Topology, Hyperbolicity, and the Shafarevich Conjecture for Complex Algebraic Varieties

Ya Deng

ABSTRACT. This survey presents recent developments concerning the Shafarevich conjecture, non-abelian Hodge theories, hyperbolicity, and the topology of complex algebraic varieties, as well as the interplay among these areas. More precisely, we present the main ideas and techniques involved in the linear versions of the following conjectures: the Shafarevich conjecture, the Chern–Hopf–Thurston conjecture, Kollár's conjecture on the holomorphic Euler characteristic, the de Oliveira–Katzarkov–Ramachandran conjecture, and Campana's nilpotency conjecture. In addition, we discuss characterizations of the hyperbolicity of complex quasi-projective varieties via representations of their fundamental groups, together with the generalized Green–Griffiths–Lang conjecture in the presence of a big local system.

Parmi les combinaisons que l'on choisira, les plus fécondes seront souvent celles qui sont formées d'éléments empruntés à des domaines très éloignés.

— HENRI POINCARÉ, *Science et méthode* (1908)

CONTENTS

1. Introduction

1.1. Overview of the Paper. The study of the fundamental groups of algebraic varieties lies at the intersection of algebraic topology and algebraic geometry. It traces back to the foundational work of Picard, Lefschetz, Hodge, Hirzebruch, and Deligne. Subsequent breakthroughs, notably the theory of variations of Hodge structures developed by the school of Griffiths, and the *non-abelian Hodge theory* due to Simpson, Corlette, Gromov–Schoen, and T. Mochizuki, have provided powerful tools for understanding the topology of complex algebraic varieties.

In this survey, I focus on the following question: how do representations of the fundamental group of the algebraic variety into $\mathrm{GL}_N(K)$, where K is a field, influence the *geometry* of such variety? More specifically, we seek to understand how such representations interact with the topology of the variety, its hyperbolicity, and various notions of positivity in algebraic geometry. This perspective is particularly appealing, as it naturally connects diverse areas of mathematics, including harmonic map theory and Nevanlinna theory.

In § 1.2, we begin with a classical problem in geometric topology, namely the *Chern–Hopf–Thurston conjecture*, which concerns the sign of the Euler characteristic of closed *aspherical* even-dimensional real manifolds. If we further assume that such an aspherical manifold carries the structure of a complex projective variety, we are naturally led to the study of algebraic varieties with *large* (or *big*) fundamental groups. We then present recent progress related to this conjecture, as well as a conjecture of Kollár regarding the positivity of Euler characteristic of the canonical bundle of projective varieties with big fundamental groups. Finally, we discuss our results on a conjecture of De Oliveira–Katzarkov–Ramachandran concerning the deformation openness of the property of having a big fundamental group.

In § 1.3, after discussing several basic examples that illustrate necessary conditions of fundamental groups for hyperbolicity, we present a positive result on the above question by showing that a smooth complex quasi-projective variety admitting a local system over an arbitrary field with semisimple algebraic monodromy group is of log general type and is pseudo Picard or Brody hyperbolic. We then turn to the *generalized Green–Griffiths–Lang conjecture*, which relates pseudo Picard or Brody hyperbolicity to some positivity in algebraic geometry. After reviewing the relatively few known results on this conjecture, we present our solution in the case of varieties admitting a *big local system.*

A central theme of this survey is developed in § 2, which is devoted to the *Shafarevich conjecture.* This conjecture predicts that the universal cover $\widetilde{X}$ of a smooth complex projective variety X is *holomorphically convex*. After providing an overview of the historical background, we outline the principal tools used in its study. These include non-abelian Hodge theory in the archimedean setting, as developed by Corlette, Simpson, and Mochizuki, as well as the non-archimedean setting initiated by Gromov–Schoen, and further developed by Katzarkov, Zuo, and Eyssidieux. We then describe our recent progress on extending the Gromov–Schoen theory to quasi-projective varieties. Building on these developments, we explain further recent progress on the Shafarevich conjecture and sketch a proof for the case of complex projective surfaces with reductive fundamental group, intended to highlight the key ideas underlying the general theory. The lecture notes by Eyssidieux [**Eys11**] and his two seminal papers with Katzarkov, Pantev, and Ramachandran [**Eys04**, **EKPR12**] represent some of the most profound work on

this problem. In this survey, we aim to provide new perspectives on the conjecture and to emphasize that the techniques arising from its study play a crucial role in the proofs of the main results discussed in §§ 1.2 and 1.3.

Finally, in §§ 3 and 4, we outline the main ideas underlying the proofs of the results presented in §§ 1.2 and 1.3, using methods developed in the study of the linear Shafarevich conjecture. In § 5, we present several applications of these techniques, including Campana's conjecture on fundamental groups of special varieties; a conjecture of Claudon–Höring–Kollár on the structure of universal covers of projective varieties; a structure theorem motivated by a conjecture of Kollár.

1.2. Algebraic varieties with big fundamental groups.

1.2.1. *Large fundamental groups: motivation, conjectures and results.* A well-known conjecture of Hopf from the 1920s states that the Euler characteristic $\chi(M)$ of a closed $2n$-dimensional Riemannian manifold of non-positive sectional curvature must satisfy the inequality

$$(-1)^n \chi(M) \geq 0.$$

This conjecture is known in dimensions $n = 1, 2$ by the Gauss–Bonnet formula, but it remains widely open for $n \geq 3$. Note that for any closed odd-dimensional real manifold, its Euler characteristic is zero by the Poincaré duality theorem.

By the Cartan–Hadamard theorem, the universal cover $\widetilde{M}$ of such a manifold is diffeomorphic to $\mathbb{R}^{2n}$; in particular, it is contractible. Therefore, M is aspherical, i.e., if its universal covering is contractible.

In the 1970s, Thurston proposed the following more general conjecture, now referred to as the *Chern–Hopf–Thurston* conjecture.

Conjecture 1.1 (Chern–Hopf–Thurston)**.** *Let M be a closed real $2n$-manifold. If M is aspherical, then it satisfies the inequality*

$$(-1)^n \chi(M) \geq 0.$$

This conjecture is particularly appealing, as both the hypothesis and the conclusion are purely topological: no metric or curvature assumptions are involved. It remains widely open for $n \geq 2$.

In this survey, we will focus on Conjecture 1.1 in the setting where M admits the structure of a complex projective variety. An easy observation is the following.

Lemma 1.2. *Let X be a smooth complex projective variety. If X is aspherical or if its universal covering $\widetilde{X}$ is a Stein manifold, then X has a large fundamental group.*

We recall the notion of large fundamental group from [**Kol95**].

Definition 1.3 (Large fundamental group)**.** A complex quasi-projective normal variety X is said to have *large fundamental group* if, for every closed irreducible positive-dimensional subvariety $Z \subset X$, the image

$$\mathrm{Im}\big[\pi_1(Z^{\mathrm{norm}}) \longrightarrow \pi_1(X)\big]$$

is an infinite group.

Remark 1.4. In alternative definitions of a *large fundamental group*, the normalization of Z is sometimes omitted. Although $\pi_1(Z^{\mathrm{norm}})$ and $\pi_1(Z)$ may differ with

respect to finiteness—for example, one may be infinite while the other is finite, as in the case of a rational nodal curve—their respective images in $\pi_1(X)$,

$$\text{Im}\big[\pi_1(Z^{\text{norm}}) \longrightarrow \pi_1(X)\big] \quad \text{and} \quad \text{Im}\big[\pi_1(Z) \longrightarrow \pi_1(X)\big],$$

are nevertheless expected to be either both infinite or both finite. This behavior was predicted as a consequence of the *Shafarevich conjecture* (cf. Conjecture 1.5) and was first observed by Gurjar [**Gur87**].

PROOF OF LEMMA 1.2. We proceed by contradiction. Assume that there exists a closed subvariety $Z \subset X$ of dimension $k > 0$ such that the image

$$\text{Im}\big[\pi_1(Z^{\text{norm}}) \longrightarrow \pi_1(X)\big]$$

is finite. Let Z' be a connected component of $\pi_X^{-1}(Z)$; note that Z' is a compact subvariety of $\widetilde{X}$. Let $f : Y \to Z'$ be a desingularization and let $g : Y \to X$ be the composite map.

If $\widetilde{X}$ is Stein, it cannot contain a compact subvariety of positive dimension, which yields an immediate contradiction.

If $\widetilde{X}$ is contractible (the aspherical case), the composite map

$$H^2(X, \mathbb{C}) \longrightarrow H^2(\widetilde{X}, \mathbb{C}) \longrightarrow H^2(Y, \mathbb{C})$$

is zero. Fix an ample line bundle L on X. Since the map on cohomology is zero, the class $c_1(g^*L) \in H^2(Y, \mathbb{C})$ vanishes. Consequently,

$$\int_Y c_1(g^*L)^k = 0.$$

However, since L is ample and g is a generically finite map onto its image, this integral must be positive, which gives a contradiction.   □

A partial converse to Lemma 1.2 is suggested by the famous conjecture of Shafarevich.

Conjecture 1.5 (Shafarevich). *Let X be a smooth complex projective variety. If X has a large fundamental group, then its universal covering $\widetilde{X}$ is Stein.*

While this conjecture remains open in full generality, a great achievement was made by Eyssidieux, Katzarkov, Pantev, Ramachandran in [**EKPR12**].

THEOREM 1.6 ([**EKPR12**]). *Let X be a smooth complex projective variety. If there exists a faithful representation $\varrho : \pi_1(X) \to \text{GL}_N(\mathbb{C})$, then $\widetilde{X}$ is holomorphically convex. In particular, if additionally X has large fundamental group, then $\widetilde{X}$ is Stein.*

We will return to Conjecture 1.5 and Theorem 1.6 later.

A stronger converse to Lemma 1.2 would ask if a large fundamental group implies asphericity. This is related to a question posed by Kollár: is the fundamental group of a projective variety commensurable (up to finite kernels) with a group G' that admits a quasi-projective $K(G', 1)$?

However, Dimca, Papadima, and Suciu provided a negative answer in [**DPS09**]. They constructed a smooth projective variety X whose universal covering is Stein (satisfying the condition of a large fundamental group), yet $\pi_1(X)$ is not commensurable to any group admitting a quasi-projective $K(\pi, 1)$. In particular, although X has a large fundamental group, its universal cover is not contractible, meaning

X is not aspherical. This demonstrates that the asphericity of projective varieties is a strictly stronger condition than having a large fundamental group.

Motivated by this viewpoint, Arapura and Wang [**AW25**, Conjecture 1.2] proposed the following conjecture.

Conjecture 1.7 (Arapura-Wang). *Let X be a complex projective n-fold with large fundamental group. Then for any perverse sheaf $\mathcal{P}$ on X, one has*

$$\chi(X, \mathcal{P}) \geq 0.$$

In particular, $(-1)^n \chi(X) \geq 0$.

Observe that the shifted constant sheaf $\underline{\mathbb{C}}[n]$ is a perverse sheaf. Consequently, we have

$$\chi(X, \underline{\mathbb{C}}[n]) = (-1)^n \chi(X).$$

Therefore, Conjecture 1.7 is broader than Conjecture 1.1 in another direction. In [**AW25**], Arapura and Wang proved their conjecture assuming the existence of a faithful and cohomologically rigid representation $\pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$. In the author's joint work with Wang [**DW24b**], based on the strategy in [**AW25**], non-abelian Hodge theories and techniques in [**Eys04**, **EKPR12**, **DYK23**].

THEOREM 1.8 ([**DW24b**]). *Let X be a smooth complex projective variety. If there exists a large representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$, where K is any field, then*

$$\chi(X, \mathcal{P}) \geq 0$$

for any perverse sheaf on X.

In particular, we proved Conjecture 1.7 when $\pi_1(X)$ is linear, i.e., there exists a faithful representation $\pi_1(X) \to \mathrm{GL}_N(K)$. Here the definition of large representation is analoguous to Definition 1.3. The representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{K})$, is called *large* if, for every closed irreducible positive-dimensional subvariety $Z \subset X$, the image

$$\varrho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \longrightarrow \pi_1(X)])$$

is an infinite group. We shall discuss the proof of Theorem 1.8 in the subsequent sections.

1.2.2. *Big fundamental groups.* Note that a variety with large fundamental group must be *minimal*; specifically, it cannot contain any rational curves or $\mathbb{A}^1$-curves. Consequently, the blow-up of an algebraic variety never has large fundamental group. To address this limitation, Kollár introduced a more natural and birationally invariant notion:

Definition 1.9 (Big fundamental group). A complex quasi-projective normal variety X is said to have a *big fundamental group* if, for every closed irreducible positive-dimensional subvariety $Z \subset X$ passing through a very general point, the image

$$\mathrm{Im}\big[\pi_1(Z^{\mathrm{norm}}) \longrightarrow \pi_1(X)\big]$$

is infinite.

This condition is birationally invariant: if $f : X \to Y$ is a proper birational morphism between quasi-projective normal varieties, then X has a big fundamental group if and only if Y does. In [**Cam94**], varieties with big fundamental groups are also referred to as being of π_1-*general type*.

We shall now provide some examples of varieties with large or big fundamental groups.

Example 1.10. Let X be a complex quasi-projective normal variety. Then:

- If X is a quasi-projective curve different from $\mathbb{C}$ and $\mathbb{P}^1$, then it has large fundamental group.
- If the universal cover of X is contractible, or a bounded symmetric domain, or more generally a Stein manifold, then X has large fundamental group.
- By the above result, if X is an abelian variety, then it has large fundamental group.
- If $f : X \to Y$ is an étale morphism, then X has a large (resp. big) fundamental group if and only if Y does.
- Let $f : X \to Y$ be a morphism to another quasi-projective normal variety Y. If Y has large (resp. big) fundamental group and if f is quasi-finite (resp. generically finite onto its image), then X has large (resp. big) fundamental group.
- As a consequence, if X has maximal quasi-Albanese dimension, i.e., the quasi-Albanese map

$$X \longrightarrow \mathrm{Alb}(X)$$

  is generically finite onto its image, then X has big fundamental group.
- Suppose that X carries a $\mathbb{Z}$-VHS. If the period map

$$p : X \longrightarrow \mathscr{D}/\Gamma,$$

  where $\mathscr{D}$ is the associated period domain and Γ is the monodromy group, is quasi-finite (resp. generically finite onto its image), then X has large (resp. big) fundamental group.

Note that Conjecture 1.7 concerns the topological aspect of varieties with large fundamental groups. In general, the statement fails if we replace the condition "large fundamental group" by "big fundamental group", since the topological Euler characteristic is not a birational invariant. However, one basic observation is that if $f : X \to Y$ is a birational morphism between two smooth projective varieties, then

$$\chi(X, K_X) = \chi(Y, K_Y),$$

where K_X and K_Y denote the canonical bundles of X and Y. Recall that

$$\chi(X, K_X) = \sum_{i=0}^{\dim X} (-1)^i \dim H^i(X, K_X).$$

Motivated by this observation, Kollár proposed in [**Kol95**] the following conjecture, which may be viewed as a sheaf-theoretic analogue of Conjecture 1.1.

Conjecture 1.11 (Kollár). *Let X be a smooth projective variety. If X has big fundamental group, then $\chi(X, K_X) \geq 0$.*

Notice that Conjecture 1.11 holds when X has maximal Albanese dimension, thanks to the generic vanishing theorem of Green–Lazarsfeld [**GL87**]. Indeed, in this case, [**GL87**] implies that for a generic topologically trivial line bundle $L \in \mathrm{Pic}^0(X)$,

$$H^i(X, K_X \otimes L) = 0 \qquad \text{for all } i \geq 1.$$

Hence,

$$\chi(X, K_X) = \chi(X, K_X \otimes L) = \sum_{i=0}^{\dim X} (-1)^i \dim H^i(X, K_X \otimes L) = \dim H^0(X, K_X \otimes L) \geq 0.$$

This proof inspired us that some vanishing theorem for cohomology will be useful to prove Conjecture 1.11. However, it seems that we do not have the generic vanishing theorem for general varieties with big fundamental group. In [**DW24a**], for the case where $\pi_1(X)$ is linear, Wang and the author established a vanishing theorem for the L^2-Dolbeault cohomology of $\widetilde{X}$, analogous to the Green–Lazarsfeld theorem.

THEOREM 1.12 ([**DW24a**]). *If there exists a big representation* $\varrho : \pi_1(X) \to$ $\mathrm{GL}_N(\mathbb{C})$, *then*

(i) $H^{p,0}_{(2)}(\widetilde{X}) = 0$ *for* $0 \leq p \leq n-1$ *and* $H^{n,q}_{(2)}(\widetilde{X}) = 0$ *for* $1 \leq q \leq n$.
(ii) *The Euler characteristic* $\chi(X, K_X) \geq 0$.
(iii) *If the strict inequality* $\chi(X, K_X) > 0$ *holds, then*
    (a) *there exists a nontrivial* L^2-*holomorphic n-form on* $\widetilde{X}$;
    (b) X *is of general type.*

The representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{K})$, is called *big* if for every closed irreducible positive-dimensional subvariety $Z \subset X$ passing through a very general point, the image

$$\varrho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \longrightarrow \pi_1(X)])$$

is infinite.

Definition 1.13 (L^2-cohomology). Let (Y, ω) be a complete Kähler manifold. Let $L^{p,q}_{(2)}(Y)$ be the space of L^2-integrable (p,q)-forms with respect to the metric ω. A section u is said to be in $\mathrm{Dom} \, \bar{\partial}$ if $\bar{\partial}u$ calculated in the sense of distributions is still in L^2. Then the L^2-Dolbeault cohomology is defined as

$$H^{p,q}_{(2)}(Y) = \ker \bar{\partial} \big/ \overline{\mathrm{Im} \, \bar{\partial} \cap \mathrm{Dom} \, \bar{\partial}}.$$

If (X, ω) is a compact Kähler manifold, we write $H^{p,q}_{(2)}(\widetilde{X})$ for the L^2-cohomology computed with respect to the metric $\pi_X^* \omega$. Note that this space is independent of the choice of the Kähler metric ω, since any two such pullback metrics on $\widetilde{X}$ are mutually bounded.

As is well-known to experts, once we have the vanishing of L^2-cohomology in Theorem 1.12.(i), one can use Atiyah's L^2-index theorem to prove that $\chi(X, K_X) \geq$. This proves Conjecture 1.11 in this case. We explain the proof below,

PROOF OF (I) $\Longrightarrow$ (II) & (III). We denote by $\Gamma = \pi_1(X)$ and $\dim_\Gamma H^{n,q}_{(2)}(\widetilde{X})$ the Von Neumann dimension of $H^{n,q}_{(2)}(\widetilde{X})$ (cf. [**Ati76**] for the definition). By Atiyah's L^2-index theorem along with Theorem 1.12.(i), we have

(1.13.1) $$\chi(X, K_X) = \sum_{q=0}^{n} (-1)^q \dim_\Gamma H^{n,q}_{(2)}(\widetilde{X}) = \dim_\Gamma H^{n,0}_{(2)}(\widetilde{X}) \geq 0.$$

Theorem 1.12.(ii) is proved.

If the strict inequality (1.13.1) holds, then $H^{n,0}_{(2)}(\widetilde{X}) \neq 0$. Thus, one can choose a non-trivial element $f \in H^{n,0}_{(2)}(\widetilde{X})$. In [**Kol95**, Chapter 13], Kollár, building on ideas of Gromov [**Gro91**], introduced the *Poincaré series*

$$P_k(f) := \sum_{\gamma \in \Gamma} \gamma^*(f^{2k})(\gamma x),$$

and showed that for each $k \in \mathbb{N}$, the series $P_k(f)$ defines a Γ-invariant holomorphic section of $2kK_{\widetilde{X}}$. Hence it descends to a pluricanonical section

$$f_k \in H^0(X, 2kK_X).$$

Kollár then considers the linear series generated by products of these sections:

$$R_{2m} := \left\{ \prod_i f_{k_i} \;\middle|\; \sum_i k_i = m \right\} \subset H^0(X, 2mK_X).$$

In [**Kol95**, Theorem 13.9], he proves that for all sufficiently large m, the rational map associated with R_{2m} is birational onto its image, assuming that X has big fundamental group. This shows that K_X is big, and thus proves Theorem 1.12.(iii).

We refer the interested reader to [**Kol95**, Chapter 13] for further details. □

Remark 1.14. Indeed, Kollár's argument above also shows that for any holomorphic line bundle L on X with big fundamental group, the existence of a non-zero L^2-section $H^0_{(2)}(\widetilde{X}, L) \neq 0$ implies that L is a big line bundle.

1.2.3. *Deformation of big fundamental groups.* From Definitions 1.3 and 1.9, one can see that the properties of having big or large fundamental groups depends essentially on the algebraic structures of the varieties, as one has to make the test for all subvarieties or all subvarieties passing to a general point. It is natural to ask whether two algebraic varieties that are homeomorphism, one has big fundamental groups if and only if the other has big fundamental group. This question holds trivially for the curves. For surfaces, it was answered positively by Benoit Claudon in [**Cla10**], based on a theorem by Siu [**Siu87**].

We first define a topological invariant $g(X)$ for any compact Kähler manifold X as follows. A vector subspace $V \subset H^1(X, \mathbb{R})$ is said to be *isotropic* if it is annihilated by the exterior product

$$\Lambda^2 H^1(X, \mathbb{R}) \longrightarrow H^2(X, \mathbb{R}),$$

that is, if $\alpha \wedge \beta = 0$ for all $(\alpha, \beta) \in V \times V$. We then set

$$g(X) = \max \left\{ \dim(V) \mid V \subset H^1(X, \mathbb{R}),\; U \text{ is isotropic} \right\}.$$

We remark that the invariant $g(X)$ is an invariant for fundamental groups. Namely, if Y is another compact Kähler manifold and $\pi_1(X) \simeq \pi_1(Y)$, then $g(X) = g(Y)$ (see [**ABC**$^+$**96**, **Py25**]).

THEOREM 1.15 (Siu). *Let X be a compact Kähler manifold and let $g \geq 2$ be an integer. Then $\pi_1(X)$ admits the surface group $\pi_1(C_g)$ as a quotient if and only if X admits a fibration onto a curve of genus $g' \geq g$. Moreover, if*

$$\rho : \pi_1(X) \longrightarrow \pi_1(C_g)$$

is a surjective homomorphism with $g = g(X)$, then there exists a fibration of X onto a curve of genus g that induces ρ.

THEOREM 1.16 (Claudon). *Let X and Y be two compact Kähler surfaces that are homeomorphic. Then X has big fundamental groups if and only if Y has big fundamental group.*

PROOF. Assume that X does not have big fundamental group. If $\pi_1(X)$ is finite, then $\pi_1(Y)$ is also finite as they are homeomorphic. Therefore, we just consider the case $\pi_1(X)$ is infinite. In this case, a theorem by Kollár [**Kol93**] and Campana [**Cam94**] states that, after replacing X by a finite étale cover, there is a fibration $f : X \to C_g$ onto a smooth projective curve C_g of genus $g \geq 1$ such that for a smooth general fiber F of f, $\mathrm{Im}[\pi_1(F) \to \pi_1(X)]$ is a finite group, and we have the following short exact sequence

$$0 \to \pi_1(F) \to \pi_1(X) \to \pi_1(C_g) \to 0.$$

This means that $\pi_1(X)$ is commensurable to a surface group $\pi_1(C_g)$. After replacing X by a finite étale cover, $\pi_1(X) \simeq \pi_1(C_{g'})$ for another projective curve $C_{g'}$ of genus $g' \geq 1$. Hence we can replace Y by a finite étale cover such that $\pi_1(Y) \simeq \pi_1(X)$.

If $g' = 1$, then $\pi_1(Y)$ is abelian, and one can see that the Albanese map $\mathrm{alb}_Y : Y \to \mathrm{Alb}(Y)$ of Y is surjective, and $\mathrm{Alb}(Y)$ is an elliptic curve. Moreover, alb_Y induces an isomorphism of fundamental groups. Therefore, Y does not have big fundamental groups, as for each fiber F of alb_Y, we have $\mathrm{Im}[\pi_1(F) \to \pi_1(Y)] = \{1\}$.

If $g' \geq 2$, by the above arguments together with Theorem 1.15, there exists a fibration $h : Y \to C'_{g'}$ to a projective curve $C'_{g'}$ with genus g' such that $h_* : \pi_1(Y) \to \pi_1(C'_{g'})$ is an isomorphism. Therefore, for each fiber F of h, we have $\mathrm{Im}[\pi_1(F) \to \pi_1(Y)] = \{1\}$. This implies that Y does not have big funmental group. The theorem is proved. □

However, it remains unknown whether Theorem 1.16 holds for smooth projective varieties of dimension greater than two. In [**dOKR02**], De Oliveira, Katzarkov, and Ramachandran proposed the following conjecture; we also refer to the monograph by Kollár [**Kol95**] for a discussion of related problems.

Conjecture 1.17. *Let $f : \mathscr{X} \to \mathbb{D}$ be a smooth projective family over the unit disk. If $X_0 := f^{-1}(0)$ has big π_1, then any $X_t := f^{-1}(t)$ also has big π_1 for small t.*

In [**Cla10**], Claudon proved Conjecture 1.17 for threefolds, assuming suitable conjectures on the fundamental groups of orbifold surfaces.

In a joint work with Mese and Wang [**DMW24**], we prove Conjecture 1.17 in the linear case.

THEOREM 1.18 ([**DMW24**]). *Let $f : \mathscr{X} \to \mathbb{D}$ be a smooth projective family over the unit disk. If there exists a big representation $\varrho : \pi_1(X_0) \to \mathrm{GL}_N(\mathbb{C})$, then there exists a representation $\tau : \pi_1(\mathscr{X}) \to \mathrm{GL}_N(\mathbb{C})$ such that for $|t|$ sufficiently small, the representation*

$$\tau_t : \pi_1(X_t) \xrightarrow{\simeq} \pi_1(\mathscr{X}) \to \mathrm{GL}_N(\mathbb{C})$$

is big. In particular, Conjecture 1.17 holds if $\pi_1(X_0)$ is linear.

To prove Theorem 1.18, one must establish the deformation continuity of equivariant harmonic mappings into symmetric spaces or Euclidean buildings (cf. Theorems 4.4 and 4.5 below).

1.3. Hyperbolicity and fundamental groups: conjectures and main results. In this subsection, we illustrate how the notions of big or large fundamental groups, introduced previously, play a pivotal role in characterizing the hyperbolicity of algebraic varieties. We then formulate a conjectural characterization (cf. Conjecture 1.34) along these lines and present our results toward this conjecture. Furthermore, we discuss our main results concerning the generalized Green–Griffiths–Lang conjecture under suitable assumptions on the fundamental group. The strategies for the proofs of the main theorems are detailed in § 3. These proofs rely on the technical machinery developed for the reductive Shafarevich conjecture in § 2.

1.3.1. *Notions of hyperbolicity.* The notion of hyperbolicity originates from Picard's great theorem and Picard's little theorem on the range of an analytic function.

THEOREM 1.19 (Little Picard theorem). *Any holomorphic map* $f : \mathbb{C} \to \mathbb{P}^1 \backslash \{0, 1, \infty\}$ *must be constant.*

This theorem is a significant strengthening of Liouville's theorem which states that the image of an entire non-constant function must be unbounded.

THEOREM 1.20 (Great Picard theorem). *Any holomorphic map* $f : \mathbb{D}^* \to \mathbb{P}^1 \backslash \{0, 1, \infty\}$ *does not have essential singularity at the origin.*

This is a substantial strengthening of the Casorati–Weierstrass theorem, which only guarantees that the range of a holomorphic function defined over $\mathbb{D}^*$ with essential singularity at the origin has image dense in $\mathbb{C}$. One can see that Theorem 1.20 implies Theorem 1.19.

The complex algebraic varieties that have the similar properties as described in Theorems 1.19 and 1.20 is called *hyperbolic*. Precisely, we have the following definition.

Definition 1.21 (Hyperbolicity). Let X be a complex quasi-projective variety.

  (i) The variety X is *pseudo Picard hyperbolic* if there is a proper Zariski closed subset $\Xi \subsetneq X$ such that any holomorphic map $f : \mathbb{D}^* \to X$ from the punctured disk $\mathbb{D}^*$ to X with $f(\mathbb{D}^*) \not\subseteq \Xi$ extends to a holomorphic map from the disk $\mathbb{D}$ to a projective compactification $\overline{X}$ of X.
 (ii) We say that the variety X is *pseudo Brody hyperbolic* if there exists a proper Zariski closed subset $\Xi \subsetneq X$ such that every non-constant holomorphic map $f : \mathbb{C} \to X$ (an *entire curve*) has image contained in Ξ.

Note that every pseudo Picard hyperbolic variety is pseudo Brody hyperbolic. While we conjecture the converse to hold true, as of now, we lack both a proof and any counter-example of our conjecture.

In the algebraic setting, we introduce the following definition.

Definition 1.22 (Strongly of log general type). Let X be a complex quasi-projective variety. We say that X is *strongly of log general type*[a] if there exists a proper Zariski closed subset $\Xi \subsetneq X$ such that every positive-dimensional closed subvariety of X not contained in Ξ is of log general type.

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[a]This terminology originates from Demailly [**Dem15**] in his strategy for the proof of Green-Griffiths-Lang conjecture. Although our notion differs from Demailly's original definition, we retain the same terminology for its descriptive convenience.

FIGURE 1. Hyperbolicity from different viewpoints

	$\deg(K_{\overline{C}} + D)$	$\pi_1(C)$	Hyperbolicity
$\mathbb{P}^1, \mathbb{C}$	< 0	$\{1\}$	no
$\mathbb{C}^*$, torus	$= 0$	Infinite, abelian	no
$\mathbb{P}^1 \backslash \{$at least three points$\}$ torus$\backslash \{$at least one point$\} \cdots$	> 0	Infinite, non-abelian	yes

Lang conjectured that a complex quasi-projective variety is strongly of log general type if it is of log general type. To my knowledge, this conjecture remains open even for complex surfaces in general.

1.3.2. *Generalized Green-Griffiths-Lang conjecture.* To characterize algebraic varieties falling into the hyperbolic category, we will start by examining cases where C is a smooth quasi-projective curve, with $\overline{C}$ as its compactification, and D representing the complement of C within $\overline{C}$. As illustrated in Figure 1, we can make the following observations. From an algebraic–geometric perspective, hyperbolic curves can be characterized as those whose logarithmic canonical bundle is positive. On the other hand, from a topological viewpoint, hyperbolic curves are precisely those with infinite and non-abelian fundamental groups.

It's worth noting that the generalized Green-Griffiths-Lang conjecture aligns with the algebraic geometric viewpoint, focusing on the positivity of the logarithmic canonical bundle.

Conjecture 1.23 (Generalized Green-Griffiths-Lang conjecture)**.** *Let X be a smooth quasi-projective variety. Then the following properties are equivalent:*

(i) *X is of log general type;*
(ii) *X is pseudo-Picard hyperbolic;*
(iii) *X is pseudo-Brody hyperbolic;*
(iv) *X is strongly of log general type.*

So far Conjecture 1.23 remains an open and challenging problem, even in situations where X is a surface. We are fascinated by this conjecture due to its analogy with the Bombieri-Lang conjecture concerning rational points.

Conjecture 1.24 (Bombieri-Lang)**.** *Let X be a smooth projective variety defined over a number field k. Then there exists a dense Zariski closed subset $\Xi \subsetneq X$ such that for all number field extensions k' of k, the set of k'-rational points in $X \backslash \Xi$ is finite.*

1.3.3. *Hyperbolicity of compactifications after taking finite étale coverings.* It is natural to ask why we are interested in the more general notion Picard hyperbolicity. It indeed enjoys the following algebraicity property.

Proposition 1.25 ([**Den23**])**.** *Let X be a smooth quasi-projective variety that is pseudo Picard hyperbolic. Then any meromorphic map $f : Y \dashrightarrow X$ from another smooth quasi-projective variety Y to X with $f(Y) \not\subset \mathrm{Sp_p}(X)$ is rational.*

A direct consequence of Proposition 1.25 is the following uniqueness of algebraic structure of pseudo Picard hyperbolic varieties.

Corollary 1.26 ([**Den23**])**.** *Let X and Y be smooth quasi-projective varieties such that there exists an analytic isomorphism $\varphi : Y^{\mathrm{an}} \to X^{\mathrm{an}}$ of associated complex spaces. Assume that X is pseudo Picard hyperbolic. Then φ is an algebraic isomorphism.*   □

Let us discuss some examples of pseudo Picard hyperbolic varieties. A classical result due to Borel [**Bor72**] and Kobayashi-Ochiai [**KO71**] is that quotients of bounded symmetric domains by torsion-free lattices are Picard hyperbolic. In [**Den23**] we proved a similar result for algebraic varieties that admit a complex variation of Hodge structures.

THEOREM 1.27 ([**Den23**, Theorem A]). *Let X be a smooth quasi-projective variety. Assume that there is a $\mathbb{C}$-VHS on X whose period mapping is injective at one point. Then X is pseudo Picard hyperbolic.*   □

A similar result was discussed in [**BB20**].

In [**Nad89**], Nadel proved the nonexistence of certain level structures on abelian varieties over complex function fields, which was refined by Aihara-Noguchi and Rousseau in [**AN91**, **Rou16**]. Precisely, they proved the following theorem:

THEOREM 1.28 ([**Nad89**, **Rou16**]). *Let X be a smooth quasi-projective variety such that $X = \Omega/\Gamma$ where Ω is a bounded symmetric domain and Γ is an arithmetic torsion free lattice acting on Ω. Then there exists a finite index subgroup $\Gamma' \subset \Gamma$, such that for the quasi-projective variety $X' := \Omega/\Gamma'$, its projective compactification $\overline{X'}$ is Brody (moreover Kobayashi) hyperbolic modulo the boundary $\overline{X'}\backslash X'$.*

In [**Den23**], we obtained the following result which incorporates previous results by Nadel, Aihara-Noguchi and Rousseau.

THEOREM 1.29 ([**Den23**, Theorem B]). *Let X be a smooth quasi-projective variety. Assume that there is a complex variation of Hodge structures on X whose period mapping is injective at one point. Then there exists a finite étale cover X' of X such that its projective compactification $\overline{X'}$ is pseudo Picard hyperbolic and strongly of general type.*

The proofs of Theorems 1.27 and 1.29 in [**Den23**] are rather involved and rely heavily on analytic techniques from non-abelian Hodge theory. Subsequently, in [**CD21**], Cadorel and the author gave a simplified proof and, moreover, established the following more general result.

THEOREM 1.30 ([**CD21**]). *Let X be a smooth quasi-projective variety. Assume that there exists a harmonic bundle (E, θ, h) on X such that the Higgs field $\theta \colon T_X \to \mathrm{End}(E)$ is injective at some point of X. Then there exists a finite étale cover X' of X whose projective compactification $\overline{X'}$ is pseudo-Picard hyperbolic.*

Nonetheless, Nevanlinna theory remains an essential ingredient in both works.

1.3.4. *How Fundamental Groups Determine Hyperbolicity.* It is natural to ask whether there is a characterization of hyperbolicity of algebraic varieties in terms of fundamental groups. As illustrated in Figure 1, such characterization requires that the fundamental group $\pi_1(X)$ be must be infinite and non-abelian. However, this topological requirement is insufficient, as demonstrated by non-hyperbolic varieties whose fundamental groups meet this criterion.

Example 1.31. Let C be a projective curve of genus $g \geq 2$. The product variety $X = C \times \mathbb{P}^1$ is clearly not pseudo Brody hyperbolic due to the $\mathbb{P}^1$ factor. Its fundamental group is $\pi_1(X) \simeq \pi_1(C)$, which is both infinite and non-abelian, showing the necessity of additional constraints.

The failure of hyperbolicity in Example 1.31 can be attributed to the fact that the non-hyperbolic factor $\mathbb{P}^1$ trivializes the fundamental group in a relative sense. This suggests that a useful condition must ensure that the fundamental group remains *infinite* when restricted to relevant subvarieties. So having *big fundamental groups* would indeed exclude the counterexample $C \times \mathbb{P}^1$.

However, simply having a big fundamental group is not enough, as shown by varieties whose non-hyperbolicity stems from a different group-theoretic defect:

Example 1.32. Let C be a projective curve of genus $g \geq 2$ and let E be an elliptic curve. The variety $X = C \times E$ is not pseudo Brody hyperbolic. Its fundamental group is $\pi_1(X) \simeq \pi_1(C) \times \pi_1(E)$. Since $\pi_1(E) \simeq \mathbb{Z}^2$, $\pi_1(X)$ is large and non-abelian.

The non-hyperbolicity of $X = C \times E$ is tied to the presence of normal abelian subgroup. More precisely, for any fixed point $x \in C$, the subgroup corresponding to the elliptic curve factor,

$$\mathrm{Im}\left[\pi_1(\{x\} \times E) \to \pi_1(C \times E)\right] \triangleleft \pi_1(C \times E),$$

is an infinite normal abelian subgroup of $\pi_1(X)$. Therefore, we should require that fundamental groups are "highly non-abelian".

Definition 1.33 (Semisimple Group). A finitely generated group G is called *semisimple* if every abelian normal subgroup of G is finite.

The variety $X = C \times E$ in Example 1.32 has fundamental group that is not semisimple. In conclusion, to give the fundamental group characterization of varieties with strong hyperbolicity, we propose the following conjecture:

Conjecture 1.34. *Let X be a quasi-projective normal variety. If there exists a quotient $\varrho : \pi_1(X) \twoheadrightarrow G$ such that G is semisimple and ϱ is big in the sense defined below, then X is strongly of log general type and pseudo Picard hyperbolic.*

We now define the precise requirement for the quotient map.

Definition 1.35. A quotient map $\varrho : \pi_1(X) \twoheadrightarrow G$ is said to be big if, for every closed irreducible positive-dimensional subvariety $Z \subset X$ passing through a very general point, the image

$$\varrho\left(\mathrm{Im}\left[\pi_1(Z^{\mathrm{norm}}) \longrightarrow \pi_1(X)\right]\right)$$

is an infinite subgroup of G.

In [**CDY25b**, **DY25**] together with Cadorel and Yamanoi, we proved Conjecture 1.34 for linear quotients.

THEOREM 1.36. *Let X be a complex quasi-projective normal variety and let G be a semisimple algebraic group defined over an algebraically closed field K. If there exists a big and Zariski dense representation $\varrho : \pi_1(X) \to G(K)$, then:*

    (i) *when $\mathrm{char}\, K = 0$, for any $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, the Galois conjugate $X^\sigma := X \times_\sigma \mathbb{C}$ is pseudo Picard hyperbolic and strongly of log general type;*

(ii) *when* char $K > 0$, X *is pseudo Picard hyperbolic and strongly of log general type.*

In [**CDY25b**], we proved Theorem 1.36 in the case char $K = 0$, and in [**DY25**], we treated the case of positive characteristic.

1.3.5. *On GGL conjecture.* We discuss the generalized Green-Griffiths-Lang conjecture in this subsection. Conjecture 1.23 is known to hold for curves. However, when dim $X \geq 2$, there has been only few progress. We summarize below four classes of varieties for which Conjecture 1.23 has been established:

- Complex projective surfaces with big cotangent bundles, proved by McQuillan [**McQ98**], and later extended to quasi-projective surfaces with big logarithmic cotangent bundles by El Goul [**EG03**].
- Subvarieties of abelian varieties, by the classical theorem of Bloch, Ochiai and Kawamata, and more generally subvarieties of semi-abelian varieties, by Noguchi [**Nog81**].
- Projective varieties of maximal Albanese dimension, by Kawamata [**Kaw81**] and Yamanoi [**Yam15a**].
- General hypersurfaces in projective space $\mathbb{P}^n$ ($n \geq 3$) of sufficiently high degree, proved in [**DMR10**] (based on the strategy of Siu [**Siu04**]), with degree bounds subsequently improved in [**Dar16**, **MT22**, **BK24**, **Cad24**]; as well as complements of general hypersurfaces of high degree in $\mathbb{P}^n$ ($n \geq 2$), proved in [**Dar16**, **BD19**], to mention only a few.

In [**CDY25a**], Cadorel, Yamanoi, and the author first established Conjecture 1.23 for quasi-projective varieties of maximal quasi-Albanese dimension. Our proof relies primarily on Nevanlinna theory, thereby generalizing the result of [**Yam15a**] to the non-compact setting. Building on the main result in [**CDY25a**], we further established a *non-abelian version* in [**CDY25b**, **DY25**].

THEOREM 1.37 ([**CDY25b**, **DY25**]). *Let X be a quasi-projective variety. If there exists a semisimple and big representation $\pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$, or a big representation $\pi_1(X) \to \mathrm{GL}_N(K)$ for some field K of positive characteristic, then Conjecture 1.23 holds for X.*

In [**CDY25b**], we proved Theorem 1.37 in the case char $K = 0$, and in [**DY25**], we addressed the case of positive characteristic. The case where X is projective and char $K = 0$ was also discussed in [**Bru22**] using different methods.

2. Non-Abelian Hodge Theories and the Shafarevich Conjecture

Let us mention that the techniques of non-abelian Hodge theory developed by Simpson [**Sim92**] in the archimedean setting and by Gromov–Schoen [**GS92**] in the non-archimedean setting were first recognized by Katzarkov [**Kat97**] as being applicable to the study of Conjecture 1.5. Together with Ramachandran, he proved Conjecture 1.5 for projective surfaces with reductive fundamental group [**KR98**]. These techniques were subsequently further developed by Eyssidieux [**Eys04**], and they proved to be highly effective in establishing the conjecture in the linear case [**EKPR12**].

On the other hand, Zuo [**Zuo96**] and Yamanoi [**Yam10**] discovered that similar methods can also be applied to questions of hyperbolicity for algebraic varieties. More recently, Mese, Wang, and the author [**DW24b**, **DW24a**, **DMW24**] realized that the techniques in studying the Shafarevich conjecture in [**Eys04**, **EKPR12**,

DYK23, DY25] can be used to investigate the topology of algebraic varieties, as discussed in § 1.2.

In this section, we review some recent progress on the Shafarevich conjecture and outline several of the essential techniques involved in its proof, which are also relevant to the other conjectures discussed in §§ 1.2 and 1.3. A particularly comprehensive survey is provided by Eyssidieux [**Eys11**], which we strongly recommend to readers interested in this conjecture.

2.1. Conjecture and some histories. Let us recall an equivalent version of Conjecture 1.5.

Conjecture 2.1 ([**Sha77**]). *The universal cover $\widetilde{X}$ of a smooth projective variety X is holomorphically convex.*

We recall the definition of holomorphic convexity.

Definition 2.2. Let Y be a complex space. For any compact subset $K \subset Y$, the *holomorphic hull* of K is defined as

$$\widehat{K} = \widehat{K}_{O(Y)} = \left\{ z \in Y \mid |f(z)| \leqslant \sup_{w \in K} |f(w)|, \, \forall f \in O(Y) \right\}.$$

We say that Y is *holomorphically convex* if $\widehat{K}$ is compact for every compact subset $K \subset Y$.

Furthermore, Y is called *Stein* if it is holomorphically convex and *holomorphically separable* (i.e., for any distinct points $x, y \in Y$, there exists $f \in O(Y)$ such that $f(x) \neq f(y)$).

We have the following criterion for the holomorphic convexity of complex spaces.

THEOREM 2.3 (Cartan-Remmert). *A complex space Y is holomorphically convex if and only if there exists a proper holomorphic fibration $Y \to S$ over a Stein space.*

If the Shafarevich conjecture holds for X, one can show that there exists a proper holomorphic fibration $\mathrm{sh}_X : X \to \mathrm{Sh}(X)$ with the property that for any subvariety $Z \subset X$, the image $\mathrm{Im}[\pi_1(Z) \to \pi_1(X)]$ is finite if and only if $\mathrm{sh}_X(Z)$ is a point. Such a holomorphic map, if it exists, is called the *Shafarevich morphism* of X. More generally, we have

Definition 2.4 (Shafarevich morphism). Let X be a quasi-projective normal variety. Let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a representation where K is any field. A dominant holomorphic map $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$ with general fibers connected is called the *Shafarevich morphism associated with ϱ* if, for any closed subvariety $Z \subset X$, the image $\varrho(\mathrm{Im}[\pi_1(Z) \to \pi_1(X)])$ is finite if and only if $\mathrm{sh}_\varrho(Z)$ is a point.

We provide a brief historical overview of the developments surrounding the Shafarevich conjecture. We note that the literature is vast, and the following list is not exhaustive.

- Gurjar [**Gur87**] and Napier [**Nap90**] initiated this investigation.
- In 1993, Kollár [**Kol93**] and Campana [**Cam94**] independently constructed a birational model of the Shafarevich morphism, known as the *Shafarevich map*, for any variety X.

- In 1998, Katzarkov and Ramachandran [**KR98**] established the Shafarevich conjecture for surfaces with reductive fundamental group, i.e. those admitting an almost faithful semisimple representation $\pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$. In the same work, they proved the holomorphic convexity of suitable intermediate Galois coverings. The latter result was recently extended by Yuan Liu [**Liu23**] to the compact Kähler setting.
- In 2004, Eyssidieux [**Eys04**] proved the reductive Shafarevich conjecture. A central and particularly deep aspect of his work is the discovery of sufficient conditions ensuring the holomorphic convexity of certain Galois coverings, associated with Simpson's absolutely constructible subsets [**Sim93b**]. This result has had a substantial impact on complex geometry: beyond resolving the reductive case, the techniques developed by Eyssidieux have inspired numerous applications by the present author and his collaborators in recent years, as will be discussed later in this paper.
- Building on the foundational deformation theory of Deligne and Goldman-Millson [**GM88**], Eyssidieux and Simpson [**ES11**] constructed a canonical variation of mixed Hodge structure associated with the formal local ring $O_{\hat{\rho}}$ of the representation variety $R(\pi_1(X), \mathrm{GL}_N)$ at a point ρ corresponding to a $\mathbb{C}$-VHS. This construction provides a crucial framework for the study of the linear Shafarevich conjecture.
- In 2012, building upon the results in [**Eys04**,**ES11**], Eyssidieux, Katzarkov, Pantev, and Ramachandran [**EKPR12**] completely proved the Linear Shafarevich conjecture (cf. Theorem 1.6), representing a significant development in the study of this conjecture.
- In 2015, based on [**EKPR12**], Campana, Claudon and Eyssidieux [**CCE15**] proved the Linear Shafarevich conjecture for compact Kähler manifolds.
- In 2023, Yamanoi, Katzarkov, and the author [**DYK23**] extended the reductive Shafarevich conjecture to the case of *singular* projective varieties. Our work also introduced new perspectives in the proof and constructed Shafarevich morphisms for quasi-projective varieties with reductive π_1, which were obtained independently by Brunebarbe [**Bru23**].
- In 2023, Green, Griffiths, and Katzarkov established the holomorphic convexity of universal coverings of quasi-projective varieties whose partial Albanese map is proper. This result was subsequently reproved by Aguilar and Campana in [**AAC25**] using different methods.
- In 2025, Yamanoi and the author [**DY25**] constructed the Shafarevich morphism for quasi-projective normal varieties with linear fundamental groups in positive characteristic. We also proved the Shafarevich conjecture for projective normal surfaces in this setting.
- More recently, Bakker, Brunebarbe, and Tsimerman addressed the Shafarevich conjecture for *quasi-projective varieties* with linear fundamental groups in their extensive work [**BBT24**].

2.2. Non-Abelian Hodge theory: archimedean setting. Non-abelian Hodge theory explores the geometry of local systems on complex algebraic varieties. The subject was initiated by the work of Siu [**Siu80**] and Sampson [**Sam86**] on harmonic maps from Kähler manifolds to non-positively curved target spaces. A major breakthrough came with the work of Corlette [**Cor88**] and Donaldson [**Don87b**],

who proved the existence of equivariant harmonic maps associated with complex semisimple local systems over compact Kähler manifolds. Building on this, Simpson developed the full framework of non-abelian Hodge theory via Higgs bundles—a notion that had also been studied earlier by Hitchin [**Hit87**] in the case of Riemann surfaces. He discovered the amazing connection between the work of Donaldson [**Don87a**] and Uhlenbeck-Yau [**UY86**] and the work by Griffiths [**Gri70**], Deligne, Schmid et. al. on the variation of Hodge structures [**Sim88, Sim92**].

The extension of the theory to the quasi-projective setting was initiated by Simpson for curves [**Sim90**], and later completed by Mochizuki [**Moc06, Moc07a, Moc07b**] in a monumental series of works spanning more than a thousand pages. As a consequence, the theory in the archimedean setting (concerning complex local systems) is now fully established.

In this subsection, we briefly recall a small portion of Simpson's non-abelian Hodge theory and its subsequent development by Mochizuki in the quasi-projective setting, providing the minimal technical framework needed for the results discussed in this paper.

2.2.1. *The Simpson correspondence.*

Definition 2.5 (Higgs bundle). A *Higgs bundle* on X is a pair (E, θ) where E is a holomorphic vector bundle, and $\theta : E \to E \otimes \Omega_X^1$ is a holomorphic one form with value in $\mathrm{End}(E)$, called the *Higgs field*, satisfying $\theta \wedge \theta = 0$.

Let (E, θ) be a Higgs bundle over a complex manifold X. Suppose that h is a smooth hermitian metric of E. Denote by ∇_h the Chern connection of (E, h), and by $\theta_h^\dagger$ the adjoint of θ with respect to h. We write $\theta^\dagger$ for $\theta_h^\dagger$ for short if no confusion arises. The metric h is *harmonic* if the connection $\nabla_h + \theta + \theta^\dagger$ is flat.

Definition 2.6 (Harmonic bundle). A harmonic bundle on X is a Higgs bundle (E, θ) endowed with a harmonic metric h.

The notion of harmonic comes from the harmonic maps that are interpreted as follows.

Let (M, g) be a closed Riemannian manifold. Assume that there exists a representation $\varrho : \pi_1(M) \to \mathrm{GL}_N(\mathbb{C})$. It corresponds to a flat bundle (V, D) on M. Then for any smooth hermitian metric h on V, it corresponds to a ϱ-equivariant smooth map

$$u_h : \widetilde{M} \to \mathrm{GL}_N(\mathbb{C})/U_N.$$

Here note that $\mathrm{GL}_N(\mathbb{C})$ acts on the Riemannian symmetric space $S := \mathrm{GL}_N(\mathbb{C})/U_N$ (which has non-positive sectional curvature). The main theorem by Corlette, proves that there exists a smooth metric h for V such that u_h is energy minimizing (so-called *harmonic map*), i.e. the energy

$$E(u_h) = \int_X |du_h|^2 d\mathrm{Vol}_g$$

is the critical point. Such a harmonic map is unique up to some ambiguity. The precise equation is given by

(2.6.1) $$d_\nabla^* du_h = 0.$$

Here $du_h \in \Gamma(\widetilde{M}, \Omega_{\widetilde{M}}^1 \otimes u_h^* T_S)$, and

$$d_\nabla : \Omega_{\widetilde{M}}^k \otimes u_h^* T_S \to \Omega_{\widetilde{M}}^{k+1} \otimes u_h^* T_S$$

is exterior covariant derivative induced by the connection

$$\nabla : u_h^* T_S \to \Omega^1_{\widetilde{M}} \otimes u_h^* T_S$$

induced by canonical metric of T_S. Here T_S is indeed the complexified vector bundle $T_S^{\mathbb{C}}$.

When (M, g) happens to be a Kähler manifold (X, ω), Siu-Sampson formula shows that u_h is moreover pluriharmonic, i.e.

$$(2.6.2) \qquad \bar{\partial}_\nabla d' u_h = 0, \quad \partial_\nabla d'' u_h = 0.$$

here $du_h = d' u_h + d'' u_h$ decompose according to $(1,0)$ and $(0,1)$-parts of $\Omega^1_{\widetilde{X}} = \Omega^{1,0}_{\widetilde{X}} \oplus \Omega^{0,1}_{\widetilde{X}}$, and also $d_\nabla = \partial_\nabla + \bar{\partial}_\nabla$ also decomposes accordingly.

Let us see how the Higgs bundle structure comes. First note that the hermitian metric h gives rise to a decomposition for the flat connection D of V by

$$D = \nabla_h + \Phi_h,$$

where ∇_h is unitary, and Φ_h is self-adjoint. Therefore, by the flatness of D, we have

$$\nabla_h^2 + [\Phi_h, \Phi_h] = 0$$
$$(2.6.3) \qquad \nabla_h \Phi_h = 0$$

Note that this holds for any choice of h!

On the other hand, one can show that

$$(2.6.4) \qquad \Phi_h = du_h, \quad \nabla = \nabla_h$$

The Siu-Sampson formula shows that

$$[\Phi_h^{1,0}, \Phi_h^{1,0}] = 0.$$

Hence we have

$$\nabla_h^2 = (\nabla^2)^{(1,1)}.$$

Let $E := u_h^* T_S$, that is a complex vector bundle (not yet a holomorphic one!). Since

$$\nabla : E \to E \otimes \Omega^1_{\widetilde{M}}$$

is unitary, it follows that $\nabla^{0,1}$ gives rise to a complex structure for E. By (2.6.2), (2.6.4) and (2.6.3), we have

$$(2.6.5) \qquad \bar{\partial}_\nabla \Phi_h^{1,0} = 0.$$

Hence $(E, \theta, h) := (E, \bar{\partial}_\nabla, \Phi_h^{1,0}, h)$ is a harmonic bundle on $\widetilde{X}$. Since it is ϱ-equivariant, one can show that it descends to a harmonic bundle on X. In summary, one has

THEOREM 2.7 (Corlette). *Let (X, ω) be a compact Kähler manifold. A representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ is reductive, if and only if there exists a harmonic bundle (E, θ, h) such that the monodromy representation $\nabla_h + \theta + \theta_h^*$ is ϱ.*

I refer the reader to the survey articles [**Lou20**,**DM23b**,**Mau15**] for the results mentioned above.

On the other hand, Simpson introduce the stability of Higgs bundles, and he proved the following result.

THEOREM 2.8 (Simpson). *Let (X, ω) be a compact Kähler manifold and let (E, θ) be a μ_ω-polystable Higgs bundle on X. If $c_1(E) \wedge \{\omega\}^{n-1} = c_2(E) \wedge \{\omega\}^{n-2} = 0$, then there exists a harmonic metric h for (E, θ).*

In other words, he established the one-to-one correspondence between semisimple local systems on X, and the polystable Higgs bundles with vanishing Chern classes. This is called the *Simpson correspondence* in the literature.

Remark 2.9.
  • While harmonic metric depends on the choice of the metric on the Riemannian manifold, for Kähler manifolds, the pluriharmonic metric is independent of the choice of the Kähler metric on X.
  • The term *non-abelian Hodge theory* originates from the following distinction: classical Hodge theory concerns the cohomology of the constant sheaf $\mathbb{C}$ on X, whereas non-abelian Hodge theory is related to the cohomology of nontrivial local systems on X, whose monodromy groups are generally non-abelian.

A straightforward application of Simpson's theorem is the following.

THEOREM 2.10 (Simpson). *Let $f: X \to Y$ be a morphism of projective varieties, where Y is smooth and X is normal. For any reductive representation $\varrho: \pi_1(Y) \to \mathrm{GL}_N(\mathbb{C})$, the pullback $f^*\varrho: \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ is also reductive.*

PROOF. Let $\mu: Z \to X$ be a desingularization. Since X is normal, the induced homomorphism on fundamental groups

$$\mu_*: \pi_1(Z) \longrightarrow \pi_1(X)$$

is surjective. Consequently, the image of the representation $f^*\varrho$ coincides with the image of $(f \circ \mu)^*\varrho$. Thus, $f^*\varrho$ is reductive if and only if $(f \circ \mu)^*\varrho$ is reductive. Therefore, after replacing X with a desingularization, we may assume without loss of generality that X is smooth.

We now apply the Simpson correspondence. Let (E, θ, h) be the harmonic bundle on Y corresponding to the reductive representation ϱ. The pullback $(f^*E, f^*\theta, f^*h)$ is also a harmonic bundle on X corresponding to the representation $f^*\varrho$. By Theorem 2.7, the existence of a harmonic metric implies that the associated representation $f^*\varrho$ is reductive. The theorem is proved. □

2.2.2. *Complex Variation of Hodge structures.* In [**Sim88**], Simpson discovered that $\mathbb{C}$-VHS in algebraic geometry is indeed a special case of harmonic bundles. Let us recall the definition.

Definition 2.11. Let Y be a complex manifold. A $\mathbb{C}$-VHS consists of $(V = \oplus_{p+q=r} V^{p,q}, \nabla, Q)$ where (V, ∇) is a flat bundle and Q is a non-definite hermitian form for V such that

  • $V = \oplus_{p+q=r} V^{p,q}$ is a direct sum of smooth vector bundles that is orthogonal with respect to Q,
  • Q is ∇-parallel such that $h := (-1)^p Q|_{V^{p,q}}$ is positively definite
  • we have the Griffiths transversality:

$$\nabla: V^{p,q} \to \underset{\bar{\theta}}{A^{0,1}(V^{p+1,q-1})} \oplus \underset{D}{A^1(V^{p,q})} \oplus \underset{\theta}{A^{1,0}(V^{p-1,q+1})}$$

Since $D''^2 = 0$, $E^{p,q} := (V^{p,q}, D'')$ has a structure of holomorphic vector bundle such that any smooth local section s of $V^{p,q}$ is a holomorphic section of $E^{p,q}$ if and only if $D''s = 0$. The pair $(E = \bigoplus_{p,q} E^{p,q}, \theta)$ is then called a *Hodge bundle*, where the

$$\theta: E^{p,q} \longrightarrow E^{p-1,q+1} \otimes \Omega_Y^1$$

satisfying the relation $\theta \wedge \theta = 0$, is the Higgs field. The Hodge decomposition is orthogonal with respect to the Hermitian metric h, which is referred to as the *Hodge metric*. One can check that $(\oplus_{p+q=r}, E^{p,q}, \theta, h)$ is a harmonic bundle.

Note that a $\mathbb{Z}$-VHS originally arises geometrically as follows: let $f : X \to Y$ be a smooth projective family over a complex manifold Y endowed with a relative ample line bundle $L \to X$. Then the primitive part of the higher direct image of the constant sheaf $\mathbb{C}$ on X, denoted by $R^k f_*(\mathbb{C})_{\mathrm{prim}}$, underlies a $\mathbb{C}$-VHS defined by

$$\left(R^k f_*(\mathbb{C})_{\mathrm{prim}} \otimes_{\mathbb{C}} O_Y \overset{C^\infty}{\simeq} \bigoplus_{p+q=k} R^q f_*(\Omega_{X/Y}^p), \nabla, Q \right)$$

where ∇ is the Gauss-Manin connection for $R^k f_*(\mathbb{C})$, and the sesquilinear form Q is determined by the Hodge-Riemann bilinear relations:

$$Q(\alpha, \beta) = i^{p-q} \int_{X_t} \alpha \wedge \overline{\beta} \wedge c_1(L_t)^{\dim X_t - k}$$

for any $\alpha, \beta \in H^k(X_t)_{\mathrm{prim}}$. Here X_t denotes the fiber $f^{-1}(t)$ for each $t \in Y$. Since the $\mathbb{C}$-VHS has an integral structure given by $R^k f_*(\mathbb{C}) = R^k f_*(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$, this constitutes a $\mathbb{Z}$-VHS. We will not verify the axioms of $\mathbb{C}$-VHS here, but refer the reader to the excellent textbook by Voisin [**Voi02**] for more details.

2.2.3. $\mathbb{C}^*$-*action on Higgs bundles.* For any Higgs bundle (E, θ) on a compact Kähler manifold Y and any $t \in \mathbb{C}^*$, we define the action by

$$t.(E, \theta) := (E, t\theta).$$

We shall use Theorems 2.7 and 2.8 to define such action on representation of $\pi_1(Y)$. Let $\varrho : \pi_1(Y) \to \mathrm{GL}_N(\mathbb{C})$ be a semisimple representation. Then there exists a harmonic bundle (E, θ, h) corresponding to ϱ. By Simpson's theorem, (E, θ) is μ_ω-polystable with vanishing Chern classes. Note that $(E, t\theta)$ is also μ_ω-polystable. Therefore, we use Simpson's theorem again to get a harmonic metric h_t for $(E, t\theta)$. Let $\varrho_t : \pi_1(Y) \to \mathrm{GL}_N(\mathbb{C})$ be the monodromy representation given by the flat connection $\nabla_t + \theta + \bar{\theta}_{h_t}$. In general, ϱ_t is not conjugate to ϱ. However, one can show that

Proposition 2.12 ([**Sim92**]). *The semisimple representation ϱ corresponds to a $\mathbb{C}$-VHS if and only if it is $\mathbb{C}^*$-invariant, i.e. ϱ_t is conjugate to ϱ for any $t \in \mathbb{C}^*$.*

Note that most of Simpson's results, except those discussed in the next subsection, have been extended to the quasi-projective setting in a series of works by Mochizuki [**Moc06**, **Moc07a**, **Moc07b**]. In particular, all of the results stated above remain valid in the quasi-projective case.

2.2.4. *Betti and Dolbeault Moduli spaces, rigid representation and ubiquity.* In this subsection, we discuss the Betti moduli spaces and rigid representations. Let X be a smooth projective variety. We have an affine scheme $\mathrm{Hom}(\pi_1(X), \mathrm{GL}_N)$ of finite type defined over $\mathbb{Z}$ that represents the functor

$$S \mapsto \mathrm{Hom}(\pi_1(X), \mathrm{GL}_N(S))$$

for any ring S. The *Betti moduli space* $M_{\mathrm{B}}(X, \mathrm{GL}_N) := \mathrm{Hom}(\pi_1(X), \mathrm{GL}_N) /\!\!/ \mathrm{GL}_N$ is the GIT quotient of $\mathrm{Hom}(\pi_1(X), \mathrm{GL}_N)$ by GL_N, where GL_N acts by conjugation. Note that it might be reducible while it is called variety. It is indeed the GL_N-*character variety* for the finitely generated group $\pi_1(X)$. Thus, such quotient $\mathrm{Hom}(\pi_1(X), \mathrm{GL}_N) \to M_{\mathrm{B}}(X, \mathrm{GL}_N)$ is surjective. For any representation

$\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$, it is a $\mathbb{C}$-point in $\mathrm{Hom}(\pi_1(X), \mathrm{GL}_N)(\mathbb{C})$. We write $[\varrho]$ to denote its image in $M_{\mathrm{B}}(X, \mathrm{GL}_N)$. We list a crucial property of GL_N-character varieties that will be used in this paper. For further background and more comprehensive accounts, we refer the reader to the survey of Sikora [**Sik12**] and to the monograph of Lubotzky–Magid [**LM85**].

Let K be any algebraically closed field of characteristic zero. By [**Sik12**, Theorem 30], the orbit of any representation ϱ in $\mathrm{Hom}(\pi_1(X), \mathrm{GL}_N)(K)$ is closed if and only if ϱ is semisimple. Two semisimple representations ϱ, ϱ' are conjugate under $\mathrm{GL}_N(K)$ if and only if $[\varrho] = [\varrho']$.

In [**Sim94a**,**Sim94b**], Simpson constructed the moduli spaces of Higgs bundles with vanishing Chern classes on a smooth projective variety X. Fix a positive integer N and a polarization L on X. Simpson constructed a quasi-projective scheme $M_{\mathrm{Dol}}(X, N)$ parametrizing μ_L-semistable Higgs bundles (E, θ) of rank N on X with vanishing Chern classes. The points of the moduli space correspond to equivalence classes defined as follows:

- Let (E, θ) be a μ_L-semistable Higgs bundle with vanishing Chern classes. Let $\mathrm{Gr}(E, \theta)$ denote the graded object associated to a Jordan–Hölder filtration of (E, θ). Then $\mathrm{Gr}(E, \theta)$ is a polystable Higgs bundle with vanishing Chern classes. It is *locally free* by [**Sim92**, Theorem 2]. This result also holds in the compact Kähler setting (see the note by the author [**Den21**]); however, it fails if the condition on the vanishing of Chern classes is dropped.
- Two μ_L-semistable Higgs bundles (E_1, θ_1) and (E_2, θ_2) with vanishing Chern classes are called *Jordan-equivalent* if there exists an isomorphism between $\mathrm{Gr}(E_1, \theta_1)$ and $\mathrm{Gr}(E_2, \theta_2)$.

THEOREM 2.13 ([**Sim94a**]). *Let X be a smooth projective variety. The points of $M_{\mathrm{Dol}}(X, N)(\mathbb{C})$ correspond to Jordan equivalence classes of μ_L-semistable Higgs bundles of rank N with vanishing Chern classes. Moreover, there is a real analytic isomorphism between $M_{\mathrm{Dol}}(X, N)(\mathbb{C})$ and $M_{\mathrm{B}}(X, \mathrm{GL}_N)(\mathbb{C})$. Moreover, the $\mathbb{C}^*$-action on $M_{\mathrm{Dol}}(X, N)$ is given by $(E, \theta) \mapsto (E, t\theta)$ for any $t \in \mathbb{C}^*$ is algebraic.*

For any Higgs bundle (E, θ), consider the characteristic polynomial of θ defined by $\det(\lambda I - \theta) = \lambda^N + \sum_{i=1}^{N} a_i \lambda^{N-i}$, where $a_i \in H^0(X, \mathrm{Sym}^i \Omega_X)$.

Definition 2.14. The map $M_{\mathrm{Dol}}(X, N) \to \oplus_{k \geq 1} H^0(X, \Omega_X^k)$ given by

$$(E, \theta) \mapsto (a_1, \ldots, a_N)$$

is called the *Hitchin morphism*.

Proposition 2.15 ([**Sim94b**]). *The Hitchin morphism $M_{\mathrm{Dol}}(X, N) \to \oplus_{k \geq 1} H^0(X, \Omega_X^k)$ is algebraic, and* proper.

Based on his theorem, he estbalished the following striking results.

Lemma 2.16. *The limit $\lim_{t \to 0} t.(E, \theta)$ exists, that corresponds to a $\mathbb{C}$-VHS.*

PROOF. Note that the inverse image of the Hitchin morphism of the origin, corresponds to Higgs bundles with nilpotent θ (including $\mathbb{C}$-VHS). The image of $t.(E, \theta)$ under the Hitchin morphism converges to the origin if $t \to 0$. Since the Hitchin morphism is proper, the limit exists. It is easy to see that the limit is $\mathbb{C}^*$-invariant, and thus it corresponds to a $\mathbb{C}$-VHS. $\qquad\square$

We then apply the real analytic isomorphism $M_{\mathrm{Dol}}(X, N)(\mathbb{C}) \to M_{\mathrm{B}}(X, \mathrm{GL}_N)(\mathbb{C})$ to obtain the $\mathbb{C}^*$-action on $M_{\mathrm{B}}(X, N)$, that is a *continuous* action, but in general not algebraic! Therefore, we have the following theorem.

Proposition 2.17 ([**Sim91**]). *In any connected component of $M_{\mathrm{B}}(X, N)$, there exists a representation that underlies a $\mathbb{C}$-VHS. Moreover, any representation can deforms to a $\mathbb{C}$-VHS.*

This result is called *Simpson's ubiquity theorem* in the literature.

Definition 2.18. A semisimple representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ is *rigid* if the irreducible component of $M_{\mathrm{B}}(X, \mathrm{GL}_N)(\mathbb{C})$ containing $[\varrho]$ is zero dimensional, otherwise it is called *non-rigid*.   □

For a rigid semisimple representation ϱ, by above arguments any continuous deformation $\varrho' : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ of ϱ which is semisimple is conjugate to ϱ under $\mathrm{GL}_N(\mathbb{C})$. By Proposition 2.17, we have the following result.

Corollary 2.19. *Any rigid representation corresponds to a $\mathbb{C}$-VHS.*   □

But the drawback is that such $\mathbb{C}$-VHS in general does not have discrete monodromy. That motivates us to consider the non-archimedean representations in § 2.3.

We note that most of the results in this subsection, except for Proposition 2.17 and Corollary 2.19, remain open in the quasi-projective setting. In particular, extending Lemma 2.16 to the quasi-projective case would be a very interesting problem.

2.3. Non-abelian Hodge theory: non-archimedean cases. While Simpson–Mochizuki's theory deals exclusively with complex local systems, *all* of the problems discussed so far naturally lead to the study of p-adic local systems, as well as $\mathbb{F}_p((t))$-local systems, on complex algebraic varieties, as we will see shortly. More precisely, given a Zariski dense representation

$$\varrho \colon \pi_1(X) \longrightarrow G(K),$$

where G is a reductive algebraic group defined over a non-archimedean local field K, Gromov–Schoen [**GS92**] proved that, when X is compact Kähler, there exists a ϱ-equivariant harmonic map from the universal cover $\widetilde{X}$ to the *Bruhat–Tits building* $\Delta(G)_K$, a CAT(0) space on which $G(K)$ acts by isometries. We refer the reader to [**BH99**] for the definition of CAT(0) spaces.

More recently, Brotbek, Daskalopoulos, Mese, and the author [**BDDM22**, **DM26**] extended the Gromov–Schoen theorem to the quasi-projective setting. In this subsection, we present this theorem and explain how one can extract algebraic and analytic structures from it.

2.3.1. *Bruhat-Tits buildings.* Note that $S := \mathrm{GL}_N(\mathbb{C})/U_N$ is the symmetric space associated with $\mathrm{GL}_N(\mathbb{C})$. It satisfies the following properties:

- it is *non-positively curved* (indeed, it has non-positive sectional curvature);
- $\mathrm{GL}_N(\mathbb{C})$ acts transitively on S;
- the stabilizer of any point under this action is a compact subgroup.

If K is a non-archimedean local field, such a naïve construction is no longer available. Indeed, a maximal compact subgroup of $\mathrm{GL}_N(K)$ is conjugate to $\mathrm{GL}_N(\mathcal{O}_K)$, which is an open subgroup. Consequently, the naïve quotient $\mathrm{GL}_N(K)/\mathrm{GL}_N(\mathcal{O}_K)$

is discrete and lacks interesting geometric structure. A natural and rich analogue of the symmetric space in the non-archimedean setting is provided by the Bruhat–Tits building.

Let K be a non-archimedean local field of characteristic zero, and let G be a reductive algebraic group defined over K. There exists a Euclidean building associated with G, called the (enlarged) Bruhat–Tits building and denoted by $\Delta(G)_K$ (or simply $\Delta(G)$). It is a non-positively curved space on which $G(K)$ acts strongly transitively by isometries, and the stabilizer of any point under this action is a precompact subgroup of $G(K)$. We refer the reader to [**KP23**, **Rou23**] for the definition and basic properties of Bruhat–Tits buildings.

Associated with $\Delta(G)_K$ is a pair (V, W), where V is a real Euclidean space endowed with its Euclidean metric, and W is an affine Weyl group acting on V. More precisely, W is a semidirect product

$$W = T \rtimes W^v,$$

where W^v is the *vectorial Weyl group*, a finite group generated by reflections of V, and T is a group of translations of V.

For any apartment A in $\Delta(G)$, there exists an isomorphism $i_A : A \to V$, which is called a chart. For two charts $i_{A_1} : A_1 \to V$ and $i_{A_2} : A_2 \to V$, if $A_1 \cap A_2 \neq \varnothing$, it satisfies the following properties:

(a) $Y := i_{A_2}(i_{A_1}^{-1}(V))$ is convex.
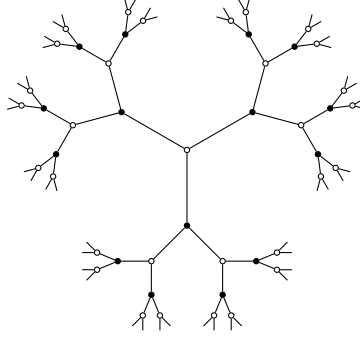(b) There is an element $w \in W$ such that $w \circ i_{A_1}|_{A_1 \cap A_2} = i_{A_2}|_{A_1 \cap A_2}$.

FIGURE 2. Bruhat-Tits building for $\mathrm{SL}_2(\mathbb{Q}_p)$ with $p = 2$.

2.3.2. *Harmonic mapping to Euclidean building.* We first present our result on the extension of Gromov-Schoen's theorem.

THEOREM 2.20 ([**BDDM22**, **DM26**]). *Let X be a smooth quasi-projective variety and let G be a reductive group defined over a non-archimedean local field K. Let $\Delta(G)$ be the Bruhat-Tits building of G. Denote by $\pi_X : \widetilde{X} \to X$ the universal covering map. If $\varrho : \pi_1(X) \to G(K)$ is a Zariski dense representation, then the following statements hold:*

(i) *There exists a ϱ-equivariant pluriharmonic map $u : \widetilde{X} \to \Delta(G)$ with logarithmic energy growth.*
(ii) *u is harmonic with respect to any Kähler metric on $\widetilde{X}$.*

(iii) *Let $f : Y \to X$ be a morphism from a smooth quasi-projective variety Y. Denote by $\tilde{f} : \tilde{Y} \to \tilde{X}$ the lift of f between the universal covers of Y and X. Then the $f^* \varrho$-equivariant map $u \circ \tilde{f} : \tilde{Y} \to \Delta(G)$ is pluriharmonic and has logarithmic energy growth.*

For the definitions appearing in the theorem, we refer the reader to [**BDDM22**] or to [**DM26**, Section 2] for more details. The above theorem was established in [**BDDM22**, Theorem A] by Brotbek, Daskalopoulos, Mese, and the author in the case where G is semisimple, building upon previous work of Daskalopoulos and Mese [**DM23a**, **DM24**, **DM23c**]. Subsequently, building on [**BDDM22**], Mese and the author extended the result to the general reductive case in [**DM26**, Theorem A]. As we will see later, this extension provides a crucial building block both for studying the hyperbolicity of (non-compact) algebraic varieties [**CDY25b**] and for addressing the linear Shafarevich conjecture in the quasi-projective case [**DYK23**, **DY25**, **Bru23**, **BBT24**].

By Theorem 2.20, there exists a ϱ-equivariant *harmonic mapping with logarithmic energy growth* $u : \tilde{X} \to \Delta(G)$. Moreover, such a map u is pluriharmonic.

We denote by $\mathcal{R}(u)$ the regular set of the harmonic map u. Explicitly, this is the set of points $x \in \tilde{X}$ for which there exists an open neighborhood Ω_x of x such that $u(\Omega_x) \subset A$ for some apartment A.

Since $G(K)$ acts transitively on the apartments of $\Delta(G)$ and u is ϱ-equivariant, the set $\mathcal{R}(u)$ is the pullback of an open subset of X. By abuse of notation, we denote this subset of X also by $\mathcal{R}(u)$.

In [**GS92**], Gromov and Schoen prove a deep *regularity theorem*: the Hausdorff codimension of the complement $X \setminus \mathcal{R}(u)$ is at least two. In [**DM26**], Mese and the author proved that $X \setminus \mathcal{R}(u)$ is contained in a proper Zariski closed subset of X.

It has long been a major open problem whether such regularity theorems for harmonic maps into Euclidean buildings hold for non-locally compact Euclidean buildings (e.g., the Bruhat-Tits building of $\mathrm{GL}_N(K)$, where $K = \mathbb{Q}((t))$). This was only recently resolved in a celebrated work by Breiner, Dees, and Mese [**BDM24**], who established the Gromov-Schoen regularity theorem for harmonic maps into non-locally compact Euclidean buildings. This is one of the most significant breakthroughs in this subject over the past two decades.

Let me also mention that, in [**DMV11**], Daskalopoulos and Mese proved a similar regularity theorem for harmonic maps into *hyperbolic buildings*, which is another highly interesting development in this subject.

We now fix orthogonal coordinates $(x_1, \ldots, x_N)$ for V. Define smooth real functions on Ω_x by setting

$$(2.20.1) \qquad\qquad u_{A,i} := x_i \circ i_A \circ u,$$

where $i_A : A \to V$ is the chart defined in the previous subsection. The pluriharmonicity of u implies that $\bar{\partial}\partial u_{A,i} = 0$ for each i; hence, $\partial u_{A,i}$ is a holomorphic 1-form. However, the multiset of holomorphic 1-forms $\{\partial u_{A,1}, \ldots, \partial u_{A,N}\}$ depends on the choice of the chart and the apartment containing $u(\Omega_x)$. We can resolve this ambiguity by considering the multiset of holomorphic 1-forms:

$$\{\partial x_i \circ w \circ i_A \circ u\}_{i=1,\ldots,N;\, w \in W^v}.$$

One can verify (see, e.g., [**BDDM22**, **CDY25b**]) that this multiset of holomorphic 1-forms is invariant under the action of W^ν and is independent of the choice of the chart. They glue together to define a multivalued 1-form on $\mathcal{R}(u)$, denoted by η_u. Since u is ϱ-equivariant, one can verify that η_u descends to a *splitting* multivalued 1-form on $\mathcal{R}(u)$.

Furthermore, using the Lipschitz property of the harmonic map, one can show that η_u extends to a multivalued 1-form on all of X. Since u has logarithmic energy growth, in [**BDDM22**] we can prove that η_u extends to a multivalued section for the logarithmic cotangent bundle $\Omega_{\overline{X}}(\log D)$, where $\overline{X}$ is a smooth projective compactification for X and $D := \overline{X}\backslash X$ is a simple normal crossing divisor. We refer the reader to [**CDY25b**, §3.1] for the formal definition of *multivalued sections* of a holomorphic vector bundle. Less formally, we have

Definition 2.21 (Multivalued section)**.** Let X be a complex manifold and let E be a holomorphic vector bundle on X. A *multivalued section* η on X is a formal sum $\sum_{i=1}^m n_i \Gamma_i$, where each $n_i \in \mathbb{Z}_{>0}$ and each Γ_i is an irreducible closed subvariety of E such that the natural projection $\Gamma_i \to X$ is a finite surjective morphism.

In [**DM26**], Mese and the author established the uniqueness of harmonic maps in a suitable setting. In particular, we showed that η_u is independent of the choice of u. Therefore, we denote this form by η_ϱ. It also extends to a multivalued section of the logarithmic bundle $\Omega_{\overline{X}}(\log D) \to \overline{X}$.

Definition 2.22 (Multivalued 1-form)**.** The multivalued 1-form η_ϱ on X described above is called the *multivalued 1-form associated with* ϱ.

On the other hand, we can construct an analytic object associated with ϱ as follows. For the open neighborhood Ω_x of x as above, we define a smooth, real, semi-positive $(1,1)$-form by

$$(2.22.1) \qquad\qquad \sqrt{-1}\sum_{i=1}^{N} \partial u_{A,i} \wedge \bar{\partial} u_{A,i}.$$

One can verify that this closed real semi-positive $(1,1)$-form is independent of the choice of A and the orthogonal coordinates $(x_1,\dots,x_N)$ for V (see [**DW24a**, §3]). Moreover, it is invariant under the $\pi_1(X)$-action. Consequently, it descends to a smooth, real, closed, semi-positive $(1,1)$-form on $\mathcal{R}(u)$. The Lipschitz property of u and elliptic regularity imply that this form extends to a positive closed $(1,1)$-current T_ϱ on X with continuous potential.

Definition 2.23 (Canonical current)**.** The closed positive $(1,1)$-current T_ϱ on X defined above is called the *canonical current* of ϱ.

Remark 2.24. Although the ϱ-equivariant harmonic map u may not be unique, the uniqueness result in [**DM26**] implies that both η_ϱ and T_ϱ are independent of the choice of u.

2.3.3. *Spectral covering and Katzarkov-Eyssidieux reduction map.* Directly manipulating such multivalued 1-forms in algebraic geometry is often impractical. A standard approach is to construct a finite (generally ramified) covering $\pi\colon X^{\mathrm{sp}} \to X$ such that $\pi^*\eta$ becomes a set of global logarithmic 1-forms on X^{sp}, with an explicitly describable ramification locus. We recall the main result from [**CDY25b**, Proposition 3.1].

Proposition 2.25 (Spectral covering)**.** *Let η_ρ be the multivalued 1-form described in Definition 2.22. There exists a finite Galois cover $\pi \colon \overline{X^{\mathrm{sp}}} \to \overline{X}$ with Galois group H, together with a multiset of holomorphic sections*

$$\{\omega_1, \ldots, \omega_m\} \subset H^0(\overline{X^{\mathrm{sp}}}, \pi^*\Omega_{\overline{X}}(\log D))$$

that is H-invariant. Furthermore, the ramification locus of π is contained in

$$\bigcup_{\omega_i \neq \omega_j} \{\omega_i - \omega_j = 0\}.$$

Such Galois covering π is called the *spectral covering of X* associated with ϱ.

Building on Proposition 2.25 and the properties of harmonic mappings into Euclidean buildings, we derive the following theorem.

THEOREM 2.26 ([**CDY25b**, Theorem E])**.** *Let X be a quasi-projective normal variety and let $\rho \colon \pi_1(X) \to \mathrm{GL}_n(K)$ be a representation, where K is a non-archimedean local field. Then there exists a dominant morphism $s_\rho \colon X \to S_\rho$ to a normal projective variety with connected general fibers, such that for any irreducible closed subvariety $Z \subset X$, the following properties are equivalent:*

(i) *The image $\rho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)])$ is bounded;*
(ii) *The image $\rho(\mathrm{Im}[\pi_1(Z) \to \pi_1(X)])$ is bounded;*
(iii) *The image $s_\rho(Z)$ is a point;*
(iv) *The restriction of the canonical current $T_\rho|_Z$ defined in Definition 2.23 is trivial.*
(v) *The restriction of the multivalued 1-form $\eta_\rho|_Z$ defined in Definition 2.22 is trivial.*

We call the map s_ρ the *Katzarkov-Eyssidieux reduction map* for ρ. When X is compact Kähler, such a map s_ρ was constructed by Katzarkov and Eyssidieux in [**Eys04**]. It plays an important role in the construction of the Shafarevich morphism in the next subsection.

2.4. Construction of the Reductive Shafarevich Morphism. Let X be a smooth quasi-projective variety. We will construct the Shafarevich morphism $\mathrm{sh}_X \colon X \to \mathrm{Sh}(X)$ in this subsection when $\pi_1(X)$ is reductive, i.e., there exists a semisimple representation $\varrho \colon \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ such that $\ker \varrho$ is finite (we say ϱ is *almost faithful*). We begin with the following definition.

Definition 2.27 (Infinite monodromy at infinity)**.** Let X be a quasi-projective normal variety and let $\overline{X}$ be a projective compactification of X. We say a subset $M \subset M_{\mathrm{B}}(X, N)(\mathbb{C})$ *has infinite monodromy at infinity* if for any holomorphic map $\gamma \colon \mathbb{D} \to \overline{X}$ with $\gamma^{-1}(\overline{X} \setminus X) = \{0\}$, there exists a reductive representation $\varrho \colon \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ such that $[\varrho] \in M$ and the restriction $\gamma^*\varrho \colon \pi_1(\mathbb{D}^*) \to \mathrm{GL}_N(\mathbb{C})$ has infinite image.

A simple consequence is the following:

Lemma 2.28. *If $\varrho \colon \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ underlies a $\mathbb{C}$-VHS $\mathcal{L}$ with discrete monodromy, then after replacing X by a partial compactification, the $\mathbb{C}$-VHS extends, and the associated period map $p \colon X \to \mathscr{D}/\Gamma$ is proper, where $\mathscr{D}$ is the period domain and Γ is the monodromy group of $\mathcal{L}$. In this case, the Shafarevich morphism $\mathrm{sh}_\varrho \colon X \to \mathrm{Sh}_\varrho(X)$ of ϱ is the Stein factorization of p.*

PROOF. For the first claim, we refer the reader to [**Gri70**]. Note that in this case, the monodromy representation of the extended $\mathcal{L}$ has infinite monodromy at infinity. The proof relies on the hyperbolicity of the period domain and a standard application of the Ahlfors-Schwarz lemma. Note that the period domain of $\mathcal{L}$ has the form $\mathscr{D} = G/V$, where G is a real reductive group and $V \subset G$ is a compact subgroup (see [**CMP17**]).

Let $Z \subset X$ be a closed subvariety and let $\mu : Y \to Z$ be a desingularization.

If $p(Z)$ is a point P, then for the lift of p to the Galois covering $\tilde{p} : \widetilde{X}_\varrho \to \mathscr{D}$ associated with $\ker \varrho$, the image $p(\widetilde{X})$ is also a point $P \in \mathscr{D}$. The image $\varrho(\mathrm{Im}[\pi_1(Z) \to \pi_1(X)])$ is contained in the stabilizer of P, which is a conjugate V' of V, and hence is compact. On the other hand, since $\mathcal{L}$ has discrete monodromy,

$$\varrho(\mathrm{Im}[\pi_1(Z) \to \pi_1(X)]) \subset \Gamma \cap V'.$$

This is a discrete subgroup of a compact group, and is therefore finite.

Conversely, if $\varrho(\mathrm{Im}[\pi_1(Z) \to \pi_1(X)])$ is finite, then the $\mathbb{C}$-VHS $\mu^*\mathcal{L}$ on Y has finite monodromy. After replacing Y by a finite étale cover, the monodromy becomes trivial. By the uniqueness of the $\mathbb{C}$-VHS structure underlying a local system (see [**Sch73**]), $\mu^*\mathcal{L}$ must be the trivial $\mathbb{C}$-VHS. In particular, its period mapping $\tilde{p}_Y : \widetilde{Y} \to \mathscr{D}$ is constant. Since we have the commutative diagram

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{\tilde{p}_Y} & \mathscr{D} \\ \downarrow & \nearrow_{\tilde{p}} & \\ \widetilde{X} & & \end{array}$$

it follows that $p(Z)$ is a point. The lemma is proved. □

Remark 2.29. It is expected that the Shafarevich morphism of a quasi-projective variety, when it exists, is algebraic. In [**Som78**], Sommese proved that $\mathrm{Sh}_\varrho(X)$ appearing in Lemma 2.28 is properly bimeromorphic to a quasi-projective variety, using Hörmander's L^2–techniques. More recently, Bakker, Brunebarbe, and Tsimerman [**BBT23**] established that $\mathrm{Sh}_\varrho(X)$ is indeed algebraic, thereby confirming a longstanding conjecture of Griffiths.

If ϱ is not a $\mathbb{C}$-VHS with discrete monodromy, the approach in [**DYK23**], following the strategy initiated in [**Eys04**], attempts to find a fibration such that the restriction of ϱ to each fiber is a direct sum of a $\mathbb{Z}$-VHS.

We now introduce an important morphism $s_{\mathrm{fac}} : X \to S_{\mathrm{Fac}}(X)$. We first start with the following lemma.

Lemma 2.30 ([**DYK23**, Lemma 1.28]). *Let V be a smooth quasi-projective variety and let $(f_\lambda : V \to S_\lambda)_{\lambda \in \Lambda}$ be a family of morphisms into quasi-projective varieties S_λ. Then there exist a quasi-projective normal variety S_∞ and a morphism $f_\infty : V \to S_\infty$ such that*

- *f_∞ is dominant and has connected general fibers,*
- *for every subvariety $Z \subset V$, $f_\infty(Z)$ is a point if and only if $f_\lambda(Z)$ is a point for every $\lambda \in \Lambda$, and*
- *there exist $\lambda_1, \ldots, \lambda_n \in \Lambda$ such that $f_\infty : V \to S_\infty$ is the Stein factorization of*

$$(f_1, \ldots, f_n) : V \to S_{\lambda_1} \times \cdots S_{\lambda_n}.$$

The above $f_\infty : X \to S_\infty$ will called the simultaneous Stein factorization *for* $(f_\lambda : V \to S_\lambda)_{\lambda \in \Lambda}$.   □

Definition 2.31. Fix some $N \in \mathbb{N}$. Consider the set $\Upsilon := \{\tau : \pi_1(X) \to \mathrm{GL}_N(K)\}$, where τ ranges over all semisimple representations with K being any non-archimedean local field of characteristic zero. Consider the set of Katzarkov-Eyssidieux reductions $\{s_\tau\}_{\tau \in \Upsilon}$. We take a *simultaneous Stein factorization* of all these s_τ, denoted by $s_{\mathrm{fac}} : X \to S_{\mathrm{Fac}}(X)$.

Note that $s_{\mathrm{fac}} : X \to S_{\mathrm{Fac}}(X)$ is a dominant morphism with connected general fibers such that for any $\tau \in \Upsilon$, we have a factorization

$$X \xrightarrow{\ s_{\mathrm{fac}}\ } S_{\mathrm{Fac}}(X)$$
$$\begin{array}{cc} & \\ {\scriptstyle s_\tau}\searrow & \downarrow \\ & S_\tau \end{array}$$

Moreover, for any subvariety Z of X, if $s_\tau(Z)$ is a point for each $\tau \in \Upsilon$, then $s_{\mathrm{fac}}(Z)$ is also a point.

This factorization possesses the following crucial property.

Proposition 2.32 ([**DYK23**, Proposition 3.10])**.** *Let X be a smooth quasi-projective variety, and let $f : Y \to X$ be a morphism from a smooth quasi-projective variety Y such that $s_{\mathrm{fac}} \circ f(Y)$ is a point. Let $\{\tau_i : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})\}_{i=1,2}$ be reductive representations such that $[\tau_1]$ and $[\tau_2]$ are in the same geometric connected component of $M(\mathbb{C})$. Then $f^*\tau_1$ is conjugate to $f^*\tau_2$. In other words, $j(M_\mathrm{B}(X, N))$ is zero-dimensional, where $j : M_\mathrm{B}(X, N) \to M_\mathrm{B}(Y, N)$ is the natural morphism of character varieties induced by f.*

The main idea of the proof proceeds as follows. Consider the morphism between character varieties $M_\mathrm{B}(X, N) \to M_\mathrm{B}(Y, N)$. If the image of $M_\mathrm{B}(X, N)$ under this morphism, denoted by M, is not zero-dimensional, then its closure is a Zariski closed subset of the affine scheme, and is hence non-compact. However, one can show that for any fixed non-archimedean local field K of characteristic zero, the set of bounded representations $\tau : \pi_1(Y) \to \mathrm{GL}_N(K)$ is a compact subset of $M_\mathrm{B}(Y, N)(K)$. Therefore, there exists some $\tau : \pi_1(X) \to \mathrm{GL}_N(K)$ such that $f^*\tau$ is unbounded. By the construction of s_{fac}, this is impossible. Hence, we conclude that M is zero-dimensional.

In [**DYK23**], we proved the following crucial result.

Proposition 2.33 ([**DYK23**, Proposition 3.13])**.** *There exists a $\mathbb{C}$-VHS $\mathcal{L}$ on X such that for any morphism $f : Y \to X$ from a smooth quasi-projective variety Y to X, if $s_{\mathrm{fac}} \circ f(Y)$ is a point, then for any semisimple representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$, $f^*\varrho$ is a direct factor of the $\mathbb{C}$-VHS $f^*\mathcal{L}$, which has discrete monodromy.*

Note that $\mathcal{L}$ itself might not have discrete monodromy.

PROOF. Fix a geometric connected component M of $M_\mathrm{B}(X, N)$. Since $M_\mathrm{B}(X, N)$ and $\mathrm{Hom}(\pi_1(X), \mathrm{GL}_N)$ are defined over $\mathbb{Z}$, we can choose a semisimple representation $\tau : \pi_1(X) \to \mathrm{GL}_N(\overline{\mathbb{Q}})$ such that $[\tau] \in M(\overline{\mathbb{Q}})$. As $\pi_1(X)$ is finitely generated, there exists a number field k such that $\tau : \pi_1(X) \to \mathrm{GL}_N(k)$.

Let $\mathrm{Ar}(k)$ be the set of all archimedean places of k, with w_1 being the identity map. By the ubiquity theorem (Proposition 2.17), and its extension to the quasi-projective setting by Mochizuki [**Moc06**], for any $w \in \mathrm{Ar}(k)$, there exists a reductive representation $\tau_w^{\mathrm{VHS}} : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ that underlies a $\mathbb{C}$-VHS, such that $[\tau_w^{\mathrm{VHS}}]$ is in the same connected component of $[\tau_w]$. Let $\mathcal{L}$ be the $\mathbb{C}$-VHS defined as the direct sum $\oplus_{w \in \mathrm{Ar}(k)} \tau_w^{\mathrm{VHS}}$.

By Theorem 2.10, both $f^*\tau_w$ and $f^*\tau_w^{\mathrm{VHS}}$ are reductive. Then, by Proposition 2.32, $f^*\tau_w^{\mathrm{VHS}}$ is conjugate to $f^*\tau_w$. Hence, each $f^*\tau_w$ underlies a $\mathbb{C}$-VHS.

Let v be any non-archimedean place of k and let k_v be the non-archimedean completion of k with respect to v. Let $\tau_v : \pi_1(X) \to \mathrm{GL}_N(k_v)$ be the representation induced by τ. By the definition of s_{fac}, it follows that $f^*\tau_v(\pi_1(Z))$ is bounded. Therefore, we have a factorization

$$f^*\tau : \pi_1(Z) \to \mathrm{GL}_N(O_k).$$

Note that the embedding $\mathrm{GL}_N(O_k) \to \prod_{w \in \mathrm{Ar}(k)} \mathrm{GL}_N(\mathbb{C})$ has a discrete image by [**Zim84**, Proposition 6.1.3]. It follows that for the product representation

$$\prod_{w \in \mathrm{Ar}(k)} f^*\tau_w : \pi_1(Z) \to \prod_{w \in \mathrm{Ar}(k)} \mathrm{GL}_N(\mathbb{C}),$$

the image is discrete.

Since $f^*\tau_w$ is conjugate to $f^*\tau_w^{\mathrm{VHS}}$ for each $w \in \mathrm{Ar}(k)$, it follows that $f^*\mathcal{L}$ has discrete monodromy. If $[\varrho]$ and $[\tau]$ both belong to M, then $f^*\varrho$ is conjugate to $f^*\tau$, and hence $f^*\varrho$ appears as a direct factor of $f^*\mathcal{L}$. Note that the local system $\mathcal{L}$ is chosen so as to be associated with M.

Repeating this construction for each geometrically connected component of $M_{\mathrm{B}}(X, N)$, and taking the direct sum of the resulting $\mathbb{C}$-VHS, we obtain the desired $\mathbb{C}$-VHS. The proposition follows.   □

We now turn to the construction of the Shafarevich morphism of X when $\pi_1(X)$ is reductive. In [**DYK23**, Proposition 3.19], it is shown that after replacing X by an étale cover, there exists a partial compactification such that ϱ extends to a representation with infinite monodromy at infinity. Let $\mathcal{L}$ be the $\mathbb{C}$-VHS from Proposition 2.33 and let $p : \widetilde{X} \to \mathscr{D}$ be the associated period map. Consider the holomorphic map

$$\Phi : \widetilde{X} \to \mathscr{D} \times S_{\mathrm{Fac}}(X)$$
$$x \mapsto (p(x), s_{\mathrm{fac}} \circ \pi_X(x)),$$

where $\pi_X : \widetilde{X} \to X$ is the universal covering map. Using Proposition 2.33 together with Lemma 2.28, one can deduce that each connected component of the fiber of Φ is compact. Then, by Theorem 2.34 below, Φ factors through a proper holomorphic fibration

$$\mathrm{sh}_{\widetilde{X}} : \widetilde{X} \to \mathrm{Sh}(\widetilde{X})$$

onto a normal complex space $\mathrm{Sh}(\widetilde{X})$. One can prove that $\mathrm{Sh}(\widetilde{X})$ does not contain any compact closed subvariety of positive dimension. Hence, the $\pi_1(X)$-action on $\widetilde{X}$ maps fibers of $\mathrm{sh}_{\widetilde{X}}$ to fibers, thereby inducing an action on $\mathrm{Sh}(\widetilde{X})$. This action is properly discontinuous, and $\mathrm{sh}_{\widetilde{X}}$ is equivariant with respect to it. We take the quotient of $\mathrm{sh}_{\widetilde{X}}$ to obtain a proper holomorphic fibration $\mathrm{sh}_X : X \to \mathrm{Sh}(X)$ such

that we have the commutative diagram

$$
\begin{array}{ccc}
\widetilde{X} & \longrightarrow & X \\
\downarrow{\scriptstyle \mathrm{sh}_{\widetilde{X}}} & & \downarrow{\scriptstyle \mathrm{sh}_X} \\
\mathrm{Sh}(\widetilde{X}) & \longrightarrow & \mathrm{Sh}(X).
\end{array}
$$

One can show that $\mathrm{sh}_X : X \to \mathrm{Sh}(X)$ is the Shafarevich morphism of X.

We need the following generalized Stein factorization proved by Henri Cartan in [**Car60**, Theorem 3].

THEOREM 2.34. *Let X and S be complex spaces and let $f : X \to S$ be a holomorphic map. Suppose that X is normal and that every connected component F of a fiber of f is compact. Then the set Y of connected components of the fibers of f can be endowed with the structure of a normal complex space such that f factors through the natural map $e : X \to Y$, which is a proper holomorphic fibration.* $\qquad\square$

Remark 2.35. The proof presented here is technically different from that of [**Eys04**] in the compact case, although we follow the essential ideas introduced by Eyssidieux. In [**Eys04**], Eyssidieux studies representations $\pi_1(X) \to \mathrm{GL}_N(k((t)))$ over non-archimedean fields, where k is a number field, arising from closed curves $C \subset \mathrm{Hom}(\pi_1(X), \mathrm{GL}_N)$. Since $k((t))$ is not locally compact, the Gromov–Schoen regularity theorem was not available at that time. In this setting, Eyssidieux used reduction modulo p arguments to work with representations $\pi_1(X) \to \mathrm{GL}_N(\mathbb{F}_q((t)))$ in order to *factor through non-rigidity*. For integrality issues, he must further consider representations $\pi_1(X) \to \mathrm{GL}_N(K)$, where K is a finite extension of some $\mathbb{Q}_p$. This makes the proof substantially more involved.

The advantage of our construction of s_{fac} is that it allows us to treat *non-rigidity* and *integrality* simultaneously, as we have seen in Proposition 2.33. Moreover, it is a significant simplification that we only work with local fields of characteristic zero.

When the faithful representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ is not reductive and X is projective, one must invoke the deep results of Eyssidieux and Simpson [**ES11**] on the construction of suitable tautological representations arising from variations of mixed Hodge structures in order to study the Shafarevich conjecture. We omit the details here and refer the reader to [**EKPR12**] for a comprehensive treatment.

2.5. Shafarevich morphism in positive characteristic. The previous subsection constructed the Shafarevich morphism for smooth quasi-projective varieties X assuming the representation of the fundamental group is reductive. In this subsection, we outline the construction of the Shafarevich morphism by Yamanoi and the author [**DY25**] for the case where there exists an almost faithful representation $\rho : \pi_1(X) \to \mathrm{GL}_N(K)$, where K is a field of characteristic $p > 0$.

2.5.1. *A lemma on finite groups.*

Lemma 2.36 ([**DY25**, Lemma 3.1]). *Let K be an algebraically closed field of positive characteristic and let Γ be a finitely generated group. Let $\varrho : \Gamma \to \mathrm{GL}_N(K)$ be a representation such that its semisimplification has finite image. Then $\varrho(\Gamma)$ is finite.*

PROOF. Since the semisimplification ϱ^{ss} of ϱ has finite image, we can replace Γ by a finite index subgroup such that $\varrho^{ss}(\Gamma)$ is trivial. Therefore, some conjugate σ of ϱ has image in the subgroup $\mathrm{U}_N(K)$ consisting of all upper-triangular matrices in $\mathrm{GL}_N(K)$ with 1's on the main diagonal.

Note $U_N(K)$ admits a central normal series whose successive quotients are isomorphic to $\mathbb{G}_{a,K}$. We remark that a finitely generated subgroup of $\mathbb{G}_{a,K}$ is a finite group, for K has positive characteristic. By [**ST00**, Proposition 4.17], any finite index subgroup of a finitely generated group is also finitely generated. Consequently, $\sigma(\Gamma)$ admits a central normal series whose successive quotients are finitely generated subgroups of $\mathbb{G}_{a,K}$, which are all finite groups. It follows that $\sigma(\Gamma)$ is finite. The lemma is proved. $\qquad\square$

2.5.2. *Character varieties in positive characteristic.* Recall that the variety of N-dimensional linear representations of $\pi_1(X)$ in characteristic zero is represented by an affine $\mathbb{Z}$-scheme $R(X, N)$ of finite type. Namely, given a commutative ring A, the set of A-points of $R(X, N)$ is:

$$R(X, N)(A) = \operatorname{Hom}(\pi_1(X), \operatorname{GL}_N(A)).$$

Let p be a prime number. Consider $R(X, N)_{\mathbb{F}_p} := R(X, N) \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{F}_p$ and note that the general linear group over $\mathbb{F}_p$, denoted by $\operatorname{GL}(N, \mathbb{F}_p)$, acts on $R(X, N)_{\mathbb{F}_p}$ by conjugation. Using Seshadri's extension of geometric invariant theory quotients for schemes of arbitrary field [**Ses77**, Theorem 3], we can take the GIT quotient of $R(X, N)_{\mathbb{F}_p}$ by $\operatorname{GL}(N, \mathbb{F}_p)$, denoted by $M_{\mathrm{B}}(X, N)_{\mathbb{F}_p}$. Then $M_{\mathrm{B}}(X, N)_{\mathbb{F}_p}$ is also an affine $\mathbb{F}_p$-scheme of finite type. For any algebraically closed field K of characteristic p, the K-points $M_{\mathrm{B}}(X, N)_{\mathbb{F}_p}(K)$ is identified with the conjugacy classes of semi-simple representations $X \to \operatorname{GL}_N(K)$.

2.5.3. *Shafarevich morphism in positive characteristic.*

THEOREM 2.37 ([**DY25**, Theorem 3.9]). *Let X be a smooth quasi-projective variety. Assume that there exists an almost faithful representation $\rho \colon \pi_1(X) \to \operatorname{GL}_N(K)$, where K is a field of characteristic $p > 0$. The Shafarevich morphism $\operatorname{sh}_X \colon X \to \operatorname{Sh}(X)$ of X is obtained through the simultaneous Stein factorization $s_\infty \colon X \to S_\infty(X)$ of the reductions $\{s_\tau \colon X \to S_\tau\}_{\tau \in \Upsilon}$, where Υ consists of all reductive representations with K a non-archimedean local field of characteristic p, and $s_\tau \colon X \to S_\tau$ is the Katzarkov-Eyssidieux reduction map defined in Theorem 2.26.*

PROOF. For simplicity, write $R = R(X, N)_{\mathbb{F}_p}$ and $M = M_{\mathrm{B}}(X, N)_{\mathbb{F}_p}$. Let $f \colon Y \to X$ be a morphism from another smooth quasi-projective variety. In the same vein as Proposition 2.32, we have the following result.

Claim 2.38. *If $s_\infty \circ f(Y)$ is a point, then $h(M)$ is zero-dimensional, where $h \colon M \to M_{\mathrm{B}}(Y, N)_{\mathbb{F}_p}$ is the natural morphism induced by f.*

PROOF. Assume by contradiction that $h(M)$ has a positive dimensional component. Then we can find any irreducible affine curve $T \subset R$ defined over $\overline{\mathbb{F}_p}$, such that its image in $M_{\mathrm{B}}(Y, N)_{\mathbb{F}_p}$ under the composite morphism

$$\Phi \colon R \to M \xrightarrow{h} M_{\mathrm{B}}(Y, N)_{\mathbb{F}_p}$$

is non a point. Take $\bar{C}$ as the compactification of the normalization C of T, and let $\{P_1, \ldots, P_\ell\} = \bar{C} \setminus C$. There exists $q = p^n$ for some $n \in \mathbb{Z}_{>0}$ such that $\bar{C}$ is defined over $\mathbb{F}_q$ and $P_i \in \bar{C}(\mathbb{F}_q)$ for each i. By the universal property of the representation scheme R, C gives rise to a representation $\varrho_C \colon \pi_1(X) \to \operatorname{GL}_N(\mathbb{F}_q[C])$, where $\mathbb{F}_q[C]$ is the coordinate ring of C. Consider the discrete valuation $v_i \colon \mathbb{F}_q(C) \to \mathbb{Z}$ defined by P_i, where $\mathbb{F}_q(C)$ is the function field of C. Let $\widehat{\mathbb{F}_q(C)}_{v_i}$ be the completion of $F_q(C)$ with respect to v_i. Then we have the isomorphism $\left(\widehat{\mathbb{F}_q(C)}_{v_i}, v_i\right) \simeq \left(\mathbb{F}_q((t)), v\right)$,

where $\big(\mathbb{F}_q((t)), v\big)$ is the formal Laurent field of $\mathbb{F}_q$ with the valuation v defined by $v(\sum_{i=m}^{+\infty} a_i t^i) = \min\{i \mid a_i \neq 0\}$. Let $\varrho_i : \pi_1(X) \to \mathrm{GL}_N(\mathbb{F}_q((t)))$ be the extension of ϱ_C with respect to $\widehat{\mathbb{F}_q(C)}_{v_i}$.

By our definition of s_∞, and the assumption that $s_\infty \circ f(Y)$ is a point, and Theorem 2.26, $f^*\varrho_i(\pi_1(X))$ is bounded for each i. Thus after we replace $f^*\varrho_i$ by some conjugate, we have $f^*\varrho_i(\pi_1(X)) \subset \mathrm{GL}_N(\mathbb{F}_q[[t]])$, where the $\mathbb{F}_q[[t]]$ is the ring of integers of $\mathbb{F}_q((t))$, i.e.

$$\mathbb{F}_q[[t]] := \{\sum_{i=0}^{+\infty} a_i t^i \mid a_i \in \mathbb{F}_q\}.$$

For any matrix $A \in \mathrm{GL}_N(K)$, we denote by $\chi(A) = T^N + \sigma_1(A)T^{N-1} + \cdots + \sigma_N(A)$ its characteristic polynomial. Since $f^*\varrho_i(\pi_1(X)) \subset \mathrm{GL}_N(\mathbb{F}_q[[t]])$ for each i, it follows that $\sigma_j(f^*\varrho_i(\gamma)) \in \mathbb{F}_q[[t]]$ for each i. Therefore, by the definition of ϱ_i, $v_i\big(\sigma_j(f^*\varrho_C(\gamma))\big) \geq 0$ for each i. It follows that $\sigma_j(f^*\varrho_C(\gamma))$ extends to a regular function on $\overline{C}$, which is thus constant. This implies that for any two representations $\eta_1 : \pi_1(X) \to \mathrm{GL}_N(K_1)$ and $\eta_2 : \pi_1(X) \to \mathrm{GL}_N(K_2)$ such that $\mathrm{char}\, K_1 = \mathrm{char}\, K_2 = p$ and $\eta_i \in C(K_i)$, we have $\chi(f^*\eta_1(\gamma)) = \chi(f^*\eta_2(\gamma))$ for each $\gamma \in \pi_1(Y)$. In other words, η_1 and η_2 has the same characteristic polynomial. It follows that $[f^*\eta_1] = [f^*\eta_2]$ by the Brauer-Nesbitt theorem. Hence $\Phi(T)$ is a point. We obtain a contradiction.   □

Since R is defined over $\overline{\mathbb{F}}_p$, there exists some representation $\eta : \pi_1(X) \to \mathrm{GL}_N(\overline{\mathbb{F}_p})$ such that η are in the same geometric connected component of ϱ. Since $\pi_1(X)$ is finitely generated, it follows that $\eta(\pi_1(X))$ is finite. Let $f : Y \to X$ be as in Claim 2.38. By Claim 2.38, we have

$$[f^*\eta] = [f^*\varrho].$$

Therefore, the semisimplification $(f^*\eta)^{ss}$ of $f^*\eta$ and the semisimplifcation $(f^*\varrho)^{ss}$ of $f^*\varrho$ are conjugate. By virtue of Lemma 2.36, we conclude that $f^*\varrho(\pi_1(Y))$ is finite. As ϱ is almost faithful, it follows that

$$\mathrm{Im}[\pi_1(Y) \to \pi_1(X)]$$

is finite.

On the other hand, if $g : Z \to X$ be a morphism such that $s_\infty \circ g(Z)$ is not a point, by Theorem 2.26, there exists some $\tau \in \Upsilon$ such that $g^*\tau$ is unbounded, thus $g^*\tau(\pi_1(Z))$ has infinite image. This proves that s_∞ is the Shafarevich morphism of X.   □

Remark 2.39. As noted in Theorem 2.37, the Shafarevich morphism in positive characteristic relies solely on Katzarkov-Eyssidieux reductions. Furthermore, by virtue of Lemma 2.36, the linear case can be handled without additional difficulty. This implies that while the problem in positive characteristic is in some sense simpler, it also exhibits a less rich structure than the characteristic zero case. Consequently, the Shafarevich conjecture for projective varieties whose fundamental groups admit a faithful representation into $\mathrm{GL}_N(K)$ with $\mathrm{char}\, K > 0$ remains open, with only the case of surfaces having been proven in [**DY25**].

It is worth noting that in [**DYK23**, **DY25**] we establish the existence of the Shafarevich morphism in a more general setting, without assuming that the fundamental group of X is linear. More precisely, we prove the following.

THEOREM 2.40 ([**DYK23**,**DY25**]). *Let X be a normal quasi-projective variety, and let $\varrho \colon \pi_1(X) \to \mathrm{GL}_N(K)$ be a representation, where K is an arbitrary field. If* $\mathrm{char}\, K = 0$, *we additionally assume that ϱ is reductive. Then the Shafarevich morphism associated with ϱ, denoted by* $\mathrm{sh}_\varrho \colon X \to \mathrm{Sh}_\varrho(X)$, *exists.*   □

We refer the interested reader to [**DYK23**,**DY25**] for further details.

2.5.4. *An application to Esnault's conjecture.* A intriguing question by Esnault asks whether a complex projective variety with infinite fundamental group must admit non-trivial symmetric differentials. This question was first answered affirmatively by Brunebarbe, Klingler, and Totaro [**BKT13**] in the case where there exists a representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$, over an arbitrary field K, such that $\varrho(\pi_1(X))$ is infinite.

More recently, Brotbek, Daskalopoulos, Mese, and the author [**BDDM22**] extended this result to the quasi-projective setting.

THEOREM 2.41 ([**BDDM22**, Theorem B]). *Let X be a smooth quasi-projective variety. If there exists a repersentation $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ for any field K such that $\varrho(\pi_1(X))$ is infinite. Then for some $k \in \mathbb{N}$, we have*

$$H^0(\overline{X}, \mathrm{Sym}^k \Omega_{\overline{X}}(\log D)) \neq 0,$$

where $\overline{X}$ is a smooth projective compactification, such that $D = \overline{X} \backslash X$ is a simple normal crossing divisor.

Our approach relies on the theorem on the existence of ϱ-equivariant harmonic maps to Euclidean buildings (see Theorem 2.20). We now briefly sketch the proof, which differs slightly from the one presented in [**BDDM22**] (though ideas are exactly the same).

PROOF OF THEOREM 2.41 (SKETCH). The existence of logarithmic symmetric differentials is preserved under finite étale covers. Therefore, throughout the proof, we may freely replace X by a finite étale cover.

Case 1: $\mathrm{char}\, K = 0$.

We may assume that $K = \mathbb{C}$. Let σ be the semisimplification of ϱ.

Case 1.1: σ has finite image.

After replacing X by a finite étale cover, the representation ϱ becomes unipotent. Its image then admits a filtration whose successive quotients are abelian. Consequently, there exists an abelian representation of $\pi_1(X)$ with infinite image. This implies that $H^1(X, \mathbb{C})$ is infinite-dimensional.

Recall that

$$H^1(X, \mathbb{C}) = H^0\big(\overline{X}, \Omega_{\overline{X}}(\log D)\big) \oplus H^{0,1}(X).$$

Hence $H^0\big(\overline{X}, \Omega_{\overline{X}}(\log D)\big) \neq 0$.

Case 1.2: σ has infinite image.

Case 1.2.1: For every representation $\tau : \pi_1(X) \to \mathrm{GL}_N(K)$, where K is a non-archimedean local field of characteristic zero, the image of τ is bounded.

In this case, the factorization map $s_{\mathrm{fac}} : X \to S_{\mathrm{Fac}}(X)$ is constant. By Proposition 2.33, σ appears as a direct factor of a complex variation of Hodge structure $\mathcal{L}$ with discrete monodromy. Since σ has infinite image, it follows that the monodromy of $\mathcal{L}$ is infinite.

By Lemma 2.28, the associated period map $p : X \to \mathscr{D}/\Gamma$ has positive-dimensional image. Let Z be a desingularization of the closure of $p(X)$. Using curvature properties of period domains in the horizontal direction [**Gri70**], Brunebarbe and Cadorel [**BC20**] proved that Z has a big logarithmic cotangent bundle. In particular, Z admits many logarithmic symmetric differentials. Pulling them back to X yields the desired conclusion.

Case 1.2.2: There exists a representation $\tau : \pi_1(X) \to \mathrm{GL}_N(K)$, where K is a non-archimedean local field of characteristic zero, whose image is unbounded.

By Theorem 2.26, there exists a nontrivial multivalued logarithmic 1-form η_τ. Taking local products of η_τ, one obtains logarithmic symmetric differentials on a Zariski open subset of $\overline{X}$. One then shows that these extend to logarithmic symmetric differentials on the pair $(\overline{X}, D)$.

Case 2: $\mathrm{char}\, K = p > 0$.

By the proof of Theorem 2.37, it follows that there exists an unbounded representation

$$\tau : \pi_1(X) \to \mathrm{GL}_N\big(\mathbb{F}_q((t))\big),$$

where $q = p^\ell$ for some $\ell \in \mathbb{N}$. Otherwise, the character variety $M_\mathrm{B}(X, N)_{\mathbb{F}_p}$ would be zero-dimensional, and it would follow from Lemma 2.36 that ϱ has finite image, contradicting the assumption. We then apply the same argument as in Case 1.2.2 to conclude that X admits a non-trivial logarithmic symmetric differential.   □

Remark 2.42. When X is projective and $K = \mathbb{C}$, the original argument in [**BKT13**] is a beautiful application of non-abelian Hodge theory. We briefly recall their proof.

If the semisimplification ϱ^{ss} has finite image, the argument proceeds exactly as above. We may therefore assume that ϱ is semisimple with infinite image. Suppose, by contradiction, that X admits no nontrivial symmetric differentials. Then, by Proposition 2.15, the Dolbeault moduli space $M_{\mathrm{Dol}}(X, N)$ is compact. By Theorem 2.13, there exists an analytic isomorphism between $M_\mathrm{B}(X, N)$ and $M_{\mathrm{Dol}}(X, N)$, and hence $M_\mathrm{B}(X, N)$ is also compact. Since $M_\mathrm{B}(X, N)$ is affine, it follows that it is zero-dimensional. Consequently, every semisimple representation $\pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ is rigid, and in particular ϱ is rigid. By Corollary 2.19, ϱ underlies a $\mathbb{C}$-variation of Hodge structure. However, such a $\mathbb{C}$-VHS may *a priori* have non-discrete monodromy.

Since $M_\mathrm{B}(X, N)$ is defined over $\mathbb{Q}$, after replacing ϱ by a conjugate we may assume that it is defined over a number field k. Let v be a non-archimedean place of k, and let k_v denote the completion of k at v. We denote by

$$\varrho_v : \pi_1(X) \longrightarrow \mathrm{GL}_N(k_v)$$

the induced representation. If ϱ_v is unbounded for some non-archimedean place v, then the same argument as in Case 1.2.2 yields the existence of a nontrivial logarithmic symmetric differential on X, contradicting our assumption.

Hence, we may assume that for every non-archimedean place v of k, the representation ϱ_v is bounded; this property is usually referred to as the *integrality* of ϱ. By the same argument as in Proposition 2.33, it follows that ϱ is a direct sum of a $\mathbb{C}$-variation of Hodge structure $\mathcal{L}$ with discrete and infinite monodromy. Applying the argument of Case 1.2.1, we conclude that X admits nontrivial symmetric differentials, which yields a contradiction.

The theorem of [**BKT13**] is in fact more general, as it also applies to the compact Kähler case. In this setting, one does not have a satisfactory theory of analytic morphisms between the moduli spaces $M_{\mathrm{B}}(X, N)$ and $M_{\mathrm{Dol}}(X, N)$. Nevertheless, Klingler [**Kli13**] observed that the analytic gauge-theoretic arguments of [**Sim92**] suffice to show that the absence of symmetric differentials still forces $M_{\mathrm{B}}(X, N)$ to be zero-dimensional. This idea also underlies the proof of the first version of [**BDDM22**].

Remark 2.43. It is worth noting that Simpson proposed in [**Sim92**] a celebrated conjecture on the integrality of rigid local systems on smooth complex projective varieties, which was proved by himself and Corlette in [**CS08**] for rank two local systems. A major breakthrough on this conjecture was achieved by Esnault and Groechenig [**EG18**, **EG20**], who proved Simpson's conjecture for cohomologically rigid local systems. Esnault also pointed out to me that, if one knows that $M_{\mathrm{B}}(X, N)$ is zero-dimensional, then their arguments still apply and imply the integrality of arbitrary rigid local systems.

2.6. Holomorphic convexity. In the previous subsections, we constructed the Shafarevich morphism for smooth quasi-projective varieties under the assumption that the fundamental group is reductive or linear in positive characteristic. However, this construction offers no insight into the holomorphic convexity of the universal covering of X when X is compact.

While the techniques developed above suffice for applications to the linear Chern–Hopf–Thurston conjecture (Theorem 1.8), the proof of the linear case of Kollár's conjecture (Theorem 1.12) relies crucially on auxiliary results established in the study of the Shafarevich conjecture, notably in [**Eys04**, **DYK23**, **EKPR12**]. In this subsection, we outline the key ideas of this approach and present the essential ingredients needed for the proof of Theorem 1.12. More precisely, we outline a proof of Conjecture 1.5 in the case of projective surfaces and sketch the main ideas underlying the general higher-dimensional case.

2.6.1. *Some analytic tools from harmonic mapping.* We shall give some analytic tools. Based on [**DM26**, Proposition 4.15], in [**DW24a**, §3], we prove that

Lemma 2.44. *Let X be a smooth projective variety endowed with a Kähler form ω. Let $\tau : \pi_1(X) \to G(K)$ be a Zariski dense representation, where G is a reductive group over a non-archimedean local field K. Let $u : \widetilde{X} \to \Delta(G)_K$ be the τ-equivariant pluriharmonic map ensured by Gromov-Schoen's theorem [**GS92**]. Fix a base point $x_0 \in \widetilde{X}$. Define a 1-form*

$$(2.44.1) \qquad \beta_\tau := \sqrt{-1}\bar{\partial} d^2_{\Delta(G)}(u(x), u(x_0)).$$

on the universal cover $\widetilde{X}$ of X. Then we have

- *β_τ has L^1-local coefficients, and is smooth almost everywhere.*
- *There exists some constant $C > 0$ such that we have*

$$|\beta_\tau|_{\pi_X^*\omega} \leq_{\mathrm{a.e.}} C(1 + d_{\widetilde{X}}(x, x_0)),$$

  where $d_{\widetilde{X}}(x, x_0)$ is the distance function on $\widetilde{X}$ induced by $\pi_X^*\omega$.
- *Define $\phi_{\tau_i} := d^2_{\Delta(G)}\big(u(x), u(x_0)\big)$, which is a continuous semipositive function on $\widetilde{X}$. Then we have*

$$d\beta_\tau = \sqrt{-1}\partial\bar{\partial}\phi_{\tau_i},$$

which is a real, closed, and positive $(1,1)$*-current, and it satisfies*

$$d\beta_\tau \geq \pi^* T_\tau.$$

In the archimedean case, Eyssidieux proved an analogous result in [**Eys00**, Proposition 4.5.1], which we recall below.

Lemma 2.45. *Let* $\mathcal{L}$ *be a* $\mathbb{C}$*-variation of Hodge structure on* X*. There exists a smooth real-valued plurisubharmonic function* $\phi_\mathcal{L}$ *on* $\widetilde{X}$*, bounded from below, such that*

$$\sqrt{-1}\partial\bar{\partial}\phi_\mathcal{L} \geq \pi_X^*\left(\sqrt{-1}\,\mathrm{tr}\,\theta \wedge \theta^*\right), \qquad |d\phi_\mathcal{L}|_{\pi_X^*\omega} \leq C.$$

Here θ *denotes the Higgs field of* $\mathcal{L}$*, and* θ^* *its adjoint with respect to the Hodge metric.*

If we introduce a smooth 1-form on $\widetilde{X}$ by setting

$$\beta_\mathcal{L} := \sqrt{-1}\bar{\partial}\phi_\mathcal{L},$$

then we have

$$(2.45.1) \qquad\qquad d\beta_\mathcal{L} \geq \pi_X^*\left(\sqrt{-1}\,\mathrm{tr}\,\theta \wedge \theta^*\right), \qquad |\beta_\mathcal{L}|_{\pi_X^*\omega} \leq C.$$

Next, we give the criterion for an infinite topological Galois covering of a compact complex normal space to be Stein.

Lemma 2.46 ([**Eys04**, Proposition 4.1.1])**.** *Let* S *be a compact complex normal space, and let* $\nu : \Sigma \to S$ *be an infinite topological Galois covering. Let* T *be a closed positive* $(1,1)$*-current on* S *with continuous local potentials, whose cohomology class* $\{T\}$ *is Kähler. Assume that there exists a real-valued continuous plurisubharmonic function* ϕ *on* Σ*, bounded from below, such that*

$$\sqrt{-1}\,\partial\bar{\partial}\phi \geq \nu^* T.$$

Then Σ *is a Stein space.* $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

2.6.2. *Two lemmas on finiteness and boundedness criteria.* We will need the following results concerning finiteness and boundedness criteria for subgroups. Their proofs rely on the geometry of Bruhat–Tits buildings.

Lemma 2.47 ([**CDY25b**, Lemma 5.3])**.** *Let* G *be an almost simple algebraic group defined over a non-archimedean local field* K*. Assume that* $\Gamma \subset G(K)$ *is an unbounded subgroup whose Zariski closure contains* $G^\circ(K)$*, where* G° *denotes the identity component of* G*. If* $N \triangleleft \Gamma$ *is a bounded normal subgroup, then* N *is finite.* $\square$

(cf. [**Bru22**] for an independent proof).

Lemma 2.48 ([**DY25**, Lemma 7.15])**.** *Let* G *be a semisimple algebraic group over a non-archimedean local field* K*. Let* $\Gamma \subset G(K)$ *be a finitely generated subgroup whose Zariski closure contains* $G^\circ(K)$*, where* G° *denotes the identity component of* G*. If its derived group* $\mathcal{D}\Gamma$ *is* bounded*, then* Γ *is also bounded.* $\square$

2.6.3. *Ideas of proof of holomorphic convexity.* By Theorem 2.40, one can show that, after passing to a finite étale cover, an almost faithful representation factors through a large local system on the Shafarevich variety $\mathrm{Sh}(X)$. Consequently, it suffices to establish the Steinness of the universal cover in the case of *large and reductive* representations. In this subsection, we outline the proof of the reductive Shafarevich conjecture for smooth projective surfaces.

THEOREM 2.49. *Let X be a smooth projective surface. If there exists a semisimple and large representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$, then the universal covering $\widetilde{X}$ of X is Stein.*

PROOF. Let $\mathcal{L}$ be the $\mathbb{C}$-VHS defined in Proposition 2.33. Let θ be its Higgs field and θ^* be the adjoint of θ with respect to the Hodge metric.

Case 1: The set Υ in Definition 2.31 contains no unbounded representation.
This means that $s_{\mathrm{fac}} : X \to S_{\mathrm{Fac}}(X)$ is a constant map. Then by Proposition 2.33, ϱ is a direct factor of a $\mathbb{C}$-VHS $\mathcal{L}$ with discrete monodromy. Since ϱ is large, the monodromy representation of $\mathcal{L}$ is thus also large. Hence, by Lemma 2.28 its period map $p : X \to \mathscr{D}/\Gamma$ is finite. One can show that for the real $(1,1)$-form $\sqrt{-1}\operatorname{tr}\theta \wedge \theta^*$, its cohomology class

$$\left\{ \sqrt{-1}\operatorname{tr}\theta \wedge \theta^* \right\}$$

is a Kähler class. By Lemma 2.45, there exists a smooth real-valued plurisubharmonic function $\phi_{\mathcal{L}}$ on $\widetilde{X}$, bounded from below, such that

$$\sqrt{-1}\partial\bar{\partial}\phi_{\mathcal{L}} \geq \pi_X^*\left(\sqrt{-1}\operatorname{tr}\theta \wedge \theta^* \right).$$

Then, by Lemma 2.46, the universal cover $\widetilde{X}$ is Stein.

From now on, we assume that Υ contains at least one unbounded representation. By Lemma 2.30, there exists unbounded and reductive representations $\boldsymbol{\tau} := \{\tau_i : \pi_1(X) \to \mathrm{GL}_N(K_i)\}_{i=1,\ldots,k}$ with K_i a non-archimedean local field of characteristic 0 such that the Stein factorization of products of Katzarkov-Eyssidieux reduction maps

$$s_{\tau_1} \times \cdots \times s_{\tau_k} : X \to S_{\tau_1} \times \cdots \times S_{\tau_k}$$

coincides with $s_{\mathrm{fac}} : X \to S_{\mathrm{Fac}}$. Let $X^{\mathrm{sp}} \to X$ be the spectral covering associated with $\boldsymbol{\tau}$. Let $\{\eta_1, \ldots, \eta_k\} \subset H^0(X^{\mathrm{sp}}, \pi^*\Omega_X^1)$ be the associated spectral 1-forms as in Proposition 2.25. We define the *rank* as follows: if there exist i and j such that $\eta_i \wedge \eta_j \neq 0$, then we say that the rank is 2; otherwise, the rank is 1.

Case 2: The spectral 1-forms have rank 2. By our construction of canonical currents in Definition 2.23, we have

$$C\pi^*(T_{\tau_1} + \cdots + T_{\tau_k}) = \sqrt{-1}\sum_{i=1}^{m} \eta_i \wedge \overline{\eta_i}$$

for some constant $C > 0$. Then

$$T_{\boldsymbol{\tau}} := T_{\tau_1} + \cdots + T_{\tau_k}$$

is strictly positive at general points since the spectral forms associated with $\boldsymbol{\tau}$ has rank 2. Since $T_{\boldsymbol{\tau}}$ has continuous local potentials, it follows that $T_{\boldsymbol{\tau}}$ is a big and nef class. By the construction of $T_{\boldsymbol{\tau}}$, one can show that there exists a closed positive $(1,1)$-current T on $S_{\mathrm{Fac}}(X)$ such that

$$s_{\mathrm{fac}}^* T = T_{\boldsymbol{\tau}}.$$

This in particular shows that s_{fac} is a birational morphism. On the other hand, as a consequence of Theorem 2.26, we can show that $\{T\}\cdot C > 0$ for any irreducible curve

C on $S_{\mathrm{Fac}}(X)$. We now apply a theorem by Demailly-Păun [**DP04**] (or [**Lam99**]) to conclude that $\{T\}$ is a Kähler class on $S_{\mathrm{Fac}}(X)$.

On the other hand, for each positive dimensional fiber F of s_{fac}, by Proposition 2.33, $\mathcal{L}|_{F^{\mathrm{norm}}}$ is a $\mathbb{C}$-VHS with discrete monodromy, and $\varrho|_{\pi_1(F^{\mathrm{norm}})}$ is a direct factor of $\mathcal{L}|_{F^{\mathrm{norm}}}$. Since ϱ is large, then the period map $F^{\mathrm{norm}} \to \mathscr{D}/\Gamma$ is finite, where Γ is the monodromy group of $\mathcal{L}|_{F^{\mathrm{norm}}}$. One can show that

$$\{(\sqrt{-1}\mathrm{tr}\theta \wedge \theta^*)|_F\}$$

is a Kähler class. Therefore,

$$\{\sqrt{-1}\mathrm{tr}\theta \wedge \theta^* + T_\tau\}$$

is a Kähler class on X. Let ϕ_{τ_i} be the continuous semipositive function associated with τ_i, as defined in Lemma 2.44, and let $\phi_{\mathcal{L}}$ be the continuous function, bounded from below, associated with $\mathcal{L}$, as defined in Lemma 2.45. Then we have

$$\sqrt{-1}\partial\bar{\partial}(\phi_{\mathcal{L}} + \sum_{i=1}^{k} \phi_{\tau_i}) \geq \pi_X^*(\sqrt{-1}\mathrm{tr}\theta \wedge \theta^* + T_\tau)$$

By Lemma 2.46, we conclude that $\widetilde{X}$ is Stein.

Case 3: Regardless of the elements $\tau_1, \ldots, \tau_m \in \Upsilon$ chosen, their spectral 1-forms have rank 1.
Case 3.1: $\dim S_{\mathrm{Fac}}(X) = 1$.

The proof is the same as Case 2. We leave it as an exercise to the readers.
Case 3.2: $\dim S_{\mathrm{Fac}}(X) = 2$.
Case 3.2.1: The dimension of spectral 1-forms is at least two:

Suppose $\dim_{\mathbb{C}}\mathrm{Span}\{\eta_1, \ldots, \eta_\ell\} \geq 2$.

Without loss of generality, we may assume that $\eta_1 \wedge \eta_2 \equiv 0$ and $\eta_1 \notin \{\mathbb{C}\eta_2\}$. According to the Castelnuovo-De Franchis theorem (cf. [**ABC$^+$96**, Theorem 2.7]), there exists a proper fibration $h : X^{\mathrm{sp}} \to C$ over a smooth projective curve C such that $\{\eta_1, \eta_2\} \subset h^*H^0(C, \Omega_C^1)$. Since s_{fac} is birational, we can choose a general fiber F of h, which is irreducible and such that $s_{\mathrm{fac}} \circ \pi(F)$ is not a point. By Theorem 2.26, we can show that there exists some i such that $\eta_i|_F \neq 0$. Given that $\eta_1|_F \equiv 0$, this implies that $\eta_i \wedge \eta_1 \neq 0$. It contradicts with our assumption that the spectral 1-forms have rank 1. Therefore, this case cannot occur.

We now turn to the final—and most difficult—case, which also relies on several deep results from non-abelian Hodge theories by Simpson.

Case 3.2.2: Regardless of the elements $\tau_1, \ldots, \tau_m \in \Upsilon$ chosen, the dimension of spectral 1-forms remains 1.

Pick any non-archimedean representation $\tau : \pi_1(X) \to \mathrm{GL}_N(K)$ that is unbounded. Let G be the Zariski closure of $\tau(\pi_1(X))$, which is reductive. Consider the isogeny $g : G \to G/Z \times G/\mathcal{D}G$ where Z is the central torus of G and $\mathcal{D}G$ is the derived group of G. As a result, $G' := G/Z$ is semisimple and $G'' := G/\mathcal{D}G$ is a torus. Let $\tau' : \pi_1(X) \to G'(\overline{K})$ be the composite of τ with the projection $G \to G'$, and $\tau'' : \pi_1(X) \to G''(\overline{K})$ be the composite of τ with the projection $G \to G''$. Then τ' and τ'' are both Zariski dense representations.

Claim 2.50. *The representation τ' is bounded.*

PROOF. Let $X^{\mathrm{sp}} \to X$ be the spectral covering associated with τ and let $\nu : Y \to X^{\mathrm{sp}}$ be a desingularization. By the assumption, the spectral 1-forms are a subset of $\mathbb{C}\eta$, where η is a 1-form in $H^0(Y, \Omega_Y^1)$. Consider the *partial Albanese morphism* $a : Y \to A$ induced by η (see [**CDY25b**, Definition 3.25] for the definition). Then there exists a one form $\eta' \in H^0(A, \Omega_A^1)$ such that $a^*\eta' = \eta$. If $\dim a(Y) = 1$, then the Stein factorization $h : Y \to C$ of a is a proper holomorphic fibration over a smooth projective curve C such that $\eta_1 \in h^*H^0(C, \Omega_C^1)$. We are now in a situation akin to Case 3.1, and we can apply the same arguments to reach a contradiction. Hence $\dim a(Y) = 2$. Let $\pi_A : \widetilde{A} \to A$ denote the universal covering map. We denote by $Y' := Y \times_{\widetilde{A}} A$ a connected component of the fiber product and let $\pi' : Y' \to Y$ be the induced étale cover. It is worth noting that $\pi'^*\eta$ is exact. Consequently, we can define the following holomorphic map:

$$h : Y' \to \mathbb{C}$$

$$y \mapsto \int_{y_0}^{y} \pi'^*\eta.$$

We then have the following commutative diagram:

The holomorphic map $\widetilde{A} \to \mathbb{C}$ in the above diagram is defined by the linear 1-form $\pi_A^*\eta'$ on $\widetilde{A}$. By Simpson's Lefschetz theorem [**Sim93a**], for any $t \in \mathbb{C}$, $h^{-1}(t)$ is connected and $\pi_1(h^{-1}(t)) \to \pi_1(Y')$ is surjective. By definition of h, $\pi_Y^*\eta|_Z \equiv 0$ where Z is any connected component of $p^{-1}(h^{-1}(t))$. Here $p : \widetilde{Y} \to Y'$ is the natural covering map.

Consider the Zariski dense representation $\tau' : \pi_1(X) \to G'(\overline{K})$ as defined previously. Let L be a finite extension of K such that G' is defined on L and $\tau' : \pi_1(X) \to G'(L)$. We denote by $\sigma : \pi_1(Y) \to G'(L)$ the pullback of τ' via the morphism $Y \to X$. The existence of a σ-equivariant harmonic mapping $u : \widetilde{Y} \to \Delta(G')_L$ is guaranteed by [**GS92**], where $\Delta(G')_L$ is the Bruhat-Tits building of G'.

We note that $\pi_Y^*\eta$ is the (1,0)-part of the complexified differential of the harmonic mapping u at general points of $\widetilde{Y}$, with $\pi_Y : \widetilde{Y} \to Y$ denoting the universal covering. For any connected component Z of $p^{-1}(h^{-1}(t))$ for a general $t \in \mathbb{C}$, since $\pi_Y^*\eta|_Z \equiv 0$, and all the spectral forms are assumed to be $\mathbb{C}$-linearly equivalent, it follows that $u(Z)$ is a point $P \in \Delta(G')_L$. Since u is σ-equivariant, it follows that $\pi'^*\sigma(\mathrm{Im}[\pi_1(h^{-1}(t)) \to \pi_1(Y')])$ is contained in the subgroup of $G'(L)$ fixing the point P, which is thus bounded. Recall that $\pi_1(h^{-1}(t)) \to \pi_1(Y')$ is surjective. Hence $\pi'^*\sigma(\pi_1(Y'))$ is a bounded subgroup of $G'(L)$. Additionally, note that the derived group

$$\mathcal{D}\pi_1(Y) \subset \mathrm{Im}[\pi_1(Y') \to \pi_1(Y)],$$

and it follows that $\sigma(\mathcal{D}\pi_1(Y))$ is bounded. Since τ' is Zariski dense, and $\mathrm{Im}[\pi_1(Y) \to \pi_1(X)]$ is a finite index subgroup of $\pi_1(X)$, the Zariski closure of $\sigma(\pi_1(Y))$ contains the identity component of G', and it is also semisimple. We apply Lemma 2.48 to conclude that $\sigma(\pi_1(Y))$ is bounded. Hence τ' is bounded. □

By Theorem 2.26, the Katzarkov-Eyssidieux reduction map $s_{\tau'}$ is constant. This implies that the reduction map s_τ is identified with $s_{\tau''}$.

Note that G'' is a torus. A key idea originally introduced by Eyssidieux in [**Eys04**] is the use of Simpson's *absolutely constructible subsets* to handle this situation. We sketch the arguments in [**DYK23**, Theorem 4.5]; the following discussion is intended as an outline rather than a rigorous proof.

Consider the composition of natural morphisms among Betti moduli spaces:

$$\Psi : M_{\mathrm{B}}(X, G) \to M_{\mathrm{B}}(Y, G) \to M_{\mathrm{B}}(Y, G'').$$

By [**Sim93b**], the image of $M_{\mathrm{B}}(X, G)$ under this composition is an *absolutely constructible subset*, denoted by M_{acs}, of $M_{\mathrm{B}}(Y, G'')$. Since G'' is a torus, $M_{\mathrm{B}}(Y, G'')$ is essentially a product of copies of $M_{\mathrm{B}}(Y, 1)$.

For simplicity, we assume that G'' is a one-dimensional torus. Then Simpson's theorem [**Sim93b**] states that the closure of M_{acs} is a finite union of torsion translates of subtori in $M_{\mathrm{B}}(Y, 1)$.

Since we assume that $\dim a(Y) = 2$, a crucial step in [**DYK23**, Theorem 4.5] shows that one can find sufficiently many representations

$$\{\sigma_i \colon \pi_1(X) \to G(K_i)\}_{i=1,\ldots,\ell},$$

where each K_i is a non-archimedean local field of characteristic zero, such that their images

$$\{\Psi(\sigma_i) \colon \pi_1(X) \to \mathrm{GL}_1(K_i)\}_{i=1,\ldots,\ell}$$

in M_{acs} are unbounded representations, and the holomorphic 1-forms on Y induced by them have rank 2. This implies that the associated spectral 1-forms have rank 2, which leads to a contradiction. □

Remark 2.51. In the general case, one must extend Simpson's Lefschetz theorem for leaves of holomorphic foliations defined by *systems of holomorphic 1-forms*. This extension is a crucial step in the proof of the reductive Shafarevich conjecture and was established by Eyssidieux in [**Eys04**]. Based on this general Lefschetz theorem, together with a suitable application of absolutely constructible subsets, we prove in [**DYK23**, Proof of Theorem 4.31] that there exist representations

$$\{\tau_i \colon \pi_1(X) \to G(K_i)\}_{i=1,\ldots,\ell},$$

where each K_i is a non-archimedean local field of characteristic zero, such that the sum of the canonical currents associated with these representations, $\sum_{i=1}^{\ell} T_{\tau_i}$, is the pullback via s_{fac} of a closed positive $(1, 1)$-current T_{fac} on $S_{\mathrm{Fac}}(X)$, whose cohomology class $\{T_{\mathrm{fac}}\}$ is *Kähler*.

Recall that for the $\mathbb{C}$-VHS $\mathcal{L}$ on X constructed in Proposition 2.33, its restriction to each fiber F of s_{fac} has discrete monodromy. If we assume that there exists a big and semisimple representation on X, then for a general fiber F of s_{fac}, the period map of $\mathcal{L}|_F$ is generically finite onto its image. Consequently, one can show

that the cohomology class

$$\left\{ \sum_{i=1}^{\ell} T_{\tau_i} + \sqrt{-1}\,\mathrm{tr}(\theta \wedge \theta^*) \right\}$$

is big and nef on X. This fact is crucial in the proof of the linear Kollár conjecture (Theorem 1.12).

Remark 2.52. In [**DYK23**, Appendix], Katzarkov, Yamanoi, and the author proved the reductive Shafarevich conjecture for projective *normal* varieties X admitting a faithful reductive representation

$$\varrho \colon \pi_1(X) \to \mathrm{GL}_N(\mathbb{C}),$$

thereby extending the results of [**Eys04**] to the singular setting. We now outline the main strategy of the proof in [**DYK23**].

Let $\mu : Y \to X$ be a desingularization. Since the induced morphism $\pi_1(X) \to \pi_1(Y)$ is surjective, the natural morphism of Betti moduli spaces

$$\iota : M_{\mathrm{B}}(X, N) \hookrightarrow M_{\mathrm{B}}(Y, N)$$

induced by μ is a closed immersion. Using a theorem of Lerer [**Ler22**], we show that the image of ι is an *absolutely constructible subset* of $M_{\mathrm{B}}(Y, N)$ in the sense of Budur–Wang [**BW20**].

Define the subgroup

$$\Gamma := \bigcap_\tau \ker \tau,$$

where $\tau : \pi_1(Y) \to \mathrm{GL}_N(\mathbb{C})$ ranges over all reductive representations lying in $\iota(M_{\mathrm{B}}(X, N))$. We then prove that the covering $\widetilde{Y}_\Gamma \to Y$ of Y is holomorphically convex.

Next, define

$$\Gamma' := \bigcap_\tau \ker \tau,$$

where $\tau : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ ranges over all reductive representations in $M_{\mathrm{B}}(X, N)$. The morphism μ lifts to a proper holomorphic map $\tilde{\mu} : \widetilde{Y}_\Gamma \to \widetilde{X}_{\Gamma'}$, fitting into the commutative diagram

$$
\begin{array}{ccc}
\widetilde{Y}_\Gamma & \longrightarrow & Y \\
\downarrow{\scriptstyle \tilde{\mu}} & & \downarrow{\scriptstyle \mu} \\
\widetilde{X}_{\Gamma'} & \longrightarrow & X.
\end{array}
$$

By Theorem 2.3, it follows that $\widetilde{X}_{\Gamma'}$ is holomorphically convex. Since the representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ is faithful and reductive, we conclude that $\widetilde{X}_{\Gamma'} = \widetilde{X}$, which completes the proof.

Let me conclude this section by presenting a recent application of the linear Shafarevich conjecture (Theorem 1.6). In [**EF25**], Eyssidieux and Funar established new constraints on algebro-geometric subgroups of mapping class groups. As a consequence, they proved that the universal cover of the smooth proper Deligne–Mumford stack $\overline{\mathcal{M}}_{g,n}[\mathbf{k}]$, the universal covering space of the stack is, in most cases, a Stein manifold. Further applications of the techniques used to study the linear Shafarevich conjecture, as presented in this section, will be discussed in the subsequent sections.

3. Hyperbolicity and funmamental groups: ideas of proofs

In this section, we sketch the main idea behind the proofs of Theorems 1.36 and 1.37.

3.1. A theorem of Nevanlinna theory on semiabelian variety.
In this subsection, we recall some notions from Nevanlinna theory. Readers who are not familiar with Nevanlinna theory are referred to the excellent survey by Yamanoi [**Yam15b**] or Demailly [**Dem97**].

In this section, A is a semi-abelian variety and Y is a Riemann surface with a proper surjective holomorphic map $\pi : Y \to \mathbb{C}_{>\delta}$, where $\mathbb{C}_{>\delta} := \{z \in \mathbb{C} \mid \delta < |z|\}$ with some fixed positive constant $\delta > 0$.

For $r > 2\delta$, define $Y(r) = \pi^{-1}\big(\mathbb{C}_{>2\delta}(r)\big)$ where $\mathbb{C}_{>2\delta}(r) = \{z \in \mathbb{C} \mid 2\delta < |z| < r\}$. In the following, we assume that $r > 2\delta$. The *ramification counting function* of the covering $\pi : Y \to \mathbb{C}_{>\delta}$ is defined by

$$N_{\mathrm{ram}\,\pi}(r) := \frac{1}{\deg \pi} \int_{2\delta}^{r} \left[\sum_{y \in Y(t)} \mathrm{ord}_{\,y}\mathrm{ram}\,\pi \right] \frac{dt}{t},$$

where $\mathrm{ram}\,\pi \subset Y$ is the ramification divisor of $\pi : Y \to \mathbb{C}_{>\delta}$.

Let L be a line bundle on X. Let $f : Y \to X$ be a holomorphic map. We define the order function $T_f(r, L)$ as follows. First suppose that X is smooth. We equip with a smooth hermitian metric h_L, and let $c_1(L, h_L)$ be the curvature form of (L, h_L).

$$T_f(r, L) := \frac{1}{\deg \pi} \int_{2\delta}^{r} \left[\int_{Y(t)} f^* c_1(L, h_L) \right] \frac{dt}{t}.$$

This definition is independent of the choice of the hermitian metric up to a function $O(\log r)$.

THEOREM 3.1 ([**CDY25a**, Theorem A]). *Let X be a smooth quasi-projective variety which is of log general type. Assume that there is a morphism $a : X \to A$ such that $\dim X = \dim a(X)$. Then there exists a proper Zariski closed set $\Xi \subsetneqq X$ with the following property: let $f : Y \to X$ be a holomorphic map such that $N_{\mathrm{ram}\,\pi}(r) = O(\log r) + o(T_f(r))\|$ and that $f(Y) \not\subset \Xi$. Then f does not have essential singularity over ∞, i.e., there exists an extension $\overline{f} : \overline{Y} \to \overline{X}$ of f, where $\overline{Y}$ is a Riemann surface such that $\pi : Y \to \mathbb{C}_{>\delta}$ extends to a proper map $\overline{\pi} : \overline{Y} \to \mathbb{C}_{>\delta} \cup \{\infty\}$ and $\overline{X}$ is a compactification of X.*

Note that Theorem 3.1 is proven by Yamanoi in [**Yam15a**] when X is compact. Its proof is based on techniques in Nevanlinna theories in [**Yam15a**]. Compared with the compact case treated in [**Yam15a**], the lack of Poincaré reducibility theorem is a major difficulty to treat the non-compact case. We use a more general "cover" than étale cover to overcome this problem. We refer the readers to [**CDY25a**, Remark 10.11] for the main difficulty and novelty in the non-compact cases. Since the proof of Theorem 3.1 is highly involved and unrelated to other aspects of the paper, we choose to omit it. Instead, we present a fundamental result from Nevanlinna theory.

Claim 3.2. *Let $f : Y \to X$ be as above. If the order function $T_f(r, L) = O(\log r)$ as $r \to \infty$, then f does not have essential singularity at infinity.*

In a nutshell, the ultimate goal in proving Theorem 3.1 is to estimate the order function $T_f(r, L)$ utilizing Nevanlinna theory tools, such as the logarithmic derivative lemma, the Second Main Theorem, jet differentials, and other related techniques.

In the context of Nevanlinna theory in [**CDY25a**, §10], another crucial result is obtained.

THEOREM 3.3 ([**CDY25a**, Corollary 10.8]). *Let X be a smooth quasi-projective variety and let $a : X \to A \times S$ be a morphism such that $\dim X = \dim a(X)$, where S is a smooth quasi-projective variety (S can be a point). Write $b : X \to S$ as the composition of a with the projection map $A \times S \to S$. Assume that b is dominant.*

    (i) *Suppose S is pseudo Picard hyperbolic. If X is of log general type, then X is pseudo Picard hyperbolic.*

    (ii) *Suppose S is strongly of log general type. If X is pseudo Brody hyperbolic, then X is strongly of log general type.* □

3.2. Hyperbolicity and non-archimedean local system. A crucial step for the proof of Theorem 1.36 is the following result.

THEOREM 3.4 ([**CDY25b**, Theorem F]). *Let X be a quasi-projective normal variety and let G be an almost simple algebraic group defined over a non-archimedean local field K. If $\varrho : \pi_1(X) \to G(K)$ is a big, Zariski dense, and unbounded representation, then X is of log general type, and pseudo Picard hyperbolic.*

We would like to sketch the idea of the proof of Theorem 3.4 since the methods are new even if X is projective (compared with [**CCE15**]).

PROOF OF THEOREM 3.4 (SKETCH). For simplicity, we assume that G is geometrically connected. Let $\pi : \overline{X^{\mathrm{sp}}} \to \overline{X}$ be the spectral covering associated with ϱ in Proposition 2.25. By Proposition 2.25, it is a finite Galois covering with the Galois group H and satisfies the following properties:

- there exists forms $\{\eta_1, \ldots, \eta_\ell\} \subset H^0(\overline{X^{\mathrm{sp}}}, \pi^* \Omega^1_{\overline{X}}(\log D))$ such that $\{\eta_1, \ldots, \eta_\ell\}$ coincides with the mutivalued one-forms $\pi^*\{\omega_1, \ldots, \omega_\ell\}$ induced by the ϱ-equivariant pluriharmonic map u with logarithmic energy at infinity constructed in Theorem 2.20.
- Let us denote by $\mathrm{Ram}(\pi)$ the ramification locus of $\pi : \overline{X^{\mathrm{sp}}} \to \overline{X}$. Then we have

(3.4.1)
$$\mathrm{Ram}(\pi) \subset \bigcup_{\eta_i \neq \eta_j} (\eta_i - \eta_j = 0).$$

- $\{\eta_1, \ldots, \eta_\ell\}$ is invariant under the Galois group H.

Claim 3.5. *The quasi-Albanese map $a : X^{\mathrm{sp}} \to A$ satisfies $\dim X^{\mathrm{sp}} = \dim a(X^{\mathrm{sp}})$.*

Let us explain the proof of Claim 3.5. Assume by contradiction that $\dim a(X^{\mathrm{sp}}) < \dim X^{\mathrm{sp}}$. Let F be a connected component of a general fiber of a. Then $\eta_i|_F \equiv 0$ for each η_i.

Let $\pi' : \widetilde{X^{\mathrm{sp}}} \to X^{\mathrm{sp}}$ be the universal covering and denote by $\tilde{\pi} : \widetilde{X^{\mathrm{sp}}} \to \widetilde{X}$ be the map between universal covering lifting $\pi : X^{\mathrm{sp}} \to X$. Denote by $\tau = \pi^* \varrho : \pi_1(X^{\mathrm{sp}}) \to G(K)$. Then $u \circ \tilde{\pi} : \widetilde{X^{\mathrm{sp}}} \to \Delta(G)$ is τ-equivariant harmonic map with logarithmic energy at infinity by Theorem 2.20.(iii). Let F' be a connected component of $\pi'^{-1}(F)$. Since $\{\eta_1, \ldots, \eta_\ell\}$ is generically the $(1, 0)$-part of complexified differentials

of $u \circ \tilde{\pi}$, it follows that $u \circ \tilde{\pi}(F')$ is a point. This implies that $\tau(\mathrm{Im}[\pi_1(F) \to \pi_1(X^{\mathrm{sp}})])$ fixes a point in $\Delta(G)$, hence is bounded.

Note that $\tau(\mathrm{Im}[\pi_1(F) \to \pi_1(X^{\mathrm{sp}})])$ is a normal subgroup of $\tau(\pi_1(X^{\mathrm{sp}}))$. Since $\tau(\pi_1(X^{\mathrm{sp}}))$ is unbounded as ϱ is unbounded, by Lemma 2.47, we conclude that $\tau(\mathrm{Im}[\pi_1(F) \to \pi_1(X^{\mathrm{sp}})])$ is finite.

Since we assume that ϱ is big, τ is also big. We obtain a contradiction. Hence $\dim a(X^{\mathrm{sp}}) = \dim X^{\mathrm{sp}}$.

Therefore, the logarithmic Kodaira dimension $\bar{\kappa}(X^{\mathrm{sp}}) \geq 0$. Assume that it is not maximal, then the logarithmic Iitaka fibration $j : X^{\mathrm{sp}} \to J$ has general fibers positive dimensional. Let F be a general fiber of j. Then $a|_F : F \to A$ is generically finite into the image and $\bar{\kappa}(F) = 0$. By the criterion of abelian variety in [**CDY25a**, Lemma 1.4], we conclude that $\pi_1(F)$ is abelian.

Note that the Zariski closure of $\tau(\pi_1(X^{\mathrm{sp}}))$ is also almost simple. Since

$$\tau(\mathrm{Im}[\pi_1(F) \to \pi_1(X^{\mathrm{sp}})])$$

is a abelian and normal subgroup of $\tau(\pi_1(X^{\mathrm{sp}}))$ and ϱ is Zariski dense, we conclude that the Zariski closure of $\tau(\mathrm{Im}[\pi_1(F) \to \pi_1(X^{\mathrm{sp}})])$, denoted by N, is a normal subgroup of G. Since G is almost simple, it follows that N is finite. Hence $\tau(\mathrm{Im}[\pi_1(F) \to \pi_1(X^{\mathrm{sp}})])$ is finite, contradicting with the assumption that ϱ is big. Therefore, j is birational, and we conclude that X^{sp} is of log general type.

We will spread the positivity from X^{sp} to X to show that X is of log general type. This step is innovative and as far as I know, the method has never appeared before. Define a section

$$\sigma := \prod_{h \in H} \prod_{\eta_i \neq \eta_j} h^*(\eta_i - \eta_j) \in H^0(\overline{X^{\mathrm{sp}}}, \mathrm{Sym}^M \pi^* \Omega_{\overline{X}}(\log D)),$$

which is non-zero. By (3.4.1), σ vanishes at $\mathrm{Ram}(\pi)$. Since it is invariant under the H-action, it descends to a section

$$\sigma^H \in H^0(\overline{X}, \mathrm{Sym}^M \Omega_{\overline{X}}(\log D))$$

so that $\pi^* \sigma^H = \sigma$. Let $R \subset X$ be the ramification locus of $\pi : \overline{X^{\mathrm{sp}}} \to \overline{X}$. By the purity we know that R is a divisor on X. Note that σ^H vanishes at R. Therefore, it induces a non-trivial morphism

$$(3.5.1) \qquad\qquad O_{\overline{X}}(R) \to \mathrm{Sym}^M \Omega_{\overline{X}}(\log D).$$

Since X^{sp} is of log general type, and π is unramified over $X - R$, it follows that $K_{\overline{X}} + D + \overline{R}$ is big. (3.5.1) together with a theorem of Campana-Păun in Theorem 3.7 below, implies that $K_{\overline{X}} + D$ is big. Therefore, X is of log general type.

Let us prove that X is pseudo Picard hyperbolic. Let $g : \mathbb{D}^* \to X$ be non-constant holomorphic map that is not contained in $\mathrm{Ram}(\pi)$. Then there exists a Riemann surface Y, a proper surjective holomorphic map $p : Y \to \mathbb{D}^*$ and a holomorphic map $f : Y \to X^{\mathrm{sp}}$ such that we have the following commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\ f\ } & X^{\mathrm{sp}} \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\
\mathbb{D}^* & \xrightarrow{\ g\ } & X
\end{array}
$$

A crucial fact is the estimation of the ramification counting function of $p : Y \to \mathbb{D}^*$ in [**CDY25b**, Lemma 4.11] together with [**CDY25a**, Lemma 11.2].

Claim 3.6. *There exists a proper Zariski closed subset $\Xi_1 \subsetneq X$ such that if $g(\mathbb{D}^*) \not\subset \Xi_1$, then we have*

$$N_{\mathrm{ram}\,\pi}(r) = O(\log r) + o(T_f(r)).$$

Recall that X^{sp} is of log general type and the quasi-Albanese map $a : X^{\mathrm{sp}} \to A$ satisfies that $\dim X^{\mathrm{sp}} = \dim a(X^{\mathrm{sp}})$. Therefore, we apply Theorem 3.1 to conclude that there exists a proper Zariski closed subset $\Xi_2 \subsetneq X$ such that g does not have essential singularity at the origin provided that $g(\mathbb{D}^*) \not\subset \Xi_1 \cup \Xi_2$. This proves that X is pseudo Picard hyperbolic. $\qquad\square$

THEOREM 3.7 ([**CP19**, Corollary 8.7]). *Let $\overline{X}$ be a smooth projective variety and let D be a simple normal crossing divisor on $\overline{X}$. Let L be a line bundle on $\overline{X}$, which admits a morphism $L \to \bigotimes^m \Omega^1_{\overline{X}}(\log D)$ for some $m > 0$, and such that the $\mathbb{Q}$-bundle $\varepsilon \left(K_{\overline{X}} + D \right) + L$ is big for some rational number $\varepsilon \geq 0$. Then $K_{\overline{X}} + D$ is big.* $\qquad\square$

3.3. Proof of Theorem 1.36 when $\mathrm{char}\,K = 0$ **(sketch).** We can assume that $K = \mathbb{C}$. For simplicity, we assume that the Zariski closure G of $\varrho(\pi_1(X))$ is *almost simple*. There are several cases that occurs.

Case 1. ϱ **is rigid.** It means that for any continuous deformation ϱ_t of ϱ, we have $[\varrho_t] = [\varrho]$, where $[\varrho]$ denotes the image of ϱ in the Betti moduli space $M_{\mathrm{B}}(X, N)(\mathbb{C}) := M_{\mathrm{B}}(\pi_1(X), \mathrm{GL}_N)(\mathbb{C})$. By Mochizuki's extension of Corollary 2.19 to the quasi-projective setting [**Moc06**], ϱ underlies a $\mathbb{C}$-variation of Hodge structure (cf. also [**CDY25b**, §6] for a more self-contained proof). Moreover, after replacing ϱ by a suitable conjugate, we may assume there exists a number field $k \subset \overline{\mathbb{Q}}$ such that

- G is defined over k;
- we have the factorization $\varrho : \pi_1(X) \to G(k)$;
- $\varrho(\pi_1(X))$ is Zariski dense in G.

Case 1.1. Assume that for each non-archimedean place v of k, the composite $\varrho_v : \pi_1(X) \to \mathrm{GL}_N(k_v)$ of ϱ and $k \hookrightarrow k_v$, is bounded. Here k_v denotes the non-archimedean completion of k with respect to v.

If this case occurs, we have a factorization $\varrho : \pi_1(X) \to \mathrm{GL}_N(O_k)$. Let us denote by $\mathrm{Ar}(k)$ the set of archimedean places of k. Note that $\mathrm{GL}_N(O_k) \to \prod_{w \in \mathrm{Ar}(k)} \mathrm{GL}_N(\mathbb{C})$ is a discrete subgroup by [**Zim84**, Proposition 6.1.3]. We denote by $\varrho_w : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ the composite of ϱ and $w : k \hookrightarrow \mathbb{C}$. Then ϱ_w is also rigid and thus underlies a $\mathbb{C}$-VHS. It follows that for the product representation

$$\prod_{w \in \mathrm{Ar}(k)} \varrho_w : \pi_1(X) \to \prod_{w \in \mathrm{Ar}(k)} \mathrm{GL}_N(\mathbb{C}),$$

its image Γ is discrete.

Let $\mathscr{D}$ be the period domain associated with the $\mathbb{C}$-VHS of $\sigma := \prod_{w \in \mathrm{Ar}(k)} \varrho_w$. Since Γ acts discretely on $\mathscr{D}$, the quotient $\mathscr{D}/\Gamma$ is a complex space. Let $p : X \to \mathscr{D}/\Gamma$ be the period map. As we assume that ϱ is big, the representation σ is also big. By Lemma 2.28, we have $\dim X = \dim p(X)$. Applying Theorem 1.27, we conclude that X is pseudo-Picard hyperbolic, and is strongly of log general type by [**BC20**,**CD21**].

Case 1.2. Assume that there exists a non-archimedean place v of k such that the composite $\varrho_v : \pi_1(X) \to G(k_v)$, obtained from ϱ via the embedding $k \hookrightarrow k_v$, is

unbounded. Note that $\varrho_v(\pi_1(X))$ is Zariski dense in G. Since ϱ is big, the representation ϱ_v is also big. Therefore, the assumptions of Theorem 3.4 are satisfied, and the theorem follows.

Case 2: ϱ is non-rigid. In the previous work like [**CS08**, **Eys04**], the authors constructed unbounded representations using curves in character varieties in positive characteristic (after taking reduction mod p). Note that this is quite natural in positive characteristic representation as we have seen in Theorem 2.37. However, once we made some reduction mod p arguments, these unbounded representations might not be Zariski dense in G nor big (hence we cannot apply Theorem 3.4). In [**CDY25b**] we introduce a completely new method to construct unbounded representations and avoid reduction mod p.

Claim 3.8 ([**CDY25b**, Proposition 6.1]). *If $\varrho : \pi_1(X) \to G(\mathbb{C})$ is non-rigid, then there exists a big, Zariski dense, and unbounded representation $\varrho' : \pi_1(X) \to G(K)$, where K is a finite extension of some $\mathbb{Q}_p$ with p prime.*

The idea of the proof of Claim 3.8 is roughly that, for the set of bounded representations R in the representation variety $R_B(\pi_1(X), G)(K)$, its image in the character variety $M_B(\pi_1(X), G)(K)$ is compact. Since ϱ is non-rigid and $M_B(\pi_1(X), G)$ is affine, the geometric connected component of $M_B(\pi_1(X), G)$ containing ϱ is non-compact. Hence there exists some unbounded representation. Moreover, since Zariski density of a representation into an almost simple algebraic group is a Zariski open condition, we may assume that such an unbounded representation is Zariski dense. To ensure that it is big, some additional work is required; we refer the reader to [**CDY25b**, Proposition 6.1] for further details.

Since ϱ is non-rigid, by Claim 3.8 we can construct a big, Zariski dense, and unbounded representation $\varrho' : \pi_1(X) \to G(K)$, where K is some non-archimedean local field. We then apply Theorem 3.4 to conclude the theorem.

3.4. On the generalized Green-Griffiths-Lang conjecture.

PROOF OF THEOREM 1.37 (SKETCH). **Case 1:** $\operatorname{char} K = 0$. We may assume that $K = \mathbb{C}$. Let G be the Zariski closure of ϱ, which is a complex reductive group as we assume that ϱ is reductive. We may assume that G is connected after we replace X by a finite étale cover. Let $\mathcal{D}G$ be the derived group of G and let $R(G)$ be the radical of G. Define $G_1 := G/R(G)$ which is semisimple and $G_2 := G/\mathcal{D}G$ which is a torus. Then $G \to G_1 \times G_2$ is an isogeny.

Consider the representation $\sigma : \pi_1(X) \to G_1(\mathbb{C})$ by composing ϱ with the quotient $G \to G_1$. Then σ is Zariski dense. One can show that, after replacing X by a finite étale cover and a birational proper modification, there exists a dominant morphism $f : X \to Y$ with connected general fibers, and a big and Zariski dense representation $\tau : \pi_1(Y) \to G(K)$ such that $f^*\tau = \sigma$ (cf. [**CDY25b**, Proposition 2.5]). Therefore, by Theorem 1.36, we conclude that Y is pseudo Picard hyperbolic and strongly of log general type, if it is not a point.

Consider the morphism $(f, \operatorname{alb}_X) : X \to Y \times A$, where $\operatorname{alb}_X : X \to A$ denotes the quasi-Albanese map of X. Since ϱ is big, we can show that $g := (f, \operatorname{alb}_X)$ is generically finite into its image. Hence we apply Theorem 3.3 to conclude Theorem 1.37.

Case 2: $\operatorname{char} K = p > 0$.

Let $\overline{X}$ be a smooth projective compactification of X such that $D := \overline{X} \setminus X$ is a simple normal crossing divisor. By the same arguments as in Theorem 2.37,

together with the assumption that ϱ is big, we can show that there exist unbounded representations

$$\{\tau_i : \pi_1(X) \to \mathrm{GL}_N(K_i)\}_{i=1,\ldots,k},$$

where each K_i is a finite extension of $\mathbb{F}_{q_i}((t))$ with $q_i = p^{n_i}$ for some $n_i \in \mathbb{N}$, such that, for the Katzarkov–Eyssidieux reduction maps $s_{\tau_i} : X \to S_{\tau_i}$ associated with τ_i, the product map

$$(s_{\tau_1}, \ldots, s_{\tau_k}) : X \longrightarrow S_{\tau_1} \times \cdots \times S_{\tau_k}$$

is generically finite onto its image.

By Proposition 2.25, there exists a spectral covering $\pi : X^{\mathrm{sp}} \to X$ of Galois group H such that

(a) there exists (spectral) forms $\{\eta_1, \ldots, \eta_m\} \subset H^0(\overline{X^{\mathrm{sp}}}, \pi^* \Omega_{\overline{X}}(\log D))$ associated with $\tau_1, \ldots, \tau_k$, which are invariant under H;
(b) π is étale outside

$$(3.8.1) \qquad R := \{x \in \overline{X^{\mathrm{sp}}} \mid \exists \eta_i \neq \eta_j \text{ with } (\eta_i - \eta_j)(x) = 0\}$$

(c) There exists a morphism $a : X^{\mathrm{sp}} \to A$ to a semi-abelian variety A with H acting on A such that a is H-equivariant.
(d) The quasi-Stein factorization of the quotient $X \to A/H$ of a by H, coincides with the quasi-Stein factorization of $(s_{\tau_1}, \ldots, s_{\tau_k})$.

Therefore, we have $\dim X^{\mathrm{sp}} = \dim a(X^{\mathrm{sp}})$.

Assume that X is of log general type. We will use notions of Nevanlinna theory in § 3.1. For any holomorphic map $f : \mathbb{C}_{>\delta} \to X$ whose image is not contained in $\pi(R)$, there exists a proper surjective holomorphic map $p : Y \to \mathbb{C}_{>\delta}$ from a connected Riemann surface Y to $\mathbb{C}_{>\delta}$ and a holomorphic map $g : Y \to X^{\mathrm{sp}}$ satisfying the following diagram:

$$(3.8.2) \qquad \begin{array}{ccc} Y & \xrightarrow{\ g\ } & X^{\mathrm{sp}} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\ \mathbb{C}_{>\delta} & \xrightarrow{\ f\ } & X \end{array}$$

By Claim 3.6, there exists a proper Zariski closed subset $\Xi \subsetneq X$ such that for any holomorphic map $f : \mathbb{C}_{>\delta} \to X$ whose image not contained in Ξ, one has

$$N_{\mathrm{ram}\,p}(r) = o(T_g(r, L)) + O(\log r)\|,$$

where L is an ample line bundle on $\overline{X^{\mathrm{sp}}}$ and $T_g(r, L)$ is the Nevanlinna order function. Note that X^{sp} of log general type as we assume that X is of log general type and $\pi : X^{\mathrm{sp}} \to X$ is a Galois cover. We apply Theorem 3.1 to conclude that f has no essential singularity at the origin, which implies that X is pseudo Picard hyperbolic.

Assume that X is pseudo Brody hyperbolic. Then X^{sp} is also pseudo Brody hyperbolic, and by applying Theorem 3.3 with S being a point, we conclude that X^{sp} is of log general type. We then use exactly the same arguments as in the proof of Theorem 3.4 to spread the positivity of X^{sp} to X, relying on Theorem 3.7 to show that X is of log general type. $\qquad \square$

3.5. Proof of Theorem 1.36 when $\operatorname{char} K = p > 0$ **(sketch).** We will still maintain the same notations as introduced in the proof of Theorem 1.37. Let $\pi : X^{\mathrm{sp}} \to X$ be the Galois covering defined therein. Consider the representation $\pi^* \varrho : \pi_1(X^{\mathrm{sp}}) \to G(K)$, which is Zariski dense. By the proof of Theorem 1.37, there exists a morphism $a : X^{\mathrm{sp}} \to A$ where A is a semiabelian variety such that $\dim X^{\mathrm{sp}} = \dim a(X^{\mathrm{sp}})$. Hence we have $\bar{\kappa}(X^{\mathrm{sp}}) \geq 0$. By the same arguments as in the proof of Theorem 3.4, one can show that X^{sp} is of log general type.

We now use Claim 3.6 together with Theorem 3.1 to conclude that X is pseudo Picard hyperbolic.

4. Topology of algebraic varieties in the presence of a big local system

In this section, we outline the proofs of the theorems stated in § 1.2, using the techniques in § 2, following a beautiful strategy initiated by Arapura–Wang [**AW25**].

4.1. Proof of linear Chern-Hopf-Thurston conjecture. In this subsection, we skech the idea of the proof of Theorem 1.8 when ϱ is semisimple and large.

For any perverse sheaf $\mathcal{P}$, its characteristic cycle is

$$CC(\mathcal{P}) = \sum_{i=1}^{m} n_i T_{Z_i}^* X,$$

where $n_i \in \mathbb{N}$, Z_i is an irreducible subvariety of X, and $T_{Z_i}^* X$ denotes the conormal bundle of Z_i in the cotangent bundle $T^* X$. Each $T_{Z_i}^* X$ is a conic Lagrangian cycle of $T^* X$. The crucial idea, initiated by Arapura and Wang [**AW25**], is the following formula:

$$(4.0.1) \qquad\qquad \chi(X, \mathcal{P}) = CC(\mathcal{P}) \cdot T_X^* X,$$

where $T_X^* X$ denotes the conormal bundle of X, namely the zero section of the cotangent bundle $T^* X \to X$. Therefore, in order to prove Theorem 1.8, it suffices to show that

THEOREM 4.1. *Let X be a smooth projective variety and let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a large representation, for any field K. Then for any closed subvariety Z of X, we always have*

$$T_Z^* X \cdot T_X^* X \geq 0$$

for any closed subvariety $Z \subset X$.

Although the cotangent bundle $T^* X$ is non-compact, the above intersection number is well defined since the zero section $T_X^* X$ is compact.

We begin with a preliminary observation regarding the strategy employed in the proof of Theorem 1.8.

Lemma 4.2. *Let X be a smooth projective n-fold. If there exists a holomorphic 1-form η on X such that its zero locus $Z(\eta) := (\eta = 0)$ is zero-dimensional, then*

$$(-1)^n \chi(X) \geq 0.$$

PROOF. Consider the graph Γ of the section η of $T^* X \to X$, which is a closed subvariety of the total space $T^* X$. Let $T_X^* X$ denote the zero section of $T^* X$. The

graph Γ is homologous to T_X^*X. Hence, computing the self-intersection of the zero section, we have

$$(-1)^n \chi(X) = T_X^*X \cdot T_X^*X = \Gamma \cdot T_X^*X.$$

Since $Z(\eta)$ is zero-dimensional, the intersection of Γ and T_X^*X is proper. Therefore, we have

$$\Gamma \cdot T_X^*X \geq 0.$$

$\square$

However, in general, we cannot guarantee the existence of holomorphic 1-forms on X in Theorem 4.1. Moreover, even if such holomorphic 1-forms exist, their zero loci need not consist of isolated points. Consequently, the cycles in T^*X used to compute the intersection—specifically the graph of the 1-form and the zero section—may not intersect properly.

To address the first issue, we use a *multivalued 1-form* (see Definition 2.22) instead and adapt the intersection theory from Lemma 4.2. To resolve the second issue, we construct a procedure that deforms the intersection progressively to achieve properness.

More precisely, using techniques analogous to deformation to the normal cone in intersection theory, one can associate to a d-valued 1-form η a map Φ_η from conic Lagrangian cycles to conic Lagrangian cycles such that, for any subvariety $Z \subset X$, one has

$$(4.2.1) \qquad \Phi_\eta(T_Z^*X) = n_0\, T_Z^*X + \sum_{1 \leq i \leq m} n_i\, T_{Z_i}^*X,$$

where n_0 is the multiplicity of the zero form in the restriction $\eta|_{Z_{\text{reg}}}$, and each Z_i is a proper closed subvariety of Z. In particular, if $\eta|_{Z_{\text{reg}}}$ is nontrivial, then $n_0 < d$. Moreover, we have

$$\Phi_\eta(T_Z^*X) \cdot T_X^*X = d T_Z^*X \cdot T_X^*X.$$

Therefore, if $\eta|_Z \not\equiv 0$, we have

$$(4.2.2) \qquad (d - n_0)T_Z^*X \cdot T_X^*X = \sum_{1 \leq i \leq m} n_i T_{Z_i}^*X \cdot T_X^*X,$$

where $Z_i \subsetneq Z$ is a proper subvariety of Z. Roughly speaking, we have

$$\dim Z = \dim(T_Z^*X \cap T_X^*X) > \dim Z_i = \dim(T_{Z_i}^*X \cap T_X^*X),$$

implying that the intersection becomes more "proper". We omit the precise definition of the map Φ_η here and instead refer the interested reader to [**DW24b**, §3.1] for further details.

PROOF OF THEOREM 4.1 (SKETCH). Case 1: char $K = p > 0$.

In Theorem 2.37, it is proved that for any linear $\sigma : \pi_1(X) \to \mathrm{GL}_N(K)$, then there are representations $\tau_i : \pi_1(X) \to \mathrm{GL}_N(K_i)$, where K_i is non-archimedean local field of characteristic p (i.e. K_i is a finite extension of $\mathbb{F}_p((t))$), such that the Shafarevich morphism of σ is the Stein factorization of

$$s_{\tau_1} \times \cdots \times s_{\tau_k} : X \to S_{\tau_1} \times \cdots \times S_{\tau_k}.$$

Therefore, when ϱ is large, its Shafarevich morphism is just the identity map, and thus for any Z, there exists some τ_i such that $s_{\tau_i}(Z)$ is not a point. By the property of the Katzarkov-Eyssidieux reduction map in Theorem 2.26, there multivalued 1-form $\eta_{\tau_i}|_Z \not\equiv 0$.

Now we perform the iteration of the above algorithm, to achieve that $\dim Z_i = 0$ for each i in right-hand side of (4.2.2). The intersection of $T^*_{Z_i} X$ and $T^*_X X$ is proper, and we have

$$T^*_Z X \cdot T^*_X X \geq 0$$

for each Z.

Case 2: $K = \mathbb{C}$ and ϱ is semisimple.

Consider the map $s_{\mathrm{fac}} : X \to S_{\mathrm{Fac}}(X)$ defined in Definition 2.31. By Lemma 2.30, there exist reductive representations $\{\tau_i : \pi_1(X) \to \mathrm{GL}_N(K_i)\}_{i=1,\ldots,m}$, where each K_i is a non-archimedean local field of characteristic zero, such that $s_{\mathrm{fac}} : X \to S_{\mathrm{fac}}$ is the Stein factorization of

$$s_{\tau_1} \times \cdots \times s_{\tau_m} : X \to S_{\tau_1} \times \cdots \times S_{\tau_m}.$$

Here $s_{\tau_i} : X \to S_{\tau_i}$ is the Katzarkov-Eyssidieux reduction map associated with τ_i.

Fix a closed subvariety $Z \subseteq X$. Assume that $s_{\mathrm{fac}}(Z)$ is not a point. Then $s_{\tau_i}(Z)$ is not a point for some i. By Theorem 2.26, this implies that $\eta_{\tau_i}|_Z \not\equiv 0$.

We employ the same strategy, using the maps $\Phi_{\eta_{\tau_1}}, \ldots, \Phi_{\eta_{\tau_m}}$ defined above, to make the intersection of $T^*_X X$ with $T^*_Z X$ sufficiently proper. In this situation, when the algorithm introduced above terminates, it means that each subvariety Z_i appearing on the right-hand side of (4.2.2) is contained in some fiber of S_{fac}. Unlike the case of positive characteristic, one may have $\dim Z_i > 0$.

We now sketch the arguments of [**DW24b**] to treat this case; the following discussion is intended as an outline rather than a rigorous proof.

Let $\mathcal{L}$ be the $\mathbb{C}$-VHS on X constructed in Proposition 2.33. For simplicity, we assume that Z_i is smooth. By Proposition 2.33, the restriction $\varrho|_{\pi_1(Z_i)}$ corresponds to a direct summand of the local system underlying $\mathcal{L}$, which has discrete monodromy. If ϱ is large and semisimple, the period map of $\mathcal{L}|_{Z_i}$ is finite by the arguments in Lemma 2.28. Consequently, certain curvature properties of the period domain imply that T^*Z_i is "almost" nef. As observed in [**AW25**], a theorem of Demailly–Peternell–Schneider [**DPS94**] then implies that $T^*_{Z_i} X \cdot T^*_X X \geq 0$. This concludes the proof for this case.

Case 3: Suppose $K = \mathbb{C}$ and ϱ is linear. The proof relies on the *tautological* variation of mixed Hodge structures introduced in [**ES11**], combined with arguments used for the linear Shafarevich conjecture in [**EKPR12**]. Due to the technical complexity of this construction, we omit the details here. □

4.2. Proof of the linear Kollár's conjecture. In this subsection, we sketch the proof of Theorem 1.12.(i) assuming that ϱ is semisimple and big.

As we see before, other two items in Theorem 1.12 follow from Theorem 1.12.(i). So it suffices to prove Theorem 1.12.(i). By the byproducts in the proof of the reductive Shafarevich conjecture [**Eys04**,**DYK23**], as we remarked in Remark 2.51, there exists

(1) a family of Zariski dense representations $\{\tau_i : \pi_1(X) \to G_i(K_i)\}_{i=1,\ldots,\ell}$, where each G_i is a reductive group over a non-archimedean local field K_i of characteristic zero;

(2) a $\mathbb{C}$-VHS $\mathcal{L}$ on X,

such that

$$\Phi := T_{\tau_1} + \cdots + T_{\tau_\ell} + \sqrt{-1}\,\mathrm{tr}(\theta \wedge \theta^*)$$

is a closed positive $(1,1)$-current on X, which is smooth and strictly positive over a non-empty analytic open subset $X^\circ \subset X$.

Here:

- T_{τ_i} denotes the canonical current on X associated with τ_i, as defined in Definition 2.23;
- θ is the Higgs field of the Hodge bundle associated with $\mathcal{L}$, and θ^* is its adjoint with respect to the Hodge metric. □

Let β_{τ_i} be the 1-form on $\widetilde{X}$ associated with τ_i defined in (2.44.1), and $\beta_{\mathcal{L}}$ be the 1-form on $\widetilde{X}$ associated with $\mathcal{L}$ defined in (2.45.1). Define $\beta := \beta_{\mathcal{L}} + \sum_{i=1}^{\ell} \beta_{\tau_i}$. Then β has L^1_{loc}-coefficients. By Lemma 2.44 and eq. (2.45.1), it satisfies

$$(4.2.3) \qquad |\beta(x)| \leq_{\mathrm{a.e.}} C(1 + d_{\widetilde{X}}(x, x_0))$$

for some constant $C > 0$, and

$$(4.2.4) \qquad d\beta \geq \pi_X^*\left(\sum_{i=1}^{\ell} T_{\tau_i} + \omega_{\mathcal{L}}\right) =: \pi_X^*\Phi.$$

Assume, for contradiction, that there exists a holomorphic $(p,0)$-form α on $\widetilde{X}$ which is L^2 with respect to ω, for some $0 \leq p \leq n-1$. A crucial step in [**DW24a**, Theorem 2.2] is that the sublinear growth condition (4.2.3) implies that a suitable "Stokes formula" holds, namely

$$\int_{\widetilde{X}} \sqrt{-1}^{p^2} d\beta \wedge \alpha \wedge \bar{\alpha} \wedge \omega^{n-p-1} = 0.$$

Since $(\sqrt{-1})^{p^2} \alpha \wedge \bar{\alpha}$ is a semipositive (p,p)-form, it follows from (4.2.4) that

$$(4.2.5) \quad 0 = \int_{\widetilde{X}} \sqrt{-1}^{p^2} d\beta \wedge \alpha \wedge \bar{\alpha} \wedge \omega^{n-p-1} \geq \int_{\widetilde{X}} \sqrt{-1}^{p^2} \alpha \wedge \bar{\alpha} \wedge \omega^{n-p-1} \wedge \pi_X^*\Phi \geq 0.$$

Since Φ is smooth and strictly positive over a non-empty analytic open subset $X^\circ \subset X$. One can show that $\alpha(x) = 0$ for any $x \in \pi_X^{-1}(X^\circ)$. Since α is holomorphic, it follows that $\alpha \equiv 0$. This proves that $H^{(p,0)}_{(2)}(\widetilde{X}) = 0$ for $p < n$. By the L^2-Lefschetz theorem [**Gro91**], the map

$$H^{(p,0)}_{(2)}(\widetilde{X}) \overset{\pi_X^* \omega^{n-p} \wedge}{\to} H^{(n,n-p)}_{(2)}(\widetilde{X})$$

is isomorphic. Hence $H^{(n,q)}_{(2)}(\widetilde{X}) = 0$ for $q > 0$. The theorem is proved when ϱ is semisimple and big.

When the representation ϱ is linear and big in the sense of [**DW24a**], the proof of the theorem relies on results regarding the linear Shafarevich conjecture established in [**EKPR12**]. Additionally, we must establish a more involved vanishing theorem than the one presented above (cf. [**DW24a**, Theorem 2.2]).

4.3. Deformation of big fundamental groups. Let (X, ω) be a compact Kähler manifold, and let $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ be a reductive representation. By Theorem 2.7, there exists a harmonic bundle (E, θ, h) whose associated flat connection $\nabla_h + \theta + \theta^*$ has monodromy representation ϱ. Here ∇_h denotes the Chern connection of (E, h), and θ^* is the adjoint of θ with respect to h. It is a standard exercise to check that the $(1,1)$-form $\sqrt{-1}\,\mathrm{tr}(\theta \wedge \theta^*)$ is real, semipositive, and closed. Moreover, it does not depend on the choice of the Kähler metric ω.

Definition 4.3 (Canonical form). Such $(1,1)$-form $\omega_\varrho := \sqrt{-1}\mathrm{tr}(\theta \wedge \theta^*)$ is called the *canonical form* associated with ϱ.

In [**DMW24**], we first establish the deformation smoothness of such canonical forms. Our main result is as follows.

THEOREM 4.4 ([**DMW24**, Remark 7.8]). *Let $\mathcal{X}$ be a Kähler manifold, and let $f : \mathcal{X} \to \mathbb{D}$ be a proper holomorphic fibration over the unit disk $\mathbb{D}$. For $t \in \mathbb{D}$, write $X_t := f^{-1}(t)$. Let $\varrho : \pi_1(X_0) \to \mathrm{GL}_N(\mathbb{C})$ be a reductive representation, and define*

$$(4.4.1) \qquad \varrho_t : \pi_1(X_t) \xrightarrow{\simeq} \pi_1(\mathcal{X}) \xrightarrow{\simeq} \pi_1(X_0) \xrightarrow{\varrho} \mathrm{GL}_N(\mathbb{C}).$$

Then the fiberwise defined canonical $(1,1)$-form ω_{ϱ_t} on X_t, associated with the representation ϱ_t, varies smoothly with respect to $t \in \mathbb{D}$.

Another main technical result, and also main difficult result, is the following consequence of the non-archimedean analogue of Theorem 4.4.

THEOREM 4.5 ([**DMW24**, Lemma 5.2]). *Let $f : \mathcal{X} \to \mathbb{D}$ be a smooth projective family of relative dimension n over the unit disk $\mathbb{D}$, and let $\varrho : \pi_1(X_0) \to G(K)$ be a Zariski-dense representation, where G is a reductive algebraic group over a non-archimedean local field K. For $t \in \mathbb{D}$, define*

$$(4.5.1) \qquad \varrho_t : \pi_1(X_t) \xrightarrow{\simeq} \pi_1(\mathcal{X}) \xrightarrow{\simeq} \pi_1(X_0) \xrightarrow{\varrho} G(K).$$

There exists a full-measure open subset $X_0^\circ \subset X_0$ such that for any $x_0 \in X_0^\circ$ there exists a coordinate system

$$(\Omega; z_1, \ldots, z_n, t; \varphi)$$

on $\mathcal{X}$ centered at x_0, together with a real $(1,1)$-form

$$T(z,t) = \sqrt{-1} \sum_{i,j} a_{ij}(z,t)\, dz_i \wedge d\bar{z}_j$$

on $\mathbb{D}^n$ (with $z = (z_1, \ldots, z_n)$) satisfying

- (i) *the map $\varphi : \mathbb{D}^n \times \mathbb{D}_\varepsilon \to \Omega$ is a biholomorphism, with $f \circ \varphi(z_1, \ldots, z_n, t) = t$.*
- (ii) *the coefficents $a_{ij}(z,t)$ are continuous function on $\mathbb{D}^n \times \mathbb{D}_\varepsilon$.*
- (iii) *For each fixed $t \in \mathbb{D}_\varepsilon$, $T_t(z) := T(z,t)$ is a smooth semi-positive closed $(1,1)$-form on $\mathbb{D}^n$.*
- (iv) *For each $t \in \mathbb{D}_\varepsilon$, one has $T_{\varrho_t}|_{\Omega \cap X_t} \geq T_t$.*
- (v) *$T_{\varrho_0}|_{\Omega \cap X_0} = T_0$.*

Here T_{ϱ_t} is the canonical current defined in Definition 2.23, associated with ϱ_t.

The proof of Theorem 4.5 relies heavily on the theory of harmonic maps into Euclidean buildings (and more generally, NPC spaces) as developed by Gromov–Schoen [**GS92**], Korevaar–Schoen [**KS93**, **KS97**] and later in [**BDDM22**]. The argument is quite involved; we refer the interested reader to [**DMW24**] for a more detailed exposition.

Let us explain the idea of the proof of Theorem 1.18 in the case where ϱ is reductive and big. Set $X := X_0 := f^{-1}(0)$. By Remark 2.51, there exist reductive representations

$$\{\tau_i : \pi_1(X) \to \mathrm{GL}_N(K_i)\}_{i=1,\ldots,\ell},$$

where each K_i is a non-archimedean local field of characteristic zero, together with a $\mathbb{C}$-VHS σ, such that the sum

$$\sum_{i=1}^{\ell} T_{\tau_i} + \omega_\sigma$$

satisfies the following properties:

- it is a closed positive $(1,1)$-current on X with continuous potential;
- it is smooth on a Zariski open subset $X^\circ \subset X$;
- its cohomology class is big and nef.

Here T_{τ_i} denotes the canonical current associated with τ_i, and ω_σ is the canonical form defined in Definition 4.3 associated with the monodromy representation $\sigma :$ $\pi_1(X) \to \mathrm{GL}_{N'}(\mathbb{C})$ of the $\mathbb{C}$-VHS $\mathcal{L}$. By Boucksom's criterion [**Bou02**], the current

$$\sum_{i=1}^{\ell} T_{\tau_i} + \omega_\sigma$$

is smooth and strictly positive on some analytic open subset $U \subset X^\circ$.

By Theorems 4.4 and 4.5, for t sufficiently small, the sum

$$(4.5.2) \qquad \sum_{i=1}^{\ell} T_{\tau_{i,t}} + \omega_{\sigma_t}$$

is strictly positive on some analytic open subset of X_t. Here $\tau_{i,t} : \pi_1(X_t) \to G(K_i)$ and $\sigma_t : \pi_1(X_t) \to \mathrm{GL}_{N'}(\mathbb{C})$ are the representations induced by τ_i and σ, respectively, as defined in (4.4.1) and (4.5.1).

On the other hand, for any subvariety $Z \subset X_t$, if $\mathrm{Im}\big[\pi_1(Z) \to \pi_1(X_t)\big]$ is finite, then the restrictions $T_{\tau_{i,t}}|_Z$ and $\omega_{\sigma_t}|_Z$ are both trivial. This follows from the functoriality of harmonic maps under pullback established in Theorem 2.20, together with the construction of canonical forms and canonical currents. Consequently, the reductive Shafarevich morphism of X_t, whose existence is guaranteed by Theorem 2.40, must be birational for t sufficiently small. This shows that X_t has a big fundamental group for all sufficiently small t.

In the general case where ϱ is linear and big, we apply the techniques developed in [**EKPR12**], together with analytic results on the variation of mixed Hodge structures, to prove Theorem 1.18. We omit the details here and refer the interested reader to [**DMW24**].

4.4. Applications to hyperbolicity. I now give an application of Theorem 1.18 to the hyperbolicity of algebraic varieties under deformation, combining Theorems 1.36 and 1.37. First, we recall the following classical result on the *openness* of Brody hyperbolicity.

THEOREM 4.6 ([**Dem20**, Proposition 1.10]). *Let $f : \mathcal{X} \to \mathbb{D}$ be a holomorphic proper submersion from a complex manifold $\mathcal{X}$ to the unit disk with connected fibers. If X_0 is Brody hyperbolic, then there exists $\varepsilon > 0$ such that X_t is Brody hyperbolic for $|t| < \varepsilon$.*

There has long been a folklore conjecture that such openness properties hold for pseudo Brody hyperbolicity.

Conjecture 4.7. *Let $f : \mathcal{X} \to \mathbb{D}$ be as in Theorem 4.6. If X_0 is pseudo Brody hyperbolic, then X_t is also pseudo Brody hyperbolic for sufficiently small t.*

This conjecture is indeed a consequence of Conjecture 1.23. Let us explain how to deduce Conjecture 4.7 from Conjecture 1.23. Assume that X_0 is pseudo Brody hyperbolic. Then X_0 is of general type by Conjecture 1.23. By Siu's invariance of plurigenera [**Siu98**, **Pau07**], it follows that X_t is of general type for any $t \in \mathbb{D}$. Applying Conjecture 1.23 again, we conclude that X_t is pseudo Brody hyperbolic. This proves Conjecture 4.7.

THEOREM 4.8 ([**DMW24**, Theorem D]). *Let $f : \mathcal{X} \to \mathbb{D}$ be a smooth projective family. Assume that there is a big and reductive representation $\varrho : \pi_1(X_0) \to \mathrm{GL}_N(\mathbb{C})$. If X_0 is pseudo Brody hyperbolic, then X_t is pseudo Picard hyperbolic for sufficiently small t.*

PROOF. We apply Theorem 1.37 to conclude that X_0 is of general type. By Siu's invariance of plurigenera [**Siu98**, **Pau07**], X_t is of general type for any $t \in \mathbb{D}$.

By [**DYK23**, Lemma 3.25], there exists another reductive representation $\tau : \pi_1(X_0) \to \mathrm{GL}_N(\mathbb{C})$ such that for any reductive representation $\sigma : \pi_1(X_0) \to \mathrm{GL}_N(\mathbb{C})$, we have

(4.8.1) $\ker \tau \subset \ker \sigma$.

Let $\tau_t : \pi_1(X_t) \to \mathrm{GL}_N(\mathbb{C})$ be the composite map of τ and the natural isomorphism $\pi_1(X_t) \to \pi_1(X_0)$ induced by f.

Claim 4.9. *There exists $\varepsilon > 0$ such that τ_t is a big and reductive representation for any $t \in \mathbb{D}_\varepsilon$.*

PROOF. We use the same notation as in the above subsection. By Theorem 2.40, the Shafarevich morphism

$$\mathrm{sh}_{\tau_t} : X_t \to \mathrm{Sh}_{\tau_t}(X_t)$$

exists. It suffices to prove that this morphism is birational for t sufficiently close to 0.

Let

$$\{\tau_i : \pi_1(X) \to \mathrm{GL}_N(K_i)\}_{i=1,\dots,\ell} \quad \text{and} \quad \sigma : \pi_1(X) \to \mathrm{GL}_{N'}(\mathbb{C})$$

be the reductive representations appearing in the above proof of Theorem 1.18. Let Z be a fiber of sh_{τ_t}. By (4.8.1), we have

$$\tau_{i,t}\big(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X_t)]\big)$$

finite for each i.

Recall that in Proposition 2.33, the $\mathbb{C}$-VHS σ is a direct sum of reductive representations of $\pi_1(X)$ into $\mathrm{GL}_N(\mathbb{C})$. By (4.8.1), it follows that

$$\sigma_t\big(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X_t)]\big)$$

is also finite. Consequently, for the closed positive $(1,1)$-current

$$\sum_{i=1}^{\ell} T_{\tau_i,t} + \omega_{\sigma,t}$$

defined in (4.5.2), its restriction to Z is trivial.

By the same arguments as in the proof of Theorem 1.18, we conclude that the morphism $\mathrm{sh}_{\tau_t} : X_t \to \mathrm{Sh}_{\tau_t}(X_t)$ is birational for t sufficiently small.   □

Since X_t is of general type, it follows from Claim 4.9 and Theorem 1.37 that X_t is pseudo Picard hyperbolic for any $t \in \mathbb{D}_\varepsilon$.   □

Corollary 4.10 ([**DMW24**, Corollary E]). *Let X be a smooth projective variety. Assume that either*

- (a) *there is $\mathbb{C}$-VHS for short $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ with discrete monodromy Γ such that the period map $X \to \mathscr{D}/\Gamma$ is generically finite onto the image, or*
- (b) *there is a big representation $\tau : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ such that the Zariski closure of $\tau(\pi_1(X))$ is a semisimple algebraic group.*

Then every small projective deformation of X is pseudo Picard hyperbolic.

PROOF. In Case (a), by the work of Griffiths-Schmid [**GS69**], the period domain $\mathscr{D}$ is equipped with a natural metric that has negative holomorphic sectional curvature along the horizontal direction. Since the period map of ϱ is assumed to be generically finite onto its image, it follows from the Ahlfors-Schwarz lemma that X is pseudo Brody hyperbolic.

On the other hand, by Lemma 2.28, ϱ is big. By Theorem 2.7, ϱ is also reductive. Hence, the conditions in Theorem 4.8 are fulfilled, allowing us to conclude that a small projective deformation of X is pseudo Picard hyperbolic.

In Case (b), by Theorem 1.36, X is pseudo Brody hyperbolic. We apply Theorem 4.8 to conclude that a small deformation of X is pseudo Picard hyperbolic.   □

5. Some further applications

In this section, we present several applications of our hyperbolicity results in Theorem 1.36.

5.1. Special and h-special varieties. We first recall the definition of special varieties by Campana [**Cam04**, **Cam11**].

Definition 5.1 (Campana's specialness). Let X be a quasi-projective normal variety.

- (i) X is *weakly special* if for any finite étale cover $\widehat{X} \to X$ and any proper birational modification $\widehat{X}' \to \widehat{X}$, there exists no dominant morphism $\widehat{X}' \to Y$ with connected general fibers such that Y is a positive-dimensional quasi-projective variety of log general type.
- (ii) X is *special* if for any proper birational modification $X' \to X$ there is no dominant morphism $X' \to Y$ to with connected general fibers over a positive-dimensional quasi-projective variety Y such that the *Campana orbifold base* (or simply orbifold base) is of log general type.
- (iii) X is *Brody special* if it contains a Zariski dense entire curve.

Campana defined X to be *H-special* if X has vanishing Kobayashi pseudo-distance. Motivated by [**Cam11**, 11.3 (5)], in [**CDY25c**, Definition 2.2] we introduce the following definition.

Definition 5.2 (h-special). Let X be a smooth quasi-projective variety. We define the equivalence relation $x \sim y$ of two points $x, y \in X$ iff there exists a sequence of holomorphic maps $f_1, \ldots, f_l : \mathbb{C} \to X$ such that letting $Z_i \subset X$ to be the Zariski closure of $f_i(\mathbb{C})$, we have

$$x \in Z_1, Z_1 \cap Z_2 \neq \emptyset, \ldots, Z_{l-1} \cap Z_l \neq \emptyset, y \in Z_l.$$

We set $R = \{(x, y) \in X \times X ; x \sim y\}$. We define X to be *hyperbolically special* (*h*-special for short) iff $R \subset X \times X$ is Zariski dense.

By definition, rationally connected projective varieties are *h*-special without referring to a theorem of Campana and Winkelmann [**CW23**], who proved that all rationally connected projective varieties contain Zariski dense entire curves.

In [**Cam04**, **Cam11**], Campana proposed the following tantalizing abelianity conjecture.

Conjecture 5.3 (Campana). *A special smooth quasi-projective variety has virtually abelian fundamental group.*

In [**CDY25c**] we discovered that Conjecture 5.3 fails for non-proper quasi-projective variety.

5.2. Campana's conjecture revisited. In [**CDY25c**, Example 4.26], we constructed a smooth quasi-projective variety such that it is both special and Brody special, yet it has nilpotent fundamental group that is not virtually abelian. Later, Aguilar-Campana [**AAC25**] also gave another simpler construction of such examples. We first recall the following definition.

Definition 5.4 (nilpotent group). A group G is *nilpotent* if it has a central series of finite length. That is, a series of normal subgroups

$$\{1\} = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

such that $G_{i+1}/G_i \leq Z(G/G_i)$. For a nilpotent group G, the smallest n such that G has a central series of length n is called the nilpotency class of G; and G is said to be *nilpotent of class n*.

The following example is given in [**AAC25**].

Example 5.5. Let L be a holomorphic line bundle L over an elliptic curve B such that $c_1(L) \neq 0$. Let X be L^*, that is the complement of the zero and infinity sections of $\mathbb{P}(L \oplus \mathcal{O}_B) \to X$. Then it is a $\mathbb{C}^*$-fibration over the elliptic curve. By the Gysin sequence, we have

(5.5.1)
$$0 \to H^1(B, \mathbb{Z}) \xrightarrow{\pi^*} H^1(X, \mathbb{Z}) \to H^0(B, \mathbb{Z}) \xrightarrow{\cdot c_1(L)} H^2(B, \mathbb{Z}) \to H^2(X, \mathbb{Z}) \to H^1(B, \mathbb{Z}) \to \cdots$$

where $\pi : X \to B$ is the projection map. Therefore, if $c_1(L) \neq 0$,

$$\pi^* : H^1(B, \mathbb{Z}) \to H^1(X, \mathbb{Z})$$

is an isomorphism. It follows that $\pi_1(X)$ is then a central extension of $\pi_1(B)$ by $\mathbb{Z}$, hence is torsionfree, and nilpotent of class 2.

Consequently, in the quasi-projective setting, we revised Conjecture 5.3 as follows.

Conjecture 5.6. *A special or h-special smooth quasi-projective variety has virtually nilpotent fundamental group.*

We propose the following stronger conjecture. Similar questions were also independently asked in [**AAC25**, **Rog25**].

Conjecture 5.7. *Let X be a smooth quasi-projective variety that is either special or h-special. Then its fundamental group $\pi_1(X)$ is virtually nilpotent of class 2.*

Shimoji has proved some interesting results on this conjecture, see [**Shi25**]. Rogov [**Rog25**] also proposed some strategy in proving this conjecture using higher Albanese maps by Hain and o-minimal geometry.

5.3. Nilpotency conjecture in the linear case. In [**CDY25c**, **DY25**], we confirm Conjecture 5.6 for quasi-projective varieties with linear fundamental groups.

THEOREM 5.8. *Let X be a special or h-special smooth quasiprojective variety. Let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a linear representation where K is any field.*

  (i) [**CDY25b**, Theorem A] *If* char $K = 0$, *then the image $\varrho(\pi_1(X))$ is virtually nilpotent.*
  (ii) [**DY25**, Theorem G] *If* char $K > 0$, *then $\varrho(\pi_1(X))$ is virtually abelian.*

By Example 5.5, Theorem 5.8 is shown to be sharp. Surprisingly, in the context of representations in positive characteristic, we obtain a stronger result.

PROOF OF THEOREM 5.8. *Step 1. We prove that $\varrho(\pi_1(X))$ is solvable.* We may assume that K is algebraically closed. Let G be the Zariski closure of $\varrho(\pi_1(X))$. By [**Cam11**], any finite étale cover of a special (resp. h-special) variety is still special (resp. h-special). After replacing X by a finite étale cover, we may assume that G is connected. Let $R(G)$ be the radical of G. Let $H := G/R(G)$, which is semisimple. If $\dim H > 0$, then ϱ induces a Zariski dense representation $\sigma :$ $\pi_1(X) \to H(K)$. We can prove that, after replacing X by a composition of birational modifications and finite étale Galois covers, there exists a a dominant morphism $f : X \to Y$ over a smooth quasi-projective variety Y with connected general fibers, and a big and Zariski dense representation $\tau : \pi_1(Y) \to H(K)$ such that $\sigma = f^*\tau$. By Theorem 1.36, Y is of log general type and pseudo Picard hyperbolic. This leads to a contradiction since X is special (thus weakly special by [**Cam11**]) or h-special. Hence $G = R(G)$.

Step 2. We prove that $\varrho(\pi_1(X))$ is virtually abelian if char $K > 0$. Note that any finite étale cover of a special (resp. h-special) variety is still special (resp. h-special) by [**Cam04**] and [**CDY25c**, Lemma 3.2]. Replacing X by a finite étale cover, we may assume that $\pi_1(X)^{ab} \to \pi_1(A)$ is an isomorphism, where $\pi_1(X)^{ab} :=$ $\pi_1(X)/[\pi_1(X), \pi_1(X)]$. Since X is special or h-special, by [**CDY25c**, Proposition 4.13], the quasi-albanese map $a : X \to A$ of X is π_1-exact, i.e., we have the following exact sequence:
$$\pi_1(F) \to \pi_1(X) \to \pi_1(A) \to 1,$$
where F is a general fiber of a. Hence $[\pi_1(X), \pi_1(X)]$ is the image of $\pi_1(F) \to$ $\pi_1(X)$, which is thus finitely generated. It implies that $[\varrho(\pi_1(X)), \varrho(\pi_1(X))] =$ $\varrho([\pi_1(X), \pi_1(X)])$ is also finitely generated. By Step 1, G is solvable. Hence we have $\mathcal{D}G \subset R_u(G)$, where $R_u(G)$ is the unipotent radical of G and $\mathcal{D}G$ is the the derived group of G. Consequently, we have
$$[\varrho(\pi_1(X)), \varrho(\pi_1(X))] \subset [G(K), G(K)] \subset R_u(G)(K).$$
Note that every subgroup of finite index in $[\pi_1(X), \pi_1(X)]$ is also finitely generated (cf. [**ST00**, Proposition 4.17]. By the same arguments in Lemma 2.36, we conclude that $[\varrho(\pi_1(X)), \varrho(\pi_1(X))]$ is finite. Hence $\varrho(\pi_1(X))$ is virtually abelian.

Step 3. We prove that $\varrho(\pi_1(X))$ is virtually nilpotent if $\operatorname{char} K = 0$. The proof is non-trivial and based on 5.9 below. □

THEOREM 5.9 ([**CDY25b**, Theorem 11.3]). *Let X be a special or h-special quasi-projective manifold. Let G be a connected, solvable algebraic group defined over $\mathbb{C}$. Assume that there exists a Zariski dense representation $\varphi : \pi_1(X) \to G$. Then G is nilpotent. In particular, $\varphi(\pi_1(X))$ is nilpotent.* □

The proof of Theorem 5.9 is involved. It is inspired by [**Cam01**] and is based on the π_1-exactness of the quasi-Albanese morphism mentioned above, together with Deligne's theorem asserting that the radical of the algebraic monodromy group of an admissible variation of mixed Hodge structures is *unipotent*. We refer the reader to [**CDY25c**, § 4] for details of the proof.

In a forthcoming paper, Cao, Hacon, Păun, and the author [**CDHP25a**] develop Hodge theory for local systems over quasi-projective varieties, extending our previous techniques on Hodge theory for rank-one local systems over quasi-compact Kähler manifolds [**CDHP25b**]. As a consequence, we refine Theorem 5.8.(i) by proving that $\varrho(\pi_1(X))$ is nilpotent of class 2. This result establishes Conjecture 5.7 in the case where the fundamental group is linear.

5.4. Algebraic varieties with compactifiable universal coverings. In the work [**CHK13**, **CH13**], Claudon, Höring and Kollár proposed the following intriguing conjecture:

Conjecture 5.10. *Let X be a complex projective manifold with infinite fundamental group $\pi_1(X)$. Suppose that the universal cover $\widetilde{X}$ is quasi-projective. Then after replacing X by a finite étale cover, there exists a locally trivial fibration $X \to A$ with simply connected fiber F onto a complex torus A. In particular we have $\widetilde{X} \simeq F \times \mathbb{C}^{\dim A}$.*

It's worth noting that assuming abundance conjecture, Claudon, Höring and Kollár proved this conjecture in [**CHK13**]. In [**CH13**], Claudon-Höring proved Conjecture 5.10 in the case where $\pi_1(X)$ is virtually abelian.

In this subsection we establish a linear version of Conjecture 5.10 without relying on the abundance conjecture.

THEOREM 5.11 ([**DY25**, Corollary G]). *Conjecture 5.10 holds if there exists a faithful representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$, where K is any field.*

PROOF. By [**CH13**], it suffices to prove that $\pi_1(X)$ is virtually abelian. Consider the core map

$$c_X : X \dashrightarrow Y := C(X)$$

defined by Campana, such that the orbifold base of c_X is of orbifold general type (see [**Cam04**]). By the orbifold version of the Kobayashi-Ochiai theorem [**Cam04**, Theorem 8.2], the composed meromorphic map

$$\widetilde{X} \xrightarrow{\pi} X \xdashrightarrow{c_X} Y$$

extends to a meromorphic map from some projective compactification $\overline{\widetilde{X}}$ of $\widetilde{X}$. This implies that the general fiber F of c_X satisfies that $\pi^{-1}(F)$ has only finitely many connected components. In particular, the induced homomorphism $\pi_1(F) \to \pi_1(X)$ has image of finite index.

According to [**Cam04**], F is a special manifold. Since we assume the existence of a faithful representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$, it follows from Theorem 5.8.(ii) when char $K > 0$ and [**Cam04**, Theorem 7.8] when char $K = 0$ that the image $\varrho([\pi_1(F) \to \pi_1(X)])$ is virtually abelian. As the image of $\pi_1(F) \to \pi_1(X)$ has finite index, $\pi_1(X)$ itself is virtually abelian. The conclusion then follows from [**CH13**, Theorem 1.5], completing the proof of Theorem 5.11. □

Remark 5.12. Our original proof of Theorem 5.11 was involved and relied essentially on the pseudo Picard hyperbolicity established in Theorem 1.36. The arguments in the above proof were pointed out to us by a referee of [**DY25**]. Similar arguments were already used in [**CH13**].

5.5. A structure theorem: on a conjecture by Kollár. In [**Kol95**, Conjecture 4.18], Kollár raised the following conjecture on the structure of varieties with big fundamental group.

Conjecture 5.13. *Let X be a smooth projective variety with big fundamental group such that $0 < \kappa(X) < \dim X$. Then X has a finite étale cover $p : X' \to X$ such that X' is birational to a smooth family of abelian varieties over a projective variety of general type Z which has big fundamental group.*

In this section we address Conjecture 5.13. Our theorem is the following:

THEOREM 5.14 ([**CDY25c**, Theorem 5.1], [**DY25**, Corollary H]). *Let X be a smooth quasi-projective variety and let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a big representation where K is any field. When char $K = 0$, we assume additionally that ϱ is reductive.*

(a) *The logarithmic Kodaira dimension $\bar{\kappa}(X) \geq 0$.*
(b) *More generally, after replacing X by a suitable finite étale cover and a birational modification, there are a semiabelian variety A, a quasi-projective manifold V, and a birational morphism $a : X \to V$ such that we have the following commutative diagram*

$$
\begin{array}{ccc}
X & \xrightarrow{\quad a \quad} & V \\
& {\scriptstyle j} \searrow \qquad \swarrow {\scriptstyle h} & \\
& J(X) &
\end{array}
$$

where j is the logarithmic Iitaka fibration and $h : V \to J(X)$ is a locally trivial fibration with fibers isomorphic to A. Moreover, for a general fiber F of j, $a|_F : F \to A$ is proper in codimension one.

PROOF. We may assume that K is algebraically closed. To prove the theorem we are free to replace X by a birational modification and by a finite étale cover since the logarithmic Kodaira dimension will remain unchanged. If char $K > 0$, we replace ϱ by its semisimplification, which remains big by Lemma 2.36. Hence we might assume that ϱ is big and semisimple. Consequently, after replacing X by a finite étale cover, the Zariski closure G of ϱ is reductive and connected. Let $\mathcal{D}G$ be the derived group of G, which is semisimple. Let $Z \subset G$ be the maximal central torus of G. Then $T := G/\mathcal{D}G$ is a torus and $S := G/Z$ is semisimple. The natural morphism $G \to S \times T$ is a central isogeny. The induced representation $\varrho' : \pi_1(X) \to S(K) \times T(K)$ by ϱ is also big. Consider the representation $\sigma : \pi_1(X) \to S(K)$, obtained by composing ϱ with the morphism $G \to S$. Then $\sigma(\pi_1(X))$ is Zariski dense.

We then prove the following factorization theorem (cf. [**DY25**, Proposition 5.9])[b]: there exist

   (i) a generically finite, proper, surjective morphism $\mu\colon X_1 \to X$ from a smooth quasi-projective variety, obtained as a composition of birational modifications and finite étale Galois covers;
  (ii) a dominant morphism $f_1\colon X_1 \to Y_1$, where Y_1 is a smooth quasi-projective variety with connected general fibers;
 (iii) a big and Zariski dense representation $\tau\colon \pi_1(Y_1) \to S(K)$;

such that $\mu^*\sigma = f_1^*\tau$. Note that Y_1 may be a point. If $\dim Y_1 > 0$, then by Theorem 1.36, Y_1 is strongly of log general type.

Claim 5.15. *For a general smooth fiber F of f_1, we have $\dim F = \dim \alpha(F)$*

PROOF. Note that $\mu^*\sigma(\mathrm{Im}[\pi_1(F) \to \pi_1(X)])$ is trivial. Since $\mu^*\varrho' : \pi_1(X_1) \to S(K) \times T(K)$ is big, by the construction of σ, we conclude that the representation $\eta : \pi_1(F) \to T(K)$ obtained by

$$\pi_1(F) \to \pi_1(X_1) \overset{\mu^*\varrho'}{\to} S(K) \times T(K) \to T(K)$$

is big. Since $T(K)$ is commutative, η factors through $\pi_1(F) \to \pi_1(A) \to T(K)$. This implies that $\dim F = \dim \alpha(F)$.   □

Let us prove item (a). By Claim 5.15, for a general smooth fiber F of f_1, $\bar\kappa(F) \geq 0$. Since Y_1 is of log general type, by the subadditivity of the logarithmic Kodaira dimension proven in [**Fuj17**, Theorem 1.9], we obtain

$$\bar\kappa(X_1) \geq \bar\kappa(Y_1) + \bar\kappa(F) \geq \bar\kappa(Y_1) = \dim Y_1 \geq 0.$$

Hence $\bar\kappa(X) = \bar\kappa(X_1) \geq 0$. The first claim is proved.

We proceed to prove the second assertion. For simplicity, we assume that the logarithmic Iitaka fibration $j : X_1 \to J(X_1)$ is regular. Let $X_t := j^{-1}(t)$ for any $t \in Y_1$.

Claim 5.16. $f_1(X_t)$ *is a point for very generic $t \in J(X_1)$.*

PROOF. Since f_1 is dominant and Y_1 is strongly of log general type, $\overline{f_1(X_t)}$ is of log general type for generic $t \in J(X)$. Now, let us take a very generic $t \in J(X)$. To show that $f_1(X_t)$ is a point, suppose for the sake of contradiction that $\dim f_1(X_t) > 0$. Then $\bar\kappa(\overline{f_1(X_t)}) = \dim \overline{f_1(X_t)}$. Since $\bar\kappa(X_t) = 0$, the general fibers of the restriction $f_1|_{X_t} : X_t \to \overline{f_1(X_t)}$ have non-negative logarithmic Kodaira dimension. By [**Fuj17**, Theorem 1.9] again, it follows that $\bar\kappa(X_t) \geq \bar\kappa(\overline{f_1(X_t)}) > 0$. This yields a contradiction. Thus, $f_1(X_t)$ is a point.   □

By Claims 5.15 and 5.16, for very generic $t \in J(X_1)$, we have $\dim X_t = \dim \alpha(X_t)$. By the birational criterion for semi-abelian varieties in [**CDY25a**, Proof of Lemma 1.4], the closure of $\alpha(X_t)$ is a semi-abelian subvariety of A. As there are at most countably many semi-abelian subvarieties of A, we conclude that the images of the very general fibers of f_1 are isomorphic. This roughly shows the second assertion.
                                                                                                  □

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[b]The main difficulty arises in the case char $K > 0$, since Selberg's lemma—asserting that a finitely generated linear group is virtually torsion-free—is no longer available.

Remark 5.17. If we assume the logarithmic abundance conjecture: a smooth quasi-projective variety is $\mathbb{A}^1$-uniruled if and only if $\bar{\kappa}(X) = -\infty$, then it predicts that $\bar{\kappa}(X) \geq 0$ if there is a big representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$, which is slightly stronger than the first claim in Theorem 5.14. Indeed, since ϱ is big, X is not $\mathbb{A}^1$-uniruled and thus $\bar{\kappa}(X) \geq 0$ by this conjecture.

Convention and notation

In this paper, we use the following conventions and notations:

- Quasi-projective varieties and their closed subvarieties are assumed to be positive-dimensional and irreducible unless specifically mentioned otherwise. Zariski closed subsets, however, may be reducible.
- Fundamental groups are always referred to as topological fundamental groups.
- If X is a complex space, we denote by X^{norm} its normalization, and by X_{reg} its smooth locus.
- $\mathbb{D}$ denotes the unit disk in $\mathbb{C}$, and $\mathbb{D}^*$ denotes the punctured unit disk.
- A finitely generated group Γ is called *linear* if it admits an almost faithful representation $\varrho : \Gamma \to \mathrm{GL}_N(\mathbb{C})$, i.e. such that $\ker \varrho$ is finite. It is called *reductive* if, moreover, ϱ is semisimple.
- For a complex algebraic variety X, unless otherwise specified, we denote by $\pi_X : \widetilde{X} \to X$ its universal covering map. More generally, for any normal subgroup $\Gamma \subset \pi_1(X)$, we denote by $\widetilde{X}_\Gamma \to X$ the Galois covering of X with Galois group $\pi_1(X)/\Gamma$. For any representation $\varrho : \Gamma \to \mathrm{GL}_N(K)$, we denote by $\widetilde{X}_\varrho \to X$ the Galois covering of X with Galois group $\pi_1(X)/\ker \varrho$.
- A proper holomorphic fibration between complex spaces, is a proper holomorphic map such that each fiber is connected.
- A $\mathbb{C}$-VHS (resp. $\mathbb{Z}$-VHS) denotes a complex (resp. integral) polarized variation of Hodge structure.
- The reductive (resp. linear) Shafarevich conjecture refers to the Shafarevich conjecture for projective varieties with reductive (resp. linear) fundamental groups.
- A group G is said to be *virtually P* if it contains a subgroup of finite index that has property P.
- For a smooth complex quasi-projective variety X, unless otherwise specified, we assume that $\overline{X}$ is a smooth compactification of X such that the boundary $D := \overline{X} \setminus X$ is a simple normal crossing divisor.

many stimulating discussions on non-abelian Hodge theories and related topics over the years.

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Email address: `ya.deng@math.cnrs.fr, deng@imj-prg.fr`

CNRS, Institut de Mathématiques de Jussieu-Paris Rive Gauche, Sorbonne Université, Campus Pierre et Marie Curie, 4 place Jussieu, 75252 Paris Cedex 05, France

URL: `https://ydeng.perso.math.cnrs.fr`