_p)$ in $\mathbb{E}_{-2}$. Finally, we define the degree zero component as

$$\mathbb{E}_0 := \mathcal{O}_{\mathcal{T}}/\mathfrak{m} \cong \mathbb{C}.$$

In conclusion, we have the following expression of the connection ∇_3 on $\mathbf{V}_3|_{X_0}$ within the smooth decomposition (5.24.2):

$$\nabla_3 := \begin{pmatrix} \nabla\otimes \mathrm{Id}_{\mathbb{E}_{-2}} & \sum_{i=1}^p z_i\eta_{i,0} + \sum_{\alpha=1}^{\ell} w_\alpha\eta_{0,\alpha} & \omega_2(z,w) \\ 0 & \nabla\otimes\mathrm{Id}_{\mathbb{E}_{-1}} & \sum_{i=1}^p z_i\eta_{i,0} + \sum_{\alpha=1}^{\ell} w_\alpha\eta_{0,\alpha} \\ 0 & 0 & \nabla\otimes\mathrm{Id}_{\mathbb{E}_0} \end{pmatrix} \tag{5.24.7}$$

Here we denote by

$$\omega_2 := \sum_{i=1}^p\sum_{\gamma=1}^{\ell} z_i w_\gamma \eta_{i,\gamma} + \sum_{i,j=1}^p z_i z_j\eta_{ij,0} + \sum_{\beta,\gamma=1}^{\ell} w_\beta w_\gamma \eta_{0,\beta\gamma} + \sum_{\alpha=\ell+1}^{q} Q_\alpha(z)\eta_{0,\alpha}, \tag{5.24.8}$$

where

$$\eta_{i,\gamma},\eta_{ij,0},\eta_{0,\beta\gamma},\eta_{0,\alpha} \in A^1(X_0,E)$$

are all defined in (5.21.1). Note that ∇_3 is flat on $\mathbf{V}_3|_{X_0}$ by Theorem 5.23.

***Theorem 5.25* (2-Step Phenomenon). —** *Let $h: Y\to X$ be a holomorphic map between compact Kähler manifolds. Let D be a simple normal crossing divisor on X such that $D_Y := h^{-1}(D)$ is also a simple normal crossing divisor. Set $X_0 := X\backslash D$ and $Y_0 := Y\backslash D_Y$ and fix a base point $x\in X_0$. Denote by $h_0: Y_0\to X_0$ the restriction of h on Y_0. Let (V,∇) be a unitary flat bundle on X with monodromy representation $\varrho: \pi_1(X,x)\to \mathrm{GL}_N(\mathbb{C})$, and set $x = h(y)$. Denote by*

$$h_0^*: R(X_0,N) \longrightarrow R(Y_0,N),$$

the induced morphism of representation varieties, given by $\tau\mapsto h_0^*\tau$ for any $\tau\in R(X_0,N)$. We denote by $i_0: X_0\hookrightarrow X$ the inclusion map, and $\varrho_0 = i_0^*\varrho: \pi_1(X_0)\to\mathrm{GL}_N(\mathbb{C})$.

(i) *Assume that $h_0^*\varrho_{\mathcal{T},3}$ is a trivial representation. Then for any representation $\tau:\pi_1(X_0)\to \mathrm{GL}_N(\mathbb{C})$ that is in the same geometric irreducible component of ϱ_0, $h_0^*\tau$ is a trivial representation.*
(ii) *Assume that $D_Y = \varnothing$, i.e. $Y = Y_0$, and ϱ is trivial. Assume moreover that $a\circ h(Y)$ is a point, where $a: X_0\to A$ denotes the quasi-Albanese map of X_0. Then for any representation $\tau:\pi_1(X_0)\to\mathrm{GL}_N(\mathbb{C})$ that is in the same irreducible component of ϱ_0, $h_0^*\tau$ is a trivial representation. In particular, for any unipotent representation $\tau:\pi_1(X_0)\to\mathrm{GL}_N(\mathbb{C})$, we have $h_0^*\tau$ is trivial.*

Proof. — It suffices to prove that $h_0^*\nabla_{z,w} = h_0^*\nabla$ for all $(z,w)\in\mathcal{T}$. Indeed, since $(h_0^*V, h_0^*\nabla)$ is assumed to be a trivial flat bundle in both items above, this equality combined with Theorem 5.23 implies that the algebraic morphism

$$h_0^*\colon R(X_0,N)\longrightarrow R(Y_0,N)$$

is constant on an analytic neighborhood of ϱ_0 in $R(X_0,N)$. Since the map is algebraic, it follows that h_0^* is constant on the entire irreducible component of $R(X_0,N)$ containing ϱ_0.

Fix an arbitrary $(z_0,w_0)\in\mathcal{T}$. We aim to show that $h_0^*\nabla_{z_0,w_0} = h_0^*\nabla$. Note that $\mathcal{Q}'$ is a quasi-homogeneous cone with weight 2 in z and weight 1 in w. We then can define a holomorphic

curve $\varphi : \mathbb{D}_{1+\varepsilon} \to \mathcal{T}$ by $\varphi(t) = (t^2 z_0, t w_0)$ for some small $\varepsilon > 0$. Then we have the following commutative diagram:
(5.25.1)

$$\begin{tikzcd}
\pi_1(X_0) \arrow[r, "\varrho_{\mathcal{T}}"] \arrow[dr, "\varrho_{\mathcal{T},3}"'] \arrow[rrr, bend left, "\sigma", shift left=3] & \mathrm{GL}_N(\mathcal{O}_{\mathcal{T},0}) \arrow[r, "\varphi^*"] \arrow[d] & \mathrm{GL}_N(\mathcal{O}_{\mathbb{D}_{1+\varepsilon},0}) \arrow[d] & \\
& \mathrm{GL}_N(\mathcal{O}_{\mathcal{T},0}/\mathfrak{m}^3) \arrow[r, "\varphi^*"] & \mathrm{GL}_N(\mathcal{O}_{\mathbb{D}_{1+\varepsilon},0}/\mathfrak{m}^3) \arrow[r, equal] & \mathrm{GL}_N(\mathbb{C}[t]/(t^3))
\end{tikzcd}$$

Consider the family of connections over the disk given by $\nabla_t := \nabla_{t^2 z_0, t w_0}$. The monodromy of this family corresponds to the representation $\varphi^* \circ \varrho_{\mathcal{T}} : \pi_1(X_0) \to \mathrm{GL}_N(\mathcal{O}_{\mathbb{D}_{1+\varepsilon},0})$. Let (E_3, ∇_3) be the corresponding flat bundle for the representation

$$\sigma : \pi_1(X_0) \to \mathrm{GL}_N(\mathbb{C}[t]/(t^3)).$$

Then the connection matrix ∇_3 is given by

$$\nabla_3 = \begin{pmatrix} \nabla \otimes \mathrm{Id}_{t^2/t^3} & t\eta_1(z_0, w_0) & t^2\eta_2(z_0, w_0) \\ 0 & \nabla \otimes \mathrm{Id}_{t/t^2} & t\eta_1(z_0, w_0) \\ 0 & 0 & \nabla \otimes \mathrm{Id}_{\mathbb{C}} \end{pmatrix}. \tag{5.25.2}$$

Proof of (i): By assumption, the pullback representation

$$h_0^* \varrho_{\mathcal{T},3} : \pi_1(Y_0) \to \mathrm{GL}_N(\mathcal{O}_{\mathcal{T},0}/\mathfrak{m}^3)$$

is trivial. Consequently, the induced representation

$$h_0^* \sigma : \pi_1(Y_0) \to \mathrm{GL}_N(\mathbb{C}[t]/(t^3))$$

is also trivial. The triviality of the connection $h_0^* \nabla_3$ implies that
- $h_0^* \nabla$ is trivial;
- we have $\eta_1(z_0, w_0) = 0$ and $\eta_2(z_0, w_0) = 0$.

By Theorem 5.14.(iii), it follows that $h_0^* \nabla_t = h_0^* \nabla$ for all $t \in \mathbb{D}_{1+\varepsilon}$. In particular, this implies $h_0^* \nabla_{z_0, w_0} = h_0^* \nabla$. Since (z_0, w_0) is an arbitrary point in $\mathcal{T}$, we conclude that $h_0^* \nabla_{z,w} = h_0^* \nabla$ for all $(z, w) \in \mathcal{T}$, that is the trivial representation. This proves item (i).

Proof of (ii): Since ϱ is trivial, it follows from (5.21.2) that

$$\eta_1(z_0, w_0) \in \mathcal{H}^1(X, E) \simeq \mathcal{H}^1(X, \mathbb{C}) \otimes \mathbb{C}^{N^2}.$$

By the assumption that $a \circ h(Y) = \{\mathrm{pt}\}$, we have

$$h^* \eta_1(z_0, w_0) = 0.$$

By Theorem 5.14, we have

$$\nabla \eta_2(z_0, w_0) + \eta_1(z_0, w_0) \wedge \eta_1(z_0, w_0) = 0.$$

Hence,

$$\nabla h^* \eta_2(z_0, w_0) = \nabla h^* \eta_2(z_0, w_0) + h^* \eta_1(z_0, w_0) \wedge h^* \eta_1(z_0, w_0) = 0.$$

Recall that $h(Y_0) \cap D = \varnothing$. It follows that $h^* \eta_2(z_0, w_0) \in A^1(X, E) \cap \ker \nabla$. By Theorem 5.14, we know that $\eta_2(z_0, w_0)$ is $\boldsymbol{\nabla}'$-exact; hence, its pullback $h^* \eta_2(z_0, w_0)$ is also $\boldsymbol{\nabla}'$-exact. By Lemma 2.16, we have

$$h^* \eta_2(z_0, w_0) = 0.$$

By Theorem 5.14.(iii), it follows that $h^* \nabla_t = h^* \nabla$ for all $t \in \mathbb{D}_{1+\varepsilon}$. In particular, this implies $h_0^* \nabla_{z_0, w_0} = h_0^* \nabla$. This proves Item (ii). □

Proof of Theorem C. — Since (E,∇) is a trivial flat bundle, the augmentation map

$$\varepsilon_x\colon H^0(\overline{X},E)\to E_x$$

is surjective. Consequently, the complex vector space W, defined in the proof of Lemma 5.22, is zero-dimensional.

By Theorem 5.23, the natural map of analytic germs

$$\begin{aligned} f\colon (\mathcal{T},0) &\to (R(X,N),\varrho)\\ z &\mapsto \mathrm{Mon}(\nabla_{z,w}) \end{aligned}$$

is an isomorphism. The theorem is proved. □

5.10. A geometric application. — As a direct consequence of Theorem 5.25.(ii), we provide a new proof of a result established by Green–Griffiths–Katzarkov [**GGK24**] and Aguilar–Campana [**AAC25**].

Theorem 5.26. — *Let X be a quasi-compact Kähler manifold such that its quasi-Albanese map $a: X\to A$ is proper. If $\pi_1(X)$ is nilpotent, then its Shafarevich morphism is the Stein factorization $g: X\to S$ of a. Moreover, the universal covering of X is holomorphically convex.*

Proof. — Note that any finitely generated nilpotent group is linear. Hence, we may assume the existence of a faithful representation $\tau\colon \pi_1(X)\to \mathrm{GL}_N(\mathbb{C})$ with nilpotent image.

After replacing τ by some conjugate, we may decompose the representation as

$$\tau=(\tau_1,\tau_2),\qquad \tau_i:\pi_1(X)\to \mathrm{GL}_{N_i}(\mathbb{C}),$$

where $\tau_1(\pi_1(X))$ is abelian and torsion-free, and $\tau_2(\pi_1(X))$ is unipotent.

Let $f: Y\to X$ be any compactifiable holomorphic map from a quasi-compact Kähler manifold Y to X such that the image $a(f(Y))$ is a point. Since a is proper, the image $f(Y)$ is contained in a compact fiber. Consequently, by replacing Y with a suitable bimeromorphic modification (or compactification), we may assume that Y is a compact Kähler manifold. Under this assumption, Theorem 5.25.(ii) implies that $f^*\tau_2$ is trivial.

On the other hand, since $\tau_1(\pi_1(X))$ is abelian, τ_1 factors through $H_1(X,\mathbb{Z})$. Moreover, the induced map $a_*: H_1(X,\mathbb{Z})/\mathrm{tor}\to H_1(A,\mathbb{Z})$ is an isomorphism. Thus, there exists a factorization:

$$\begin{tikzcd} \pi_1(X) \arrow[r] \arrow[dr, "\tau_1"'] & \pi_1(A) \arrow[d] \\ & \mathrm{GL}_{N_1}(\mathbb{C}). \end{tikzcd}$$

Since $a(f(Y))$ is a point, the image of $\pi_1(Y)$ in $\pi_1(A)$ is trivial. It follows that $f^*\tau_1(\pi_1(Y))=\{1\}$. Consequently, $f^*\tau(\pi_1(Y))=\{1\}$. Since τ is faithful, this implies that $f_*\pi_1(Y)$ is trivial.

Next, by the characterization of the quasi-Albanese map, for any compactifiable holomorphic map $f: Y\to X$ from a quasi-compact Kähler manifold, if $a(f(Y))$ is not a point, then the image of $f^*H^1(X,\mathbb{C})$ is non-trivial. This implies that $f_*\pi_1(Y)$ is infinite. Combining this with the contracting property established above, we conclude that g is the Shafarevich morphism of X.

We now show the last assertion. By Selberg's lemma, any finitely generated linear group is virtually torsion-free. Hence, up to replacing X by a finite étale cover, we may assume that $\pi_1(X)$ is torsion-free. Consequently, we have

$$\mathrm{Im}[\pi_1(F)\to\pi_1(X)]=\{1\}\quad\text{for each fiber } F \text{ of } g. \tag{5.26.1}$$

Let $\widetilde{X}^{\mathrm{ab}}$ be a connected component of the fiber product $X\times_A\widetilde{A}$, where $\widetilde{A}\to A$ denotes the universal covering of A, and let $\varphi:\widetilde{X}^{\mathrm{ab}}\to W$ be the Stein factorization of the induced map $\widetilde{X}^{\mathrm{ab}}\to\widetilde{A}$.

Let $\pi:\widetilde{X}\to\widetilde{X}^{\mathrm{ab}}$ be the universal cover of $\widetilde{X}^{\mathrm{ab}}$ (which is also the universal cover of X), and consider the composition $\Phi:=\varphi\circ\pi:\widetilde{X}\to W$. By (5.26.1), for any $y\in W$, its preimage $\Phi^{-1}(y)=\pi^{-1}(\varphi^{-1}(y))$ in $\widetilde{X}$ is a disjoint union of copies of the compact connected space $\varphi^{-1}(y)$. In particular, the connected components of the fibers of Φ are compact.

By a theorem of Cartan [Car60], the set Z of connected components of the fibers of Φ can be endowed with the structure of a normal complex space such that Φ factors as $h \circ \psi$, where $\psi : \widetilde{X} \to Z$ is a proper holomorphic fibration and $h : Z \to W$ is a map with discrete fibers. This yields the following commutative diagram:

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\pi"] \arrow[d, "\psi"'] \arrow[dr, "\Phi"] & \widetilde{X}^{\mathrm{ab}} \arrow[r] \arrow[d, "\varphi"] & X \arrow[dd, "g"] \\
Z \arrow[r, "h"'] & W \arrow[d] & \\
& \widetilde{A} \arrow[r] & A
\end{tikzcd}$$

Let $G := \mathrm{Aut}(\widetilde{X}/\widetilde{X}^{\mathrm{ab}})$ be the Galois group of the covering π. By (5.26.1) again, the preimage $\pi^{-1}(\varphi^{-1}(y))$ is exactly the G-orbit of a single connected component in $\widetilde{X}$, and thus the group G acts transitively and freely on the set of connected components of the fibers of Φ. Consequently, G induces a free action on the complex space Z. Furthermore, this action is properly discontinuous (see, e.g., [DYK23, Lemma 3.34]). This implies that the natural action of G on Z is free and properly discontinuous.

By construction, the quotient Z/G is canonically isomorphic to W, meaning $h : Z \to W$ is a topological covering map with Galois group G. Note that the map $W \to \widetilde{A}$ is finite and proper, and since $\widetilde{A} \cong \mathbb{C}^{\dim A}$ is Stein, W is Stein. Since $h : Z \to W$ is a topological covering of a Stein space, Z is also Stein.

Finally, since $\psi : \widetilde{X} \to Z$ is a proper holomorphic map to the Stein space Z, it follows from the Cartan-Remmert theorem that $\widetilde{X}$ is holomorphically convex. The theorem is proved. □

6. Two-step nilpotent monodromy of local systems on special varieties

6.1. Two-step phenomenon for trivial local systems. — It is natural to ask when the condition in Theorem 5.25.(i) is satisfied, enabling the application of the 2-step phenomenon. In Theorem 6.1, we show that for the trivial flat bundle (V, ∇) of rank N on X, the associated representation

$$\varrho_{\mathcal{T},3} : \pi_1(X_0) \to \mathrm{GL}_N(\mathcal{O}_{\mathcal{T},0}/\mathfrak{m}^3),$$

defined in (5.24.1), becomes trivial when restricted to any fiber F of the second iterated quasi-Albanese map; that is, the quasi-Albanese map of a fiber of the quasi-Albanese map of X_0. Consequently, the 2-step phenomenon applies to these fibers F. As an application, we prove Theorem A.

Throughout this subsection, we will keep the same notations as in § 5.9, and we will only recall their meaning when necessary.

Theorem 6.1. **—** *Let X, Y, and Z be compact Kähler manifolds. Let $f : Y \to X$ and $g : Z \to Y$ be holomorphic maps such that $D_Y := f^{-1}(D)$ and $D_Z := g^{-1}(D_Y)$ are simple normal crossing divisors in Y and Z, respectively. Set $Y_0 := Y \setminus D_Y$, and $Z_0 := Z \setminus D_Z$, and denote the restrictions by $f_0 := f|_{Y_0}$ and $g_0 := g|_{Z_0}$.*

Assume that the images of Y_0 and Z_0 under the respective quasi-Albanese maps are points; that is,

$$\mathrm{alb}_{X_0}(f(Y_0)) = \{\mathrm{pt}\} \quad \textit{and} \quad \mathrm{alb}_{Y_0}(g(Z_0)) = \{\mathrm{pt}\}, \tag{6.1.1}$$

where $\mathrm{alb}_{X_0} : X_0 \to \mathrm{Alb}(X_0)$ and $\mathrm{alb}_{Y_0} : Y_0 \to \mathrm{Alb}(Y_0)$ denote the quasi-Albanese maps.

Then the pullback representation

$$(f_0 \circ g_0)^* \varrho_{\mathcal{T}} : \pi_1(Z_0) \to \mathrm{GL}_N(\mathcal{O}_{\mathcal{T},0})$$

is trivial. In particular, for any representation $\tau : \pi_1(X_0, x) \to \mathrm{GL}_N(\mathbb{C})$ belonging to the irreducible component of the representation variety $R(X_0, N)$ that contains the trivial representation, the pullback representation $(f_0 \circ g_0)^* \tau$ is trivial.

Proof. — Let $h_0 := f_0 \circ g_0$. We proceed analogously to the proof of Theorem 5.25.(i) and adopt the notation used therein. It suffices to show that $h_0^*\nabla_{z,w} = h_0^*\nabla$ for all $(z, w) \in \mathcal{T}$.

Fix an arbitrary point $(z_0, w_0) \in \mathcal{T}$. For some small $\varepsilon > 0$, define a holomorphic curve $\varphi : \mathbb{D}_{1+\varepsilon} \to \mathcal{T}$ by $\varphi(t) = (t^2 z_0, tw_0)$. Consider the family of connections parametrized by the disk, given by $\nabla_t := \nabla_{t^2 z_0, tw_0}$. Let (E_3, ∇_3) be the flat bundle associated with the induced representation modulo t^3:

$$\sigma : \pi_1(X_0) \to \mathrm{GL}_N(\mathbb{C}[t]/(t^3))$$

defined in (5.25.1), where the connection matrix ∇_3 is defined in (5.25.2). Since (E, ∇) is a trivial flat bundle, by (5.21.2),

$$\eta_1(z_0, w_0) \in \mathcal{H}^1(X, E) \simeq \mathcal{H}^1(X, \mathbb{C}) \otimes \mathbb{C}^{N^2}.$$

By (6.1.1), we have

$$f^*\eta_1(z_0, w_0) = 0.$$

By Theorem 5.14, we have

$$\nabla\eta_2(z_0, w_0) + \eta_1(z_0, w_0) \wedge \eta_1(z_0, w_0) = 0.$$

This implies that, $\nabla f^*\eta_2(z_0, w_0) = 0$. Hence we have

$$f^*\eta_2(z_0, w_0) \in A^1(Y, D_Y, E) \cap \ker\nabla \simeq \Big(A^1(Y, D_Y, \mathbb{C}) \cap \ker d\Big) \otimes \mathbb{C}^{N^2}$$

Let $(f^*\eta_2(z_0, w_0))^{1,0}$ denote the $(1,0)$-component in $A^1(Y, D_Y, E)$. Then

$$(f^*\eta_2(z_0, w_0))^{1,0} \in H^0(Y, \Omega_Y(\log D_Y) \otimes E) \simeq H^0(Y, \Omega_Y(\log D_Y)) \otimes \mathbb{C}^{N^2}. \tag{6.1.2}$$

By (6.1.1),

$$g^*(f^*\eta_2(z_0, w_0))^{1,0} = 0.$$

Consequently, the form $g^*f^*\eta_2(z_0, w_0)$ is purely of type $(0,1)$ and is ∇-closed:

$$g^*f^*\eta_2(z_0, w_0) \in A^{0,1}(Z, D_Z, E) \cap \ker\nabla \simeq \Big(A^{0,1}(Z) \cap \ker d\Big) \otimes \mathbb{C}^{N^2}. \tag{6.1.3}$$

By Theorem 5.14, we know that $\eta_2(z_0, w_0)$ is $\boldsymbol{\nabla}'$-exact; hence, its pullback $g^*f^*\eta_2(z_0, w_0)$ is also $\boldsymbol{\nabla}'$-exact. Combining this with (6.1.3), we conclude that it vanishes identically.

By Theorem 5.14.(iii), it follows that $h_0^*\nabla_t = h_0^*\nabla$ for all $t \in \mathbb{D}_{1+\varepsilon}$. In particular, this implies $h_0^*\nabla_{z_0,w_0} = h_0^*\nabla$. Since (z_0, w_0) is an arbitrary point in $\mathcal{T}$, we conclude that $h_0^*\nabla_{z,w} = h_0^*\nabla$ for all $(z, w) \in \mathcal{T}$.

Consequently, the pullback representation

$$(f_0 \circ g_0)^*\varrho_{\mathcal{T}} : \pi_1(Z_0) \to \mathrm{GL}_N(\mathcal{O}_{\mathcal{T},0})$$

is trivial. By Theorem 5.23, for any representation $\tau\colon \pi_1(X_0, x) \to \mathrm{GL}_N(\mathbb{C})$ belonging to the irreducible component of the representation variety $R(X_0, N)$ containing the trivial representation, the pullback representation $(f_0 \circ g_0)^*\tau$ is trivial. The proposition is proved. □

6.2. Proof of Theorem A. — We are now in a position to prove Theorem A, assuming Theorem E.

The proof of the latter is deferred to the next section.

Theorem 6.2 **(=Theorem A).** — *Let X be smooth quasi-projective variety that is special. Then for any linear representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$, its image $\varrho(\pi_1(X))$ is virtually nilpotent of class 2.*

Proof. — By Theorem 2.11, after replacing X by a finite étale cover, we may assume that the Zariski closure of $\varrho(\pi_1(X))$ decomposes as a direct product of its central torus and its unipotent radical. This induces a natural decomposition of the representation

$$\varrho = (\tau_1, \tau_2)\colon \pi_1(X) \to \mathrm{GL}_{N_1}(\mathbb{C}) \times \mathrm{GL}_{N_2}(\mathbb{C}),$$

where $\tau_1(\pi_1(X))$ is abelian and torsion-free, and $\tau_2(\pi_1(X))$ is unipotent. It then suffices to show that $\tau_2(\pi_1(X))$ is nilpotent of class 2.

Let $\mathrm{alb}_X\colon X \to \mathrm{Alb}(X)$ be the quasi-Albanese map of X. By Proposition 2.13, it is a dominant morphism with a connected, smooth general fiber Y, yielding the exact sequence

$$\pi_1(Y) \to \pi_1(X) \to \pi_1(\mathrm{Alb}(X)) \to \{1\}.$$

Applying τ_2, we see that the quotient group

$$\tau_2(\pi_1(X))\big/\tau_2(\mathrm{Im}[\pi_1(Y) \to \pi_1(X)])$$

is therefore abelian.

By Theorem E, Y is also special. Applying Proposition 2.13 again, the quasi-Albanese morphism $\mathrm{alb}_Y\colon Y \to \mathrm{Alb}(Y)$ is a dominant morphism with a connected general fiber F. Since $\tau_2(\pi_1(X))$ is unipotent, it follows from Claim 4.13 that τ_2 lies in the irreducible component of the representation variety $R(X, N_2)(\mathbb{C})$ that contains the trivial representation. Letting $\iota\colon Y \hookrightarrow X$ denote the inclusion map, Theorem 6.1 yields

$$\iota^*\tau_2(\mathrm{Im}[\pi_1(F) \to \pi_1(Y)]) = \{1\}.$$

Furthermore, by Proposition 2.13 we have the exact sequence

$$\pi_1(F) \to \pi_1(Y) \to \pi_1(\mathrm{Alb}(Y)) \to \{1\}.$$

This implies that $\iota^*\tau_2\colon \pi_1(Y) \to \mathrm{GL}_{N_2}(\mathbb{C})$ factors through the abelian group $\pi_1(\mathrm{Alb}(Y))$. Consequently, the image $\iota^*\tau_2(\pi_1(Y)) = \tau_2(\mathrm{Im}[\pi_1(Y) \to \pi_1(X)])$ is abelian. It follows that $\tau_2(\pi_1(X))$ is virtually nilpotent of class 2. The theorem is proved. □

6.3. A structure theorem. — As a consequence of Theorem 6.2, we obtain the following structure theorem.

Corollary 6.3. — *Let X be a special smooth quasi-projective variety, and let $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ be a big representation. Then, after replacing X by a finite étale cover, for a very general fiber Y of the quasi-Albanese map $a : X \to A$, the quasi-Albanese map $\mathrm{alb}_Y : Y \to \mathrm{Alb}(Y)$ is birational.*

Proof. — By Selberg's lemma, after replacing X by a finite étale cover, we may assume that $\varrho(\pi_1(X))$ is torsion free. Let Y be a very general fiber of the quasi-Albanese map $a : X \to A$, and let $f : Y \to X$ denote the inclusion. By the proof of Theorem 6.2, the image $f^*\varrho(\pi_1(Y))$ is abelian and torsion free. Since ϱ is big and Y is a very general fiber of a, it follows that the restricted representation

$$f^*\varrho : \pi_1(Y) \to \mathrm{GL}_N(\mathbb{C})$$

is also big. By [**CDY25c**, Theorem B.(iv)], the quasi-Albanese map $\mathrm{alb}_Y : Y \to \mathrm{Alb}(Y)$ is birational. □

7. Quasi-Albanese map of special varieties

In this section, we establish Theorem E. To help the reader navigate the ensuing technicalities, we first provide a rough idea of the proof in [**CC16**]. Suppose Y is a smooth special projective variety, and let $a : Y \to A$ be its Albanese map. To show that the general fiber of a is special, Campana and Claudon [**CC16**] argue by contradiction. They consider the relative core map $c : Y \to X$ of the Albanese map $a : Y \to A$, meaning the general fiber of the induced map $f : X \to A$ is the core of the corresponding fiber of a. By replacing c with a suitable neat model, the properties of the core map ensure that for the orbifold base (X, D_X) of c, the general fibers $f : (X, D_X) \to A$ are of general type. By a result of $C_{n,m}$ conjecture proved by Birkar-Chen, the Kodaira dimension of the orbifold (X, D_X) is at least the relative dimension of f, and is therefore strictly positive.

One then considers the Iitaka fibration $j : X \to J$ induced by a suitable birational model of (X, D_X). A crucial step in [**CC16**] is showing that (X, D_X) admits a birational model $(\bar{X}, D_{\bar{X}})$ whose Iitaka fibration $\bar{j} : \bar{X} \to J$ is the quotient of a locally trivial fibration of abelian varieties by a finite group. This structure implies that the orbifold base (J, D_J) of the map $(\bar{X}, D_{\bar{X}}) \to J$ is of general type. Furthermore, by carefully choosing the birational models for c and j, one can show that the orbifold base of the composition $j \circ c : Y \to J$ is neat and coincides exactly with (J, D_J).

This shows $j \circ c : Y \to J$ is a fibration of general type, which contradicts the assumption that Y is special.

When Y is quasi-projective, we follow a similar strategy, but additional technical difficulties arise. In particular, the delicate orbifold structure of the Iitaka fibration $\bar{j}\colon \bar{X} \to J$ forces us to make supplementary reductions compared to the arguments in [**CC16**]. To bypass these technical obstructions, we do not prove the theorem directly for Y. Instead, our strategy crucially relies on making an *a priori* base change: we establish the result for a carefully chosen étale cover of Y and its suitable birational models. Passing to this specific étale cover from the very beginning ensures that the resulting Iitaka fibration is a locally trivial fibration of toroidal compactifications of semiabelian varieties over the relevant open locus. The goal of §§ 7.4 and 7.5 is to show that passing to an étale cover and a suitable birational model is a valid reduction that does not introduce any new obstacles for the subsequent arguments in the proof of Theorem E.

7.1. Some results on semi-abelian varieties. —

Lemma 7.1. — *Let A be a semi-abelian variety and let $f : C \to A$ be a morphism from another semi-abelian variety such that $f : C \to f(C)$ is an isogeny. Then there exists a surjective isogeny $i : A' \to A$ such that $i^{-1}(B) = C$, where $B := f(C)$. Moreover, the natural morphism $A'/C \to A/B$ is an isomorphism.*

Proof. — Let $B := f(C)$, which is a semi-abelian subvariety of A. Write $A = \mathbb{C}^n/\Gamma_A$, where $\Gamma_A \cong \pi_1(A)$ is a discrete subgroup of $\mathbb{C}^n$. It follows that there exists a complex subspace $V \subset \mathbb{C}^n$ such that $B = V/(V \cap \Gamma_A)$. Write $\Gamma_B := V \cap \Gamma_A$. Since $C \to B$ is an isogeny, there exists a subgroup $\Gamma_C \subset \Gamma_B$ of finite index such that $C = V/\Gamma_C$.

Since B is a closed semi-abelian subvariety of A, the quotient $A/B \cong (\mathbb{C}^n/V)/(\Gamma_A/\Gamma_B)$ is also a semi-abelian variety. Consequently, the quotient group Γ_A/Γ_B is a free abelian group (thus torsion-free). Therefore, there exists an isomorphism

$$g : \Gamma_A \xrightarrow{\sim} \Gamma_B \oplus \Gamma_A/\Gamma_B$$

given by a (non-canonical) splitting of the exact sequence

$$0 \to \Gamma_B \to \Gamma_A \to \Gamma_A/\Gamma_B \to 0.$$

Consider the subgroup

$$\Gamma'_A := g^{-1}(\Gamma_C \oplus \Gamma_A/\Gamma_B)$$

of Γ_A. Because Γ_C has finite index in Γ_B, Γ'_A has finite index in Γ_A. It follows directly from the construction that

$$\Gamma'_A \cap V = \Gamma'_A \cap \Gamma_B = \Gamma_C.$$

Therefore, for the semi-abelian variety $A' := \mathbb{C}^n/\Gamma'_A$, the natural morphism

$$i : \mathbb{C}^n/\Gamma'_A \to \mathbb{C}^n/\Gamma_A$$

is a surjective isogeny $i : A' \to A$ such that

$$V/(\Gamma'_A \cap V) = i^{-1}(B)$$

is a semi-abelian subvariety of A' that is canonically isomorphic to C. It follows that the induced map on the quotients $A'/C \to A/B$ is an isomorphism. □

Lemma 7.2. — *Let A be a semi-abelian variety and let $0 \to T \to A \to A_0 \to 0$ be its Chevalley decomposition, where $T \cong (\mathbb{C}^*)^d$ is an algebraic torus and A_0 is an abelian variety. Let Y be any smooth projective toric compactification of T. Then there exists a smooth compactification $\bar{A}$ of A with a simple normal crossing boundary divisor $D_A := \bar{A}\backslash A$ such that:*

1. *$\bar{A}$ is a fiber bundle over A_0 with fiber Y (specifically, the associated bundle $\bar{A} = A \times^T Y$).*
2. *$K_{\bar{A}} + D_A \sim 0$.*
3. *$\bar{A}$ is a semi-toric variety; specifically, the translation action $A \times A \to A$ extends to an action $A \times \bar{A} \to \bar{A}$.*

A compactification satisfying the above properties will be called a toroidal compactification *for A.*

Furthermore, let $f : B \to A$ be a surjective morphism from another semi-abelian variety B. If f is an isogeny, then f extends to a morphism $\bar{f} : \bar{B} \to \bar{A}$ between some toroidal compactifications $\bar{B}$ and $\bar{A}$ with simple normal crossing boundary divisors $D_B := \bar{B}\backslash B$ and $D_A := \bar{A}\backslash A$, such that $\bar{f}$ is a generically finite surjective morphism, and the log-canonical divisors satisfy the pullback relation:

$$\bar{f}^*(K_{\bar{A}} + D_A) \cong K_{\bar{B}} + D_B.$$

If f is surjective with connected kernel C, then there exist toroidal compactifications $\bar{B}$ and $\bar{A}$ with simple normal crossing boundary divisors $D_B := \bar{B}\backslash B$ and $D_A := \bar{A}\backslash A$ such that:

(i) *the morphism f extends to $\bar{f} : \bar{B} \to \bar{A}$, that is isotrivial over A.*

(ii) *Moreover, every fiber of $\bar{f}$ over A is isomorphic to a toroidal compactification $\bar{C}$ of C, satisfying*

$$K_{\bar{C}} + D_{\bar{C}} \sim 0,$$

where $D_{\bar{C}} := \bar{C}\backslash C$ is the boundary divisor.

Proof. — We begin with the *Chevalley decomposition* of A:

$$0 \to T \to A \xrightarrow{\pi} A_0 \to 0,$$

where A_0 is an abelian variety and $T \cong (\mathbb{C}^*)^d$. Analytically, A is a principal T-bundle over A_0.

Step 1: Construction of $\bar{A}$. Let Y be the given smooth projective toric compactification of T. The torus T acts on Y via the standard toric action. We define $\bar{A}$ as the associated fiber bundle:

$$\bar{A} := A \times^T Y = (A \times Y)/\sim,$$

where T acts by $t(a, y) = (t \cdot a, t^{-1} \cdot y)$ for $t \in T$. Since $A \to A_0$ is locally trivial in the analytic topology, $\bar{A} \to A_0$ is a smooth toric fiber bundle with fiber Y.

The boundary $D_A := \bar{A}\backslash A$ corresponds to the sub-bundle associated with the toric boundary $D_Y = Y\backslash T$. Since Y is smooth, D_Y is a simple normal crossing divisor, and consequently, D_A is a simple normal crossing divisor on $\bar{A}$.

Set $D_Y := Y\backslash T$. Since $K_Y + D_Y \sim 0$, it follows that the relative log-canonical bundle $K_{\bar{A}/A_0} + D_A$ is trivial. Since A_0 is an abelian variety then $K_{A_0} \sim 0$ and it follows that

$$K_{\bar{A}} + D_A \sim 0.$$

Step 2: Extension Setup. Let $0 \to T' \to B \to B_0 \to 0$ be the Chevalley decomposition of B. The morphism $f : B \to A$ induces a morphism of exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & T' \arrow[r] \arrow[d, "g"] & B \arrow[r] \arrow[d, "f"] & B_0 \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & T \arrow[r] & A \arrow[r] & A_0 \arrow[r] & 0 \end{tikzcd} \tag{7.2.1}$$

where h is a morphism of abelian varieties and g is a homomorphism of algebraic tori. Let N' and N be the co-character lattices of T' and T. The morphism g induces a linear map $g_* : N' \to N$. Y is determined by a smooth fan Σ_Y in $N_{\mathbb{R}}$. To construct $\bar{B}$, we must construct a suitable smooth fan Σ_X in $N'_{\mathbb{R}}$.

Step 3: The Isogeny Case. Assume f is an isogeny. Then $g : T' \to T$ is an isogeny, implying that $g_* : N' \to N$ is an injection of lattices with finite cokernel, inducing an isomorphism of the underlying vector spaces $g_{*,\mathbb{R}} : N'_{\mathbb{R}} \xrightarrow{\sim} N_{\mathbb{R}}$.

We define a fan Σ'_X in $N'_{\mathbb{R}}$ as the exact pullback of the smooth fan Σ_Y under this vector space isomorphism. Since Σ'_X may not be regular with respect to the lattice N', we let Σ_X be a smooth subdivision of Σ'_X.

Let X be the smooth toric variety associated to Σ_X. The lattice injection and the fan refinement induce a toric morphism $\bar{g} : X \to Y$ which is proper, generically finite, and surjective. We define $\bar{B} := B \times^{T'} X$. Then we have a morphism $\bar{f} : \bar{B} \to \bar{A}$ induced by the map $\bar{g}$. Note that $\bar{g}$ pulls back the logarithmic volume form of Y to the logarithmic volume form of X (up to a scalar factor).

Since $K_{\bar{A}} + D_A$ and $K_{\bar{B}} + D_B$ are trivialized by these forms extended over the base, we obtain the isomorphism $\bar{f}^*(K_{\bar{A}} + D_A) \cong K_{\bar{B}} + D_B$.

Step 4: The Surjective Case. Let N' and N be the co-character lattices of T' and T. The surjective group homomorphism g induces a linear map of lattices $g_* : N' \to N$. Its kernel $N_C := \ker(g_*)$ is necessarily a saturated sublattice of N', i.e. $N' \cap (N_C)_{\mathbb{R}} = N_C$, which is exactly the co-character lattice of the maximal torus $T_C \subset C$.

Let Σ_Y be any smooth projective fan in $N_{\mathbb{R}}$ compactifying T. We choose Σ_X to be any smooth regular fan in $N'_{\mathbb{R}}$ that refines the preimage $g_{*,\mathbb{R}}^{-1}(\Sigma_Y)$. Because Σ_X is a refinement, the induced toric morphism $\bar{g} : X \to Y$ is well-defined.

We define $\Sigma_C := \Sigma_X \cap (N_C)_{\mathbb{R}}$. Since Σ_X is a smooth regular fan, its restriction Σ_C is also a smooth regular fan, which defines a smooth toric compactification $\bar{T}_C$ of T_C.

We define the compactifications as associated fiber bundles: $\bar{A} := A \times^T Y$ and $\bar{B} := B \times^{T'} X$. The compatible maps of bundles and toric varieties induce the extended morphism $\bar{f} : \bar{B} \to \bar{A}$.

We show that $\bar{f}$ is isotrivial over A. The open stratum $A \subset \bar{A}$ corresponds to the trivial cone $\{0\} \in \Sigma_Y$ by our construction. Its preimage in Σ_X is exactly the sub-fan Σ_C. Since the fan Σ_C is contained in the real subspace $(N_C)_{\mathbb{R}}$ generated by the saturated sublattice $N_C \subset N'$, the toric variety it defines with respect to the larger ambient lattice N' naturally decomposes as the associated bundle $T' \times^{T_C} \bar{T}_C$. Thus, the restriction of the toric variety X over the open torus T is precisely $X|_T = T' \times^{T_C} \bar{T}_C$.

Consequently, the restriction of the total space over A is precisely computed by the associated bundle cancellation:

$$\bar{B}|_A = B \times^{T'} (X|_T) = B \times^{T'} (T' \times^{T_C} \bar{T}_C) = B \times^{T_C} \bar{T}_C.$$

This implies that, the fibers of $\bar{f}$ over A is isomorphic to $C \times^{T_C} \bar{T}_C =: \bar{C}$. Note that $\bar{C}$ is a toroidal compactification of C. This shows the last assertion. The lemma is proved. □

Convention: Throughout this section, unless stated otherwise, any compactification $\bar{A}$ of a semi-abelian variety A is assumed to be a smooth toroidal compactification as in Lemma 7.2 such that $K_{\bar{A}} + D_A$ is trivial, where $D_A := \bar{A} \backslash A$ is a simple normal crossing divisor.

7.2. Subadditivity of the logarithmic Kodaira dimension. —

Theorem 7.3. — *Let (X, D) be a smooth projective log pair, where D is a simple normal crossing divisor with rational coefficients ≤ 1, $\bar{A}$ a smooth toroidal compactification of a semi-abelian variety A, and*

$$f : X \to \bar{A}$$

be a morphism with connected fibers, such that

$$\lfloor D \rfloor \geq f^{-1}(D_A)_{\mathrm{red}}.$$

If a general fiber (X_a, D_a) of $f : (X, D) \to \bar{A}$ is of log general type, then $\kappa(K_X + D) \geq \dim X_a$.

This is an immediate consequence of the following more general statement:

Theorem 7.4. — *Let $f : (X, D) \to (Y, E)$ be a surjective morphism of projective log canonical pairs with both (X, D_{red}) and (Y, E) log-smooth. Assume that $\lfloor D \rfloor$ contains the support of $f^*(E)$, and the generic fiber (X_η, D_η) is of log-general type. Then*

$$\kappa(X, D) \geq \kappa(X_\eta, D_\eta) + \kappa(Y, E).$$

Proof. — Note that this theorem is stated in [**Wei20**, Theorem 12], however the proof is omitted and it is stated that it can be easily deduced from [**KP17**, Theorems 9.5, 9.6]. We include the details for the benefit of the reader but claim no originality.

We follow [**KP17**, Section 9]. Note that if D is reduced, then this follows from [**KP17**, Theorem 9.6]. Clearly we may assume $E = E_{\mathrm{red}}$ and we may assume that $D^v \leq \lfloor D \rfloor$ where $D = D^v + D^h$ and each component of D^v is vertical over Y and each component of D^h dominates Y. We now follow the proof of [**KP17**, Proposition 9.12]. In Step 1, we define $D' = \rho_*^{-1} D^h + (\rho^* D^v)_{\mathrm{red}}$. All the required properties are easily seen to hold (that is $\kappa(K_{X_\eta} + D_\eta) = \kappa(K_{X'_\eta} + D'_\eta)$, $\rho_*(K_{X'} + D') = K_X + D$, and $\kappa(K_X + D) \geq \kappa(K_{X'} + D')$). Thus, we may assume that the consequences of [Lemma 9.10, KP16] hold for $f : (X, D) \to Y$. In Step 2, we apply [**KP17**, Lemma 9.10] to $f : (X, G := D^v) \to Y$ to obtain $f' : (X', G') \to Y'$. We let $\rho : X' \to X$ and set $D' := G' + \rho_*^{-1} D^h$. Since $\rho_*^{-1} D^h \le \rho^* D^h$, it follows easily from the last part of the proof of [KP17, Lemma 9.11] that

$$h^0(m(K_{X'/Y'} + D' + f'^*\tau^*(K_Y + E))) \ge h^0(\zeta^*(m(K_X + D))). \tag{7.4.1}$$

Recall that $\zeta : X^n \to X$ where X^n is the normalization of $X \times_Y Y'$. We then have

$$\kappa(K_X + D) = \kappa(\zeta^*(m(K_X + D))) \ge \kappa(K_{X'/Y'} + D' + f'^*\tau^*(K_Y + E)) \ge$$
$$\kappa(K_{X'_{\eta'}} + D'_{\eta'}) + \kappa(\tau^*(K_Y + E)) = \kappa(K_{X_\eta} + D_\eta) + \kappa(K_Y + E)$$

where the first (in)-equality follows as ζ is finite, the second from (7.4.1), the third by [KP17, Theorem 9.9], and the fourth as $(X'_{\eta'}, D'_{\eta'}) = (X_\eta, D_\eta)_{\eta'}$ and τ is finite. □

Lemma 7.5. — *Let $f : (X, D_X) \to (A, D_A)$ be an orbifold morphism of smooth orbifolds such that $f : X \to A$ is birational, where (A, D_A) is a toroidal compactification of a semi-abelian variety $A^0 := A \backslash D_A$. If $\kappa(K_X + D_X) = 0$, then $f_* D_X = D_A$, in other words; $f : (X, D_X) \to (A, D_A)$ is an elementary birational orbifold morphism (cf. Definition 2.2).*

Proof. — Since $f : (X, D_X) \to (A, D_A)$ is an orbifold morphism, by Definition 2.2, we have

$$D := f^{-1}(D_A) \le \lfloor D_X \rfloor.$$

Note that

$$0 = \bar{\kappa}(X \backslash f^{-1}(D_A)) \le \bar{\kappa}(X \backslash \lfloor D_X \rfloor) = \kappa(K_X + \lfloor D_X \rfloor) \le \kappa(K_X + D_X) = 0.$$

By [CDY25a, Lemma 1.3], it follows that, $X \backslash \lfloor D_X \rfloor \to A^0$ is proper over a big open subset. This implies that

$$\lfloor D_X \rfloor = D + R$$

where R is some f-exceptional effective divisor.

Note that $K_X + D = f^*(K_A + D_A) + F$ where F is f-exceptional. Note that $K_A + D_A \cong 0$. Write $G := D_X - \lfloor D_X \rfloor$. It follows that

$$\kappa(F + R + G) = \kappa(K_X + D_X) = 0$$

where $F + R$ is some f-exceptional and effective divisor. Hence we have

$$\kappa(G) = \kappa(F + R + G) = 0.$$

Therefore, we have

$$\kappa(K_X + \lceil D_X \rceil) = \kappa(F + R + \lceil G \rceil) = \kappa(\lceil G \rceil) = \kappa(G) = 0.$$

By [CDY25a, Lemma 1.3] again, it follows that $X \backslash \lceil D_X \rceil \to A^0$ is proper over a big open subset. Therefore, we have $f_*(D_X) = D_A$. The lemma is proved. □

7.3. A structure theorem. —

Proposition 7.6. — *Let $f : (X, D_X) \to (A, D_A)$ be a surjective morphism of smooth projective orbifold pairs with connected fibers, where D_A is reduced and (A, D_A) is a toroidal compactification of the semi-abelian variety $A^0 := A \backslash D_A$. Define $X_0 := f^{-1}(A^0)$. Suppose that $j : X \to J$ is the Iitaka fibration of (X, D_X) and that for a general point $y \in J$, the restricted morphism $f_y := f|_{X_y}$ is birational onto its image, where $X_y := j^{-1}(y)$.*

Then after replacing (X, D_X) and J with higher birational models $(X', D_{X'})$ and J', where $\nu : X' \to X$ is a birational morphism and $D_{X'} := \nu_*^{-1} D_X + \mathrm{Ex}(\nu)$, there exists a birational morphism $(X, D_X) \to (\bar{X}, D_{\bar{X}})$ such that the following diagram commutes:

$$\begin{tikzcd} & J \\ X \arrow[ur, "j"] \arrow[r, "h"] \arrow[d, "f"'] & \bar{X} \arrow[u, "\bar{j}"'] \arrow[dl, "p"] \\ A & \end{tikzcd}$$

such that the following properties hold:

– *For $\bar{X}_0 := p^{-1}(A^0)$, there exists a Zariski dense open subset $J^0 \subset J$ satisfying $\bar{j}(\bar{X}_0) = J^0$, and the restriction $\bar{j}|_{\bar{X}_0} : \bar{X}_0 \to J^0$ is a locally trivial fibration whose fibers are isomorphic to a fixed semi-abelian subvariety $B^0 \subset A^0$.*
– *The composition $\bar{j} \circ h$ is the Iitaka fibration j of (X, D_X).*
– *We define the divisor $D_{\bar{X}} := h_*(D_X)$ as push-forward of D_X via h. Then the orbifold base $(J, \Delta(\bar{j}, D_{\bar{X}}))$ of $j : (\bar{X}, D_{\bar{X}}) \to J$ is of general type.*

Proof. — Throughout the proof, we will frequently replace (X, D_X) with a higher birational model $(X', D_{X'})$, where $\nu : X' \to X$ is a birational morphism and $D_{X'} := \nu_*^{-1} D_X + \mathrm{Ex}(\nu)$.

Set $D_1 := f^{-1}(D_A)$. For a general $y \in J$, let $D_{1,y} := D_1|_{X_y}$ and $D_{X_y} := D_X|_{X_y}$. Since j is the Iitaka fibration of (X, D_X), we have $\kappa(K_{X_y} + D_{X_y}) = 0$. Since we assume that f_y is birational onto its image, we have:

$$0 \le \kappa(K_{X_y} + D_{1,y}) \le \kappa(K_{X_y} + D_{X_y}) = \kappa((K_X + D_X)|_{X_y}) = 0.$$

Let $X^0 := f^{-1}(A^0)$ and let $f_0 : X^0 \to A^0$ be the restricted morphism. For a general $y \in J$, let $X_y^0 := X_y \cap X^0$. Then the logarithmic Kodaira dimension satisfies $\bar{\kappa}(X_y^0) = \kappa(K_{X_y} + D_{1,y}) = 0$. By [CDY25a, Lemmas 1.3, 1.4], this implies that $f_0|_{X_y^0} : X_y^0 \to A^0$ factors through a proper birational morphism $X_y^0 \to B_y^0$ and a closed immersion $B_y^0 \hookrightarrow A^0$, where B_y^0 is a translate of a semi-abelian subvariety of A^0. Since A^0 contains at most countably many semi-abelian subvarieties, B_y^0 must be a translate of a fixed semi-abelian subvariety $B^0 \subset A^0$. Consequently, there exists a Zariski dense open subset $J^* \subset J$ together with a morphism $J^* \to A^0/B^0$ satisfying the following commutative diagram:

$$\begin{tikzcd} X^* \arrow[r] \arrow[d] & A^0 \arrow[d] \\ J^* \arrow[r] & A^0/B^0 \end{tikzcd}$$

where $j_0 := j|_{X^0}$ and $X^* := j_0^{-1}(J^*)$.

Let $\mathcal{A}^0 := A^0/B^0$, and let $\mathcal{A}$ be a toroidal compactification of $\mathcal{A}^0$. Up to replacing J with a smooth birational model, we may assume that the morphism $J^* \to \mathcal{A}$ extends to a regular morphism $q : J \to \mathcal{A}$. Write $J^0 := q^{-1}(\mathcal{A}^0)$. We may further assume that the boundary divisor $\partial J := J \backslash J^0$ has simple normal crossings. By Lemma 7.2, we can modify the toroidal compactification of A^0 so that the projection $A^0 \to \mathcal{A}^0$ extends to a morphism $\pi : A \to \mathcal{A}$ with the property that the family $(A, D_A) \to \mathcal{A}$ is isotrivial over $\mathcal{A}^0$. Furthermore, the fibers of this family over $\mathcal{A}^0$ are isomorphic to (B, D_B), where B is a toroidal compactification of B^0 satisfying $K_B + D_B \sim 0$, with boundary divisor $D_B := B \backslash B^0$.

Then $X \to J \times_{\mathcal{A}} A$ is a birational morphism, and

$$X^0 \to J \times_{\mathcal{A}} A^0 = J^0 \times_{\mathcal{A}^0} A^0 =: \bar{X}^0$$

is a proper birational morphism, where the latter is a locally trivial fibration of semi-abelian varieties over J^0 with fibers isomorphic to B^0. Applying Lemma 7.2 again, the projection $J^0 \times_{\mathcal{A}} A \to J^0$ is a smooth isotrivial fibration whose fibers are isomorphic to B. We take a smooth projective compactification $\bar{X}$ for $J^0 \times_{\mathcal{A}} A$ with boundary a simple normal crossing divisor such that $\bar{X} \to J$ and $p : \bar{X} \to A$ are regular. We replace X by a higher birational model and assume that the birational map $h : X \to \bar{X}$ is regular. Set $D_{\bar{X}} := h_*(D_X)$. Note that for a very general fiber (X_y, D_{X_y}), we have

$$\kappa(K_{X_y} + D_{X_y}) = 0.$$

We let $D_{\bar{X}} = D_{\bar{X}}^{\mathrm{vert}} + D_{\bar{X}}^{\mathrm{hor}}$ be its vertical and horizontal components over J, and for a general fiber $\bar{X}_y$ of $\bar{j}$, we denote by $D_{\bar{X}_y} := D_{\bar{X}}|_{\bar{X}_y}$. Note that by applying Lemma 7.5, $D_{\bar{X}}^{\mathrm{hor}} \subset p^{-1}(D_A)$ where $p : \bar{X} \to A$ is the induced morphism (and these divisors agree on a general fiber $\bar{X}_y$).

We now analyze $D_{\bar{X}}^{\mathrm{vert}}$. Since $\mathrm{Supp}(D_A) \supset \mathrm{Supp}(\pi^* D_{\mathcal{A}})$ and $(X, D_X) \to (A, D_A)$ is an orbifold morphism, then we have

$$\mathrm{Supp}(\lfloor D_{\bar{X}} \rfloor) \supset (\pi \circ p)^* D_{\mathcal{A}}.$$

Note that

$$(\pi \circ p)^{-1}(\mathcal{A}^0) = J^0 \times_{\mathcal{A}} A,$$

we thus have $\mathrm{Supp}(\lfloor D^{\mathrm{vert}}_{\bar{X}} \rfloor) \supset \bar{j}^*(\partial J)$. If P is a component of the support of $D^{\mathrm{vert}}_{\bar{X}}$ not contained in $\bar{j}^*(\partial J)$, then P intersects $\bar{X} \times_J J^0$ and hence, over J^0, we have $P = \bar{j}^{-1}(Q)$ for some divisor Q on J not contained in ∂J. Since $j^0 : J^0 \times_{\mathcal{A}} A \to J^0$ is a smooth proper fibration, we may write $D_{\bar{X}}$ as:

$$D_{\bar{X}} = D^{\mathrm{hor}}_{\bar{X}} + \bar{j}^{-1}(\partial J) + \sum_{i \in I} (1 - \frac{1}{m_i}) P_i$$

where each P_i is a prime divisor on $\bar{X}$ such that $\bar{j}(P_i) = Q_i$ for some prime divisor $Q_i \subset J$ not contained in ∂J. We then have

$$P_i = \bar{j}^* Q_i + R_i,$$

where R_i is a $\bar{j}$-exceptional divisor supported within $(\bar{j}^*\partial J)_{\mathrm{red}}$. The orbifold base of $\bar{j} : (\bar{X}, D_{\bar{X}}) \to J$, is given by

$$D_J := \Delta(\bar{j}, D_{\bar{X}}) = \sum_{i \in I} (1 - \frac{1}{m_i}) Q_i + \partial J.$$

Claim 7.7. — *We have $\kappa(K_{\bar{X}} + D_{\bar{X}}) = \kappa(K_J + D_J)$.*

Proof. — We begin by observing that $(\bar{X}, D_{\bar{X}})$ has a good minimal model over J. To see this we will apply [HX13, Theorem 1.1]. To this end, note that if $G = \sum_{i \in I} Q_i + \partial J$, then $(X, D_X - \epsilon j^* G)$ is dlt for any $0 < \epsilon \ll 1$ and all log canonical centers of $(\bar{X}, D_{\bar{X}} - \epsilon \bar{j}^* G)$ dominate J. By [HX13, Theorem 1.1] $(\bar{X}, D_{\bar{X}} - \epsilon \bar{j}^* G)$ has a good minimal model over J. Since $K_{\bar{X}} + D_{\bar{X}} - \epsilon \bar{j}^* G \equiv_J K_{\bar{X}} + D_{\bar{X}}$, this is also a good minimal model over J for $(\bar{X}, D_{\bar{X}})$. We denote this by $\phi : \bar{X} \dashrightarrow \bar{X}'$ where $\bar{j}' : \bar{X}' \to J'$ is the induced morphism and $\psi : J' \to J$ is a birational morphism. We now have $K_{\bar{X}'} + D_{\bar{X}'} \sim_{\mathbb{Q}, J'} 0$ so that we can apply the canonical bundle formula. Note that ϕ is an isomorphism over J^0 (which we henceforth identify with $\psi^{-1}(J^0)$) and $(\bar{X}', D_{\bar{X}'}) \to J'$ is isotrivial (over the open subset J^0). Therefore, the moduli part is trivial and so, by the canonical bundle formula, we have $K_{\bar{X}'} + D_{\bar{X}'} = (j')^*(K_{J'} + B_{J'})$ where $B_{J'}$ is the boundary divisor whose coefficients over codimension 1 points Q are computed by $1 - \mathrm{lct}_Q(\bar{X}', D_{\bar{X}'}; (\bar{j}')^* Q)$. Here

$$\mathrm{lct}_Q(\bar{X}', D_{\bar{X}'}; (\bar{j}')^* Q) := \sup\{t \geq 0 | (\bar{X}', D_{\bar{X}'} + t \bar{j}^* Q) \text{ is lc over } \eta_Q\}.$$

Since any divisor contained in the inverse image of $\partial J' = \psi^{-1}(\partial J)$ has coefficient 1 in $D_{\bar{X}'}$, it follows immediately that $\mathrm{lct}_Q(\bar{X}', D_{\bar{X}'}; (\bar{j}')^* Q) = 0$ for any divisor in $\partial J'$. If Q is not contained in $\partial J'$ then we can identify it with a divisor on J^0 and hence (over η_Q) $(\bar{j}')^* Q = P$ for some prime divisor P on $\bar{X}'$. If m is the multiplicity of $D_{\bar{X}}$ along P (which is the same as the multiplicity of $D_{\bar{X}'}$ along P), then, the log canonical threshold is

$$\mathrm{lct}_Q(\bar{X}', D_{\bar{X}'}; (\bar{j}')^* Q) = \sup\{t \geq 0 | 1 - \frac{1}{m} + t \leq 1\} = \frac{1}{m}.$$

Therefore the coefficient of $B_{J'}$ along Q is $1 - \frac{1}{m}$. Thus $B_{J'} = \Delta(g, D_{\bar{X}})$ where $g : \bar{X} \to J'$ is the induced map.

Finally note that since $(\bar{X}, D_{\bar{X}})$ is log canonical and ϕ is a $K_{\bar{X}} + D_{\bar{X}}$ minimal model, then

$$\begin{aligned} \dim J = \kappa(K_X + D) = \kappa(K_{\bar{X}} + D_{\bar{X}}) = \kappa(K_{\bar{X}'} + D_{\bar{X}'}) = \\ \kappa(K_{J'} + B_{J'}) \geq \kappa(K_J + B_J) = \kappa(K_J + D_J) \geq \dim J. \end{aligned} \tag{7.7.1}$$

So $\kappa(K_{\bar{X}} + D_{\bar{X}}) = \kappa(K_J + D_J)$. □

It follows from (7.7.1) that the orbifold base (J, D_J) of $\bar{j} : (\bar{X}, D_{\bar{X}}) \to J$ is of general type. The proposition is proved. □

7.4. Some reductions. — To prove Theorem E, we will replace Y_0 with suitable finite étale covers and some birational models.

We start with the following lemma, whose proof is standard (cf. [CDY25b, Lemma 4.8]) so we omit it.

Lemma 7.8. — *Let $\nu : (\widehat{X}, D_{\widehat{X}}) \to (X, D_X)$ be a generically finite surjective orbifold morphism between smooth orbifolds. Let $D \subset \lfloor D_X \rfloor$ be such that $\widehat{X} \backslash \hat{D} \to X \backslash D$ is a finite étale morphism, where $\hat{D} := \nu^{-1}(D)$. Write $\widehat{X}_0 := \widehat{X} \backslash \hat{D}$ and $X_0 := X \backslash D$. If $\nu^* D_X \cap \widehat{X}_0 = D_{\widehat{X}} \cap \widehat{X}_0$, then we have*

$$\kappa(K_X + D_X) = \kappa(K_{\widehat{X}} + D_{\widehat{X}}).$$

Lemma 7.9. — *Let $c : Y \to X$ be an algebraic fiber space between smooth projective varieties. Let D_Y be a reduced simple normal crossings divisor on Y. Let (X, D_X) be the orbifold base of (Y, D_Y), and assume that the support of D_X has simple normal crossings. Let $\nu : \widehat{X} \to X$ be a generically finite surjective morphism from a smooth projective variety $\widehat{X}$ such that $\nu^{-1}(D_X)$ has simple normal crossings support, and let $D \subset \lfloor D_X \rfloor$ be a divisor such that the restriction $\widehat{X} \backslash \hat{D} \to X \backslash D$ is a finite étale morphism, where $\hat{D} := \nu^{-1}(D)$. Assume further that $c^{-1}(D) \subset D_Y$. Let $\mu : \widehat{Y} \to Y$ be the natural morphism from a strong resolution of singularities $\widehat{Y} \to \widehat{X} \times_X Y$, chosen such that $D_{\widehat{Y}} := \mu^{-1}(D_Y)$ is also a simple normal crossings divisor. Then, letting $(\widehat{X}, D_{\widehat{X}})$ be the orbifold base of the natural morphism $\hat{c} : (\widehat{Y}, D_{\widehat{Y}}) \to \widehat{X}$, we have*

$$\kappa(K_X + D_X) = \kappa(K_{\widehat{X}} + D_{\widehat{X}}).$$

Proof. — Write $Y_0 := Y \backslash D_Y$. Since $c^{-1}(D) \subset D_Y$, the morphism $\widehat{X} \backslash \hat{D} \to X \backslash D$ is finite étale, and $\widehat{Y} \to \widehat{X} \times_X Y$ is a strong resolution of singularities, it follows that $\widehat{Y}_0 := \mu^{-1}(Y_0)$ is étale over Y_0. Furthermore, because the induced map $\pi_1(Y) \to \pi_1(X)$ is surjective, $\widehat{Y}$ is connected. Since we have the Cartesian square

$$\begin{array}{ccc} \widehat{Y}_0 & \longrightarrow & \widehat{X}_0 \\ \big\downarrow & & \big\downarrow \\ Y_0 & \longrightarrow & X_0 \end{array}$$

where the vertical morphisms are étale, and since $(\widehat{X}, D_{\widehat{X}})$ is the orbifold base of $\hat{c} : (\widehat{Y}, D_{\widehat{Y}}) \to \widehat{X}$, it follows that

$$\nu^* D_X \cap \widehat{X}_0 = D_{\widehat{X}} \cap \widehat{X}_0.$$

Note that since $\hat{c}^{-1}(\hat{D}) \subset D_{\widehat{Y}}$, it follows that $\hat{D} \subset \lfloor D_{\widehat{X}} \rfloor$. Hence, $\nu : (\widehat{X}, D_{\widehat{X}}) \to (X, D_X)$ is an orbifold morphism. The lemma then follows from Lemma 7.8. □

Lemma 7.10. — *Let $\mu_0 : X_0 \to Y_0$ be a birational morphism between smooth quasi-projective varieties that is proper over a big open subset of Y_0. If X_0 is special, then Y_0 is special.*

Proof. — Let X and Y be smooth projective compactifications of X_0 and Y_0, respectively, such that $D_X := X \backslash X_0$ and $D_Y := Y \backslash Y_0$ are simple normal crossings divisors, and such that μ_0 extends to a morphism $\mu : X \to Y$. By assumption, there exists a Zariski closed subset $Z \subset Y_0$ of codimension at least two such that μ_0 is proper over $Y_0 \backslash Z$. Then, for any prime divisor E contained in D_X, if $\mu(E)$ is not contained in D_Y, it follows that $\mu(E)$ must be contained in the closure of Z in Y, and hence has codimension at least two. Therefore, we have $\mu_*(D_X) = D_Y$. Thus, $\mu : (X, D_X) \to (Y, D_Y)$ is an elementary birational morphism between orbifolds (see Definition 2.2). Since specialness is defined via birational models of orbifolds (see Definition 2.9), it follows that (X, D_X) is special if and only if (Y, D_Y) is special. □

Lemma 7.11. — *Let $\mu_0 : X_0 \to Y_0$ be a birational morphism between smooth quasi-projective varieties that is proper over a big open subset of Y_0. Then to prove Theorem E for Y_0, it suffices to prove Theorem E for X_0.*

Proof. — By assumption, the induced map $\pi_1(X_0) \to \pi_1(Y_0)$ is an isomorphism. Thus, their quasi-Albanese varieties coincide, yielding the commutative diagram

$$\begin{tikzcd} X_0 \arrow[r, "\mu_0"] \arrow[dr, "a"'] & Y_0 \arrow[d, "b"] \\ & A \end{tikzcd}$$

where a and b are their respective quasi-Albanese maps. If Y_0 is special, then by Lemma 7.10, X_0 is also special. By Proposition 2.13, a and b are both dominant morphisms with connected general fibers. Let Z be the Zariski closed subset of Y_0 of codimension at least two such that μ_0 is proper over $Y_0 \backslash Z$. It follows that for a general fiber $Y_{0,t} := b^{-1}(t)$ of b, the intersection $Y_{0,t} \cap Z$ has codimension at least two in $Y_{0,t}$. Therefore, the restriction of μ_0 to the general fiber $X_{0,t} := a^{-1}(t)$ is a birational morphism $X_{0,t} \to Y_{0,t}$ that is proper over a big open subset. Hence, by Lemma 7.10, $X_{0,t}$ is special if and only if $Y_{0,t}$ is special. □

This lemma enables us to take birational models of orbifolds in the proof of Theorem E.

Lemma 7.12. — *Let $c : Y \to X$ be an algebraic fiber space between smooth projective varieties. Let $f : (Y, D_Y) \to (A, D_A)$ be a morphism between smooth orbifolds with $D_A = (D_A)_{\text{red}}$. Assume that we have a commuting tower of algebraic fiber spaces*

$$\begin{tikzcd} Y \arrow[rr, "c"] \arrow[dr, "f"'] & & X \arrow[dl, "g"] \\ & A & \end{tikzcd}$$

Consider the orbifold divisor $D_X := \Delta(c, D_Y)$ of $c : (Y, D_Y) \to X$, which we assume has simple normal crossings support. If $a \in A$ is a general point, let $c_a : Y_a \to X_a$ denote the restriction of c to the fibers Y_a and X_a, and let $D_{X,a}$ denote the restriction of D_X to X_a. We then have:

$$D_{X,a} \geq D_{c_a},$$

where $D_{c_a} = \Delta(c_a, D_{Y,a})$ is the orbifold divisor of $c_a : (Y_a, D_{Y,a}) \to X_a$. In particular, if c is the relative core of $(Y, D_Y) \to A$, then the general fibers of $(X, D_X) \to A$ are of general type.

Proof. — Let D_i be a component of D_X such that $g(D_i) = A$. We write the decomposition:

$$c^*(D_i) = \sum_{j \in J} m_j^{(i)} D_j^{(i)} + R_i$$

where the divisors $D_j^{(i)}$ (contained in Y) map surjectively onto D_i, and where R_i is c-exceptional. By restricting this equality to the general fiber Y_a, it follows that

$$c_a^*(D_i|_{X_a}) = c^*(D_i)|_{Y_a} = \sum_{j \in J} m_j^{(i)} D_j^{(i)}|_{Y_a} + R_i|_{Y_a}$$

where $c_a(D_j^{(i)}|_{Y_a}) = D_i|_{X_a}$. We observe that the set of divisors over which the infimum is taken to calculate the multiplicity of $D_i|_{X_a}$ in D_{c_a} is greater than or equal to the set corresponding to J. Indeed, certain components of $R_i|_{Y_a}$ may no longer be c_a-exceptional. This yields the stated inequality. □

7.5. Suitable base changes. —

Lemma 7.13. — *Let (Y, D_Y) be a smooth orbifold with $D_Y = (D_Y)_{\text{red}}$. Assume that we have an orbifold morphism $a : (Y, D_Y) \to (A, D_A)$ with connected fibers, where A is a toroidal compactification of a semiabelian variety $A^0 := A \backslash D_A$. Let $c : (Y, D_Y) \to X$ be an algebraic fiber space such that its orbifold base (X, D_X) is smooth. We assume the following:*

- *There exists an algebraic fiber space $f : X \to A$ such that $f \circ c = a$.*

– *There exists a birational morphism $g : Y \to Y'$ onto a smooth projective variety such that c is neat relative to g. Moreover, there exists a morphism $a' : Y' \to A$ such that the following diagram commutes:*

$$\begin{tikzcd} Y' \arrow[dr, "a'"'] & Y \arrow[l, "g"'] \arrow[r, "c"] \arrow[d, "a"] & X \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

– *The Iitaka fibration $j : X \to J$ of (X, D_X) is a regular morphism, where J is a smooth projective variety.*
– *For a very general fiber $(X_y, D_{X_y}) := (X, D_X)|_{X_y}$ of $j : X \to J$, the restriction $f_y := f|_{X_y} : X_y \to A$ is generically finite onto its image.*

Then there exists a surjective isogeny $A_1^0 \to A^0$ of semi-abelian varieties together with suitable extended morphism between their toroidal compactifications $A_1 \to A$ as in Lemma 7.2 such that the following holds: let $X^0 := X \backslash f^{-1}(D_A)$ and $Y^0 := Y \backslash D_Y$. Let $\widehat{X}^0 := X^0 \times_{A^0} A_1^0$ and let $\widehat{X}$ and $\widehat{Y}$ be suitable strong desingularizations of $X \times_A A_1$ and $Y \times_A A_1$ such that $\widehat{X} \backslash \widehat{X}^0$ and $D_{\widehat{Y}} := \widehat{Y} \backslash \mu^{-1}(Y^0)$ are both simple normal crossing divisors, admitting a natural morphism $\hat{c} : \widehat{Y} \to \widehat{X}$. Let $(\widehat{X}, D_{\widehat{X}})$ be the orbifold base of $\hat{c} : (\widehat{Y}, D_{\widehat{Y}}) \to \widehat{X}$. Let $\widehat{X} \xrightarrow{\hat{j}} \hat{J} \to J$ be the Stein factorization of $j \circ \nu : \widehat{X} \to J$. Then $\hat{j} : \widehat{X} \to \hat{J}$ is the Iitaka fibration of $(\widehat{X}, D_{\widehat{X}})$, and for a general $y \in \hat{J}$, the induced morphism $\widehat{X}_y \to A_1$ is birational onto its image, where A_1 is a suitable toroidal compactification of A_1^0.

$$\begin{tikzcd} \widehat{Y} \arrow[r, "\hat{c}"] \arrow[d] \arrow[dd, bend right=60, "\mu"'] & \widehat{X} \arrow[r, "\hat{j}"] \arrow[d] \arrow[dd, bend left=60, "\nu"] & \hat{J} \arrow[dd] \\ Y \times_A A_1 \arrow[d] & X \times_A A_1 \arrow[d] & \\ Y \arrow[r, "c"] & X \arrow[r, "j"] & J \end{tikzcd}$$

Proof. — Write $D := f^{-1}(D_A)$. Since a is a morphism of orbifolds, it follows that $D_Y \geq a^{-1}(D_A)$. Since (X, D_X) is the orbifold base of $c : (Y, D_Y) \to X$, we have

$$D_X \geq f^{-1}(D_A) =: D.$$

Let $D_y := D|_{X_y}$ and $D_{X_y} := D_X|_{X_y}$ for a general $y \in J$. Since f_y is generically finite onto its image, and $j : X \to J$ is the Iitaka fibration of (X, D_X), we have

$$0 \leq \kappa(K_{X_y} + D_y) \leq \kappa(K_{X_y} + D_{X_y}) = 0.$$

Then $\bar{\kappa}(X_y^0) = \kappa(K_{X_y} + D_y) = 0$, where X_y^0 is the fiber of $X^0 \to J$ at y. By [**CDY25a**, Lemmas 1.3, 1.4], it follows that the Stein factorization of $X_y^0 \to A^0$ factors through a proper birational morphism $X_y^0 \to C_y^0$, and a surjective isogeny $C_y^0 \to B_y^0$ between semi-abelian varieties, where B_y^0 is a semi-abelian subvariety of A^0. Since A^0 contains at most countably many semi-abelian subvarieties, it follows that B_y^0 is a translate of a fixed semi-abelian subvariety B^0 of A^0. In this case, C_y^0 does not depend on y, and we shall denote by C^0. By Lemma 7.1, there exists a surjective isogeny $i : A_1^0 \to A^0$ such that

$$i^{-1}(B^0) = C^0,$$

and the induced map $A_1^0/C^0 \to A^0/B^0$ is an isomorphism.

By Lemma 7.2, after changing the toroidal compactification A for A^0, there exists a toroidal compactification A_1 for A_1^0 such that the surjective isogeny $A_1^0 \to A^0$ extends to a morphism $A_1 \to A$. Consider the fiber product $X \times_A A_1$, which is connected since $\pi_1(X) \to \pi_1(A)$ is surjective. Let $\widehat{X}$ be a strong resolution of the singularities of $X \times_A A_1$ and let $\nu : \widehat{X} \to X$ be the induced natural morphism. Set $\widehat{X}^0 := \nu^{-1}(X^0)$. Then the restriction $\widehat{X}^0 \to X^0$ is finite étale.

The fiber of $\widehat{X}^0 \to J$ over a general point $y \in J$ is exactly the fiber product $X^0_y \times_{A^0} A^0_1$. Since the image of X^0_y in A^0 is a translate B^0_y of B^0, and the Stein factorization of $X^0_y \to B^0_y$ is the composition of a proper birational morphism $X^0_y \to C^0$ and the isogeny $C^0 \to B^0 \simeq B^0_y$, it follows that the fiber product $X^0_y \times_{A^0} A^0_1$ decomposes into a disjoint union of connected components, each canonically isomorphic to X^0_y. Hence, for each connected component, its image under the natural morphism $\widehat{X}^0 \to A^0_1$ is mapped proper birationally to a translate of C^0.

Let $\widehat{X} \xrightarrow{\hat{j}} \hat{J} \to J$ be the Stein factorization of $j \circ \nu : \widehat{X} \to J$. The general fibers of $\widehat{X} \to J$ are disjoint unions of the connected general fibers $\widehat{X}_{y'}$ of $\hat{j}$. Recall that a general fiber $\widehat{X}^0_{y'}$ of $\widehat{X}^0 \to \hat{J}$ is exactly a general fiber of $X^0 \to J$. Consequently, the restricted morphism $\hat{f}|_{\widehat{X}_{y'}} : \widehat{X}_{y'} \to A_1$ is birational onto its image, where $\hat{f} : \widehat{X} \to A_1$ is the natural morphism. This concludes the first assertion of the lemma.

Finally, we show that $\hat{j} : \widehat{X} \to \hat{J}$ is the Iitaka fibration of $(\widehat{X}, D_{\widehat{X}})$. By Lemma 7.9, the orbifold base $(\widehat{X}, D_{\widehat{X}})$ of $\hat{c} : (\widehat{Y}, D_{\widehat{Y}}) \to \widehat{X}$ has the same Kodaira dimension as (X, D_X). It follows that a general fiber of $(\widehat{X}, D_{\widehat{X}}) \to \hat{J}$ has zero Kodaira dimension, and thus it is the Iitaka fibration of $(\widehat{X}, D_{\widehat{X}})$. The lemma is proved. □

Lemma 7.14. — *Let (X, D_X) be a smooth orbifold and let A^0 be a semi-abelian variety. Let A be a toroidal compactification of A^0. Assume that $f : (X, D_X) \to (A, D_A)$ is an orbifold morphism such that $f_* \mathcal{O}_X = \mathcal{O}_A$ and*

$$\lfloor D_X \rfloor \geq f^{-1}(D_A) =: D.$$

If a general fiber of $f : (X, D_X) \to A$ is of general type, then after replacing (X, D_X) by a birational model, the Iitaka fibration $j : (X, D_X) \to J$ of (X, D_X) is regular, and for a very general fiber X_y of j, the restriction $f|_{X_y} : X_y \to A$ is generically finite onto its image.

Proof. — By Theorem 7.3, we have

$$\kappa(K_X + D_X) \geq \dim X - \dim A > 0.$$

Therefore, we consider the Iitaka fibration $j : (X, D_X) \dashrightarrow J$ of (X, D_X), where $\dim J = \kappa(K_X + D_X)$. By [**Cam11**, Théorème 3.9], after replacing (X, D_X) by a birational model, j is a regular morphism, and a very general fiber (X_y, D_{X_y}) of j is a smooth orbifold of Kodaira dimension zero.

$$\begin{array}{ccc} (X, D_X) & \xrightarrow{j} & J \\ \big\downarrow{\scriptstyle f} & & \\ A & & \end{array}$$

Let $h_y : \Gamma_y \to f(X_y)$ be a desingularization such that $D_{\Gamma_y} := h_y^{-1}(D_A)$ is a simple normal crossing divisor. We replace (X_y, D_{X_y}) by a birational model such that the induced rational map $X_y \dashrightarrow \Gamma_y$ becomes regular. Note that $f_y : (X_y, D_{X_y}) \to (\Gamma_y, D_{\Gamma_y})$ is an orbifold morphism.

Assume by contradiction that the lemma fails. For a general $a \in \Gamma_y$, we denote by $(X_{y,a}, D_{X_{y,a}})$ the fiber of $f_y = f|_{X_y} : (X_y, D_{X_y}) \to \Gamma_y$. Note that a general fiber (X_a, D_{X_a}) of $f : (X, D_X) \to A$ is of general type, and that (X_a, D_{X_a}) is covered by $\bigcup_{\{y | a \in f(X_y)\}} (X_{y,a}, D_{X_{y,a}})$. Then by [**Cam11**, Théorème 9.12], for a fixed general $y \in J$, we have that $(X_{y,a}, D_{X_{y,a}})$ is of general type for a general $a \in \Gamma_y$.

Note that $\Gamma_y \backslash D_{\Gamma_y}$ is birational to a subvariety of the semi-abelian variety A^0. It follows that

$$\kappa(K_{\Gamma_y}, D_{\Gamma_y}) \geq 0.$$

Applying Theorem 7.4 to $f_y : (X_y, D_{X_y}) \to (\Gamma_y, D_{\Gamma_y})$, we conclude that

$$0 = \kappa(X_y, D_{X_y}) \geq \dim X_{y,a} + \kappa(K_{\Gamma_y} + D_{\Gamma_y}),$$

and hence $\dim X_{y,a} = 0$, so that $f_y : X_y \to f(X_y)$ is generically finite. The lemma is proved. □

7.6. Proof of Theorem E. —

Theorem 7.15. — *Let Y_0 be smooth quasi-projective variety. If Y_0 is special, then a general fiber of its quasi-Albanese map is also special.*

Proof. — We adapt the proof of [CC16, Théorème 2.4], with suitable modifications to the quasi-projective setting. Let $a_0 : Y_0 \to A^0$ be the quasi-Albanese map. By Proposition 2.13, a_0 is dominant with connected fibers. We let (A, D_A) be a toroidal compactification of A^0. We may also pick a smooth compactification Y with $D_Y := Y \backslash Y_0$ being simple normal crossing such that the induced map $a : Y \to A$ is an algebraic fiber space and $D_Y = Y \backslash Y_0$ is a reduced divisor with simple normal crossings. In particular $a : (Y, D_Y) \to (A, D_A)$ is an orbifold morphism.

We proceed by contradiction and assume that general fibers of a_0 are not special. Then for any surjective isogeny $\varphi : A_1^0 \to A^0$ from another semi-abelian variety, the general fibers of $Y_0 \times_{A^0} A_1^0 \to A_1^0$ are also not special. Let A_1 be a toroidal compactification for A_1^0 and we take a smooth projective compactification Y_φ for $Y_0 \times_{A^0} A_1^0 \to A_1^0$ such that the boundary $D_{Y_\varphi} := Y_\varphi \backslash Y_0 \times_{A^0} A_1^0$ is a simple normal crossing divisor.

By Theorem 2.10, we may consider the relative core map of $(Y_\varphi, D_{Y_\varphi})$ over A_1:

$$c_\varphi : Y_\varphi \dashrightarrow X_\varphi := C(Y_\varphi, D_{Y_\varphi}/A_1).$$

Replacing c_φ by an appropriate birational model, we may assume that c_φ is a regular, neat, and high fibration. Let $(X_\varphi, D_{X_\varphi})$ be the orbifold base of $c_\varphi : (Y_\varphi, D_{Y_\varphi}) \to X_\varphi$. In this case, by Lemma 2.7, we have

$$\kappa(X_\varphi, D_{X_\varphi}) = \kappa(c_\varphi, D_{Y_\varphi}). \tag{7.15.1}$$

Note that the equivalence class of the fibration c_φ is uniquely determined by φ up to birational equivalence.

We define a bimeromorphic invariant associated with Y_0 by

$$\mathcal{K}(Y_0) := \inf_{\varphi : A_1^0 \to A^0} \{\kappa(c_\varphi, D_{Y_\varphi})\}, \tag{7.15.2}$$

where $\kappa(c_\varphi, D_{Y_\varphi})$ is the canonical dimension of c_φ defined in Definition 2.6, and the infimum is taken over all surjective isogenies $\varphi : A_1^0 \to A^0$ from semi-abelian varieties to A^0.

Claim 7.16. — *We have $\mathcal{K}(Y_0) > 0$.*

Proof. — By Theorem 2.10 and Lemma 7.12, the general orbifold fibers $(X_{\varphi,y}, D_{X_{\varphi,y}}) := (X_\varphi, D_{X_\varphi})|_{X_{\varphi,y}}$ of the natural morphism $(X_\varphi, D_{X_\varphi}) \to A_1$ are of general type. Hence, by Theorem 7.3 and (7.15.1), we obtain

$$\kappa(c_\varphi, D_{Y_\varphi}) = \kappa(X_\varphi, D_{X_\varphi}) \geq \dim X_\varphi - \dim A_1.$$

By our assumption and construction, for each φ, the general fibers of $(Y_\varphi, D_{Y_\varphi}) \to A_1$ are not special. Since c_φ is its relative core map, the dimension of its base must be strictly greater than the dimension of A_1, yielding

$$\dim X_\varphi - \dim A_1 > 0.$$

The claim is thus proved. □

Since $Y_\varphi \setminus D_{Y_\varphi}$ is birational to a finite étale cover of Y_0 via a birational morphism that is proper over a big open subset, the pair $(Y_\varphi, D_{Y_\varphi})$ remains special by [Cam11, Proposition 10.11] and Lemma 7.10. Furthermore, the general fibers of the morphism $Y_\varphi \setminus D_{Y_\varphi} \to A_1^0$ are birational to those of $Y_0 \to A^0$, and this birational equivalence is also proper over a big open subset. Thus, by Lemma 7.11, the proof of the theorem reduces to showing that the general fibers of $(Y_\varphi, D_{Y_\varphi}) \to A_1$ are special. To this end, we choose a specific isogeny $\varphi : A_1^0 \to A^0$ that attains the infimum of $\mathcal{K}(Y_0)$, that is, we have

$$\kappa(c_\varphi, D_{Y_\varphi}) = \mathcal{K}(Y_0).$$

To simplify notation, we replace the original $(Y, D_Y) \to A$ with this minimizing $(Y_\varphi, D_{Y_\varphi}) \to A_1$. Thus, we have

$$\kappa(c, D_Y) = \mathcal{K}(Y_0). \tag{7.16.1}$$

Furthermore, by Proposition 2.8, we may assume that:

(a) $c : (Y, D_Y) \longrightarrow X$ is *neat and high*,
(b) the orbifold base $(X, D_X := \Delta(c, D_Y))$ is smooth,
(c) the Iitaka fibration of (X, D_X) is a regular morphism $j : X \to J$.

By Lemma 2.7 and our minimizing assumption, we have

$$\kappa(X, D_X) = \kappa(c, D_Y) = \mathcal{K}(Y_0). \tag{7.16.2}$$

By Theorem 2.10 and Lemma 7.12, the general orbifold fibers $(X_y, D_{X_y}) := (X, D_X)|_{X_y}$ of $f : (X, D_X) \to A$ are of general type. By Lemma 7.14, for a general fiber X_y of j, the restricted morphism $f|_{X_y} : X_y \to A$ is generically finite onto its image.

We now apply Lemma 7.13. There exists a new surjective isogeny $A_1^0 \to A^0$ of semi-abelian varieties such that if $X^0 := X \setminus f^{-1}(D_A)$ and $Y^0 := Y \setminus D_Y$, $\widehat{X}^0 := X^0 \times_{A^0} A_1^0$, $\widehat{X}$ a suitable smooth projective compactification of $\widehat{X}^0$ such that if $\widehat{X} \setminus \widehat{X}^0$ is a simple normal crossing divisor and $\widehat{Y} \to \widehat{X} \times_X Y$ a strong desingularization, admitting natural morphisms $\nu : \widehat{X} \to X$, $\hat{c} : \widehat{Y} \to \widehat{X}$, and $\mu : \widehat{Y} \to Y$ such that $D_{\widehat{Y}} := \widehat{Y} \setminus \mu^{-1}(Y^0)$ is a simple normal crossing divisor, $(\widehat{X}, D_{\widehat{X}})$ the orbifold base of $\hat{c} : (\widehat{Y}, D_{\widehat{Y}}) \to \widehat{X}$, $\widehat{X} \xrightarrow{\hat{j}} \hat{J} \to J$ the Stein factorization of $j \circ \nu : \widehat{X} \to J$, then $\hat{j} : \widehat{X} \to \hat{J}$ is the Iitaka fibration of $(\widehat{X}, D_{\widehat{X}})$, and for a general point $y \in \hat{J}$, the induced morphism $\widehat{X}_y \to A_1$ is birational onto its image, where A_1 is a suitable toroidal compactification of A_1^0.

Crucially, because $\mathcal{K}(Y_0)$ was defined as the absolute infimum over all isogenies, and by utilizing (7.16.2), we obtain:

$$\kappa(\hat{c}, D_{\widehat{Y}}) = \kappa(X, D_X) = \kappa(\widehat{X}, D_{\widehat{X}}). \tag{7.16.3}$$

This fact is essential; as we will see later (by Proposition 2.5.(ii) and Definition 2.6), for any further high birational model of $(\widehat{Y}, D_{\widehat{Y}}) \to \widehat{X}$, the Kodaira dimension of the resulting orbifold base remains equal to $\kappa(X, D_X)$.

To simplify the presentation in what follows, we will drop the hats and rename $(\widehat{Y}, D_{\widehat{Y}}) \to (\widehat{X}, D_{\widehat{X}}) \to \hat{J}$ and $(\widehat{X}, D_{\widehat{X}}) \to A_1$ simply as $(Y, D_Y) \to (X, D_X) \to J$ and $f : (X, D_X) \to A$.

We also note that $f : (X, D_X) \to (A, D_A)$ is an orbifold morphism. To see this, let Q be any prime divisor in the support of D_A and P a prime divisor in the support of f^*Q. If E is a prime divisor contained in the support of c^*P, then it is contained in the support of a^*Q, and so $m_E(D_Y) = +\infty$. This implies that $m_P(D_X) = +\infty$ and so $(X, D_X) \to (A, D_A)$ is an orbifold morphism.

Consider the following commutative diagram where each vertical map is a birational modification and Y', X' and J' are smooth,

$$\begin{tikzcd} Y' \arrow[r, "c'"] \arrow[d, "\mu"] & X' \arrow[r, "j'"] \arrow[d, "\nu"] & J' \arrow[d] \\ Y \arrow[r, "c"] & X \arrow[r, "j"] & J \end{tikzcd} \tag{7.16.4}$$

Set $D_{Y'} := \mu^{-1}(D_Y) + \mathrm{Ex}(\mu)$. Then $\mu : (Y', D_{Y'}) \to (Y, D_Y)$ is an elementary orbifold morphism. By Proposition 2.5.(ii), we have

$$\kappa(c, D_Y) = \kappa(c', D_{Y'}) \leq \kappa(X', \Delta(c', D_{Y'})) \leq \kappa(X, \Delta(c, D_Y)) \overset{(7.16.3)}{=} \kappa(c, D_Y) \tag{7.16.5}$$

This implies that the map $j' : X' \to J'$ is the Iitaka fibration of the orbifold base $(X', D_{X'} = \Delta(c', D_{Y'}))$ of c'. Based on this remark, we are free to replace J by birational models and perform base changes as in (7.16.4).

We can therefore apply Proposition 7.6 to the morphism $f : (X, D_X) \to A$, and we will use the notation introduced there without recalling it. We deduce that we may assume that there is a birational morphism $h : X \to \bar{X}$ constructed as in the proof of Proposition 7.6. In this case, we have a morphism $\bar{j} : \bar{X} \to J$ together with a simple normal crossing divisor ∂J on J such that $\bar{j}$ is a locally trivial fibration over $J^0 := J \backslash \partial J$. Moreover, we have

$$\bar{f}^{-1}(D_A) \supset \bar{j}^{-1}(\partial J). \tag{7.16.6}$$

If we let if $D_{\bar{X}} = h_*(D_X)$, and $D_J = \Delta(\bar{j}, D_{\bar{X}})$, then we have

$$\kappa(K_J + D_J) = \dim J.$$

Moreover, $f : X \to A$ factors through $\bar{f} : \bar{X} \to A$.

To prove that Y is not special, we need to construct a neat model for $j \circ c$. By the Raynaud-Hironaka theorem, there exists a smooth birational modification $i : J' \to J$ such that the main component of $Y \times_J J'$ dominating J' is flat over J'. Let $\mu : \tilde{X} \to \bar{X}' = \bar{X} \times_J J'$ be a strong resolution of singularities. Let $\tilde{Y}$ be a desingularizaton of the main component $(Y \times_J J')_{\mathrm{main}}$ of $Y \times_J J'$, such that the induced map $\tilde{Y} \to \tilde{X}$ is regular.

$$\begin{tikzcd}
\tilde{Y} \arrow[rr, "\tilde{c}"] \arrow[d, "\nu_0"] \arrow[dd, bend right=60, "\nu"'] & & \tilde{X} \arrow[d, "\mu"] \arrow[dr, "\tilde{j}"] \arrow[ddd, bend left=80, "\tilde{f}"] & \\
(Y \times_J J')_{\mathrm{main}} \arrow[rr] \arrow[d] & & \bar{X}' \arrow[r, "\bar{j}'"] \arrow[d, "\mu'"] & J' \arrow[d, "i"] \\
Y \arrow[r, "c"] \arrow[drr, "a"'] & X \arrow[r, "h"] \arrow[dr, "f"] & \bar{X} \arrow[r, "\bar{j}"] \arrow[d, "\bar{f}"] & J \\
& & A &
\end{tikzcd}$$

We remark that $\tilde{j} \circ \tilde{c}$ is neat. Indeed, for any $(\tilde{j} \circ \tilde{c})$-exceptional divisor P in $\tilde{Y}$, its image in $(Y \times_J J')_{\mathrm{main}}$ has codimension at least two since $(Y \times_J J')_{\mathrm{main}}$ of $Y \times_J J' \to J'$ is flat. Hence $\nu(P)$ has codimension at least two, and thus $\tilde{j} \circ \tilde{c}$ is neat.

We set

$$D_{\tilde{Y}} := \nu_*^{-1} D_Y + \mathrm{Ex}(\nu). \tag{7.16.7}$$

and $D_{\tilde{X}} := \Delta(\tilde{c}, D_{\tilde{Y}})$ the orbifold divisor of $\tilde{c}$. By Proposition 2.5.(ii) and (7.16.5), $\tilde{j}$ is the Iitaka fibration of $K_{\tilde{X}} + D_{\tilde{X}}$. Hence by the same arguments in the proof of Claim 7.7, if we let $D_{J'} := \Delta(\tilde{j}, D_{\tilde{X}})$, then we have

$$\kappa(K_{J'} + D_{J'}) = \dim J'. \tag{7.16.8}$$

Claim 7.17. — *We have the following equality of orbifold base divisors*

$$\Delta(\tilde{j} \circ \tilde{c}, D_{\tilde{Y}}) = D_{J'}.$$

Proof. — Write $\partial J' := J' \backslash i^{-1}(J^0)$ and let $\tilde{f} : \tilde{X} \to A$ be the composition $\bar{f} \circ \mu' \circ \mu$. By (7.16.6), we have

$$\tilde{D} := \tilde{f}^{-1}(D_A) \geq \tilde{j}^{-1}(\partial J'). \tag{7.17.1}$$

Let Q be any prime divisor on J'. We first assume that Q is not contained in $\partial J'$. Since $\bar{j} : \bar{X} \to J$ is a smooth fibration over J^0, and $\mu : \tilde{X} \to \bar{X}' = \bar{X} \times_J J'$ is a strong resolution of singularities, it follows that $\tilde{j}$ is smooth over $J' \backslash \partial J'$. Then

$$\tilde{j}^* Q = P + R$$

for some prime divisor P on $\tilde{X}$ and some $\tilde{j}$-exceptional divisor R contained in $\tilde{D}$ by (7.17.1), such that $\tilde{j}(P) = Q$. Recall that

$$D_Y \geq a^{-1}(D_A).$$

Hence we have

$$D_{\tilde{Y}} \geq (a \circ \nu)^{-1}(D_A) = \tilde{c}^{-1}(\tilde{D}). \tag{7.17.2}$$

This implies that

$$(\tilde{j}\circ\tilde{c})^*Q = \tilde{c}^*(P+R) = \tilde{c}^*(P) + \sum_i a_i E_i \tag{7.17.3}$$

where each E_i is supported in $D_{\tilde{Y}}$. Hence the multiplicity of $D_{\tilde{Y}}$ along E_i is

$$m_{E_i}(D_{\tilde{Y}}) = +\infty.$$

We remark that for any $\tilde{c}$-exceptional prime divisor E, one has

$$m_E(D_{\tilde{Y}}) = +\infty.$$

Indeed, since $(Y\times_J J')_{\mathrm{main}} \to \bar{X}'$ is flat and $(\mu\circ\tilde{c})(E)$ has codimension at least two, it follows that $\nu_0(E)$ has codimension at least two in $(Y\times_J J')_{\mathrm{main}}$. Consequently, $\nu(E)$ also has codimension at least two. Hence $m_E(D_{\tilde{Y}}) = +\infty$ by (7.16.7).

Therefore, by Definition 2.1, the multiplicity of the orbifold divisor $\Delta(\tilde{j}\circ\tilde{c}, D_{\tilde{Y}})$ along Q is

$$m_Q\left(\Delta(\tilde{j}\circ\tilde{c}, D_{\tilde{Y}})\right) = \inf_{G_i}\{m_i m_{G_i}(D_{\tilde{Y}}) \mid (\tilde{j}\circ\tilde{c})(G_i) = Q\},$$

where G_i ranges over all prime divisors in $\tilde{Y}$ such that $(\tilde{j}\circ\tilde{c})(G_i) = Q$, and we set

$$m_i = m(\tilde{j}\circ\tilde{c}, G_i)$$

that is the *multiplicity* of $\tilde{j}\circ\tilde{c}$ along G_i defined in Definition 2.1. Note that if G_i does not dominate P via $\tilde{c}$, we have already seen that $m_{G_i}(D_{\tilde{Y}}) = +\infty$. It follows that

$$m_Q\left(\Delta(\tilde{j}\circ\tilde{c}, D_{\tilde{Y}})\right) = \inf_{G_i}\{m_i m_{G_i}(D_{\tilde{Y}}) \mid \tilde{c}(G_i) = P\},$$

where G_i now ranges over all prime divisors in $\tilde{Y}$ satisfying $\tilde{c}(G_i) = P$. By (7.17.3), for such G_i, we have

$$m_i = m(\tilde{j}\circ\tilde{c}, G_i) = m(\tilde{c}, G_i).$$

Consequently, we obtain

$$m_Q\left(\Delta(\tilde{j}\circ\tilde{c}, D_{\tilde{Y}})\right) = m_P\left(\Delta(\tilde{c}, D_{\tilde{Y}})\right).$$

We now assume that Q is contained in $\partial J'$. By (7.17.2), we have

$$\lfloor \Delta(\tilde{c}, D_{\tilde{Y}})\rfloor \geq \tilde{D} := \tilde{f}^{-1}(D_A) \geq \tilde{j}^{-1}(\partial J').$$

Together with $D_{\tilde{Y}} \geq \tilde{c}^{-1}(\tilde{D})$, this implies that

$$m_Q\left(\Delta(\tilde{j}\circ\tilde{c}, D_{\tilde{Y}})\right) = m_Q\left(\Delta(\tilde{j}, D_{\tilde{X}})\right) = +\infty.$$

The claim is thus proved. □

Therefore, we have

$$\kappa(J', \Delta(\tilde{j}\circ\tilde{c}, D_{\tilde{Y}})) = \kappa(J', D_{J'}) \overset{(7.16.8)}{=} \dim J' > 0.$$

Since $\tilde{j}\circ\tilde{c}$ is neat, by Definition 2.9, $(\tilde{Y}, D_{\tilde{Y}})$ is not special. This implies that, Y_0 is not special, and this is a contradiction. Therefore the general fibers of a_0 are special. The theorem is proved. □

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