LINEAR SINGER-HOPF CONJECTURE

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ABSTRACT. If X is a closed $2d$-dimensional aspherical manifold, i.e., the universal cover of X is contractible, then the Singer-Hopf conjecture predicts that $(-1)^d\chi(X) \geq 0$. We prove this conjecture when X is a complex projective manifold whose fundamental group admits an almost faithful linear representation over any field. In fact, we prove a much stronger statement that if X is a complex projective manifold with large fundamental group and $\pi_1(X)$ admits an almost faithful linear representation, then $\chi(X, \mathcal{P}) \geq 0$ for any perverse sheaf $\mathcal{P}$ on X.

To prove the main result, we introduce a vanishing cycle functor of multivalued one-forms. Then using techniques from non-abelian Hodge theories in both archimedean and non-archimedean settings, we deduce the desired positivity from the geometry of pure and mixed period maps.

CONTENTS

1. Introduction

In the 1930s, Hopf conjectured that a closed $2n$-dimensional compact Riemannian manifold X with non-positive sectional curvature satisfies $(-1)^n\chi(X) \geq 0$ (see [Yau82, Problem 10]). Singer proposed to study Hopf's conjecture using L^2-cohomology groups of the universal cover of X, which naturally lead to the following more general conjecture.

Conjecture 1.1 (Singer-Hopf)**.** *Let X be a closed $2n$-dimensional manifold. If X is aspherical, i.e., the universal cover of X is contractible, then*

$$(-1)^n\chi(X) \geq 0.$$

The above conjecture was also considered by Thurston for 4-manifolds ([Kir97, Problem 4.10]). When $n = 1$, Conjecture 1.1 follows from the easy fact that a Riemann surface is aspherical if and only if its genus is at least 1. If we know that X admits a Riemannian metric with non-positive sectional curvature, then the conjecture follows from the Gauss-Bonnet theorem when $n = 2$ (see [Che55]). Beyond this case, the conjecture is widely open.

If we assume X is a complex projective manifold, then some progresses have been made in [Gro91, JZ00, CX01, DCL24, LMW21, AW21, LIP24], to quote only a few. In [LMW21], it was noticed that for projective manifolds, a natural generalization of being aspherical is having large fundamental group. Here we recall that a projective manifold X is said to *have large fundamental group* if for any irreducible subvariety $Z \subset X$, the image of the composition $\pi_1(Z_{\mathrm{norm}}) \to \pi_1(Z) \to \pi_1(X)$ is infinite, where Z_{norm} denotes the normalization of Z. Moreover, the assertion that $(-1)^n\chi(X) \geq 0$ was strengthened to

$$\chi(X, \mathcal{P}) \geq 0 \text{ for any perverse sheaves } \mathcal{P} \text{ on } X. \tag{1}$$

These ideas were further pursued in [AW21]. Even though not explicitly stated, the following conjecture was expected to hold.

Conjecture 1.2. *If X is a complex projective manifold with large fundamental group, then $\chi(X, \mathcal{P}) \geq 0$ for any perverse sheaves $\mathcal{P}$ on X.*

In [AW21], the above conjecture is proved when there exists a semisimple, almost faithful, cohomologically rigid representation $\pi_1(X) \to \mathrm{GL}(r, \mathbb{C})$. Here we recall that a linear representation is called *almost faithful* if the kernel of the representation is finite, and it is called *cohomologically rigid*, if it has no nontrivial first order deformation.

In this paper, we prove Conjecture 1.2 within a much broader framework: we merely require the existence of an almost faithful linear representation $\pi_1(X) \to \mathrm{GL}(r, K)$ over a field K of arbitrary characteristic, without assuming semisimplity or cohomological ridigity of the representation.

Given a normal projective variety X and a field K, a representation $\rho : \pi_1(X) \to \mathrm{GL}(r, K)$ is called *large*, if for any irreducible subvariety $Z \subset X$, the composition $\pi_1(Z_{\mathrm{norm}}) \to \pi_1(X) \xrightarrow{\rho} \mathrm{GL}(r, K)$ has infinite image.

Theorem 1.3. *Let X be a projective manifold. If there is a large representation $\rho : \pi_1(X) \to \mathrm{GL}(r, K)$ where K is any field, then $\chi(X, \mathcal{P}) \geq 0$ for any perverse sheaves $\mathcal{P}$ on X. In particular, $(-1)^{\dim X}\chi(X) \geq 0$.*

Notice that if X has large fundamental group, and if ρ is almost faithful, then ρ is large. Therefore, the following corollary is immediate.

Corollary 1.4. *Let X be a projective manifold with large fundamental group (e.g. X is aspherical). If there exists an almost faithful representation $\rho : \pi_1(X) \to \mathrm{GL}(r, K)$, then for any perverse sheaf $\mathcal{P}$ on X, $\chi(X, \mathcal{P}) \geq 0$. In particular, Singer-Hopf conjecture holds for X.*

A slightly stronger formulation of Theorem 1.3 is the following.

Theorem 1.5. *Let X be a projective manifold and let $\rho : \pi_1(X) \to \mathrm{GL}(r, K)$ be a large representation, where K is any field. Then, for any closed irreducible subvariety $Z \subset X$,*

$$T^*_Z X \cdot T^*_X X \geq 0 \tag{2}$$

where $T^*_Z X$ denotes the conormal variety of Z and $\cdot$ denotes the intersection number in T^*X.

Proof of Theorem 1.3 assuming Theorem 1.5. We need to use the notions of characteristic cycle and conormal variety which will be reviewed in section 2.2.

Since the characteristic cycle of a perverse sheaf is always effective (cf. Proposition 2.2), we have

$$CC(\mathcal{P}) = \sum_{1 \leq i \leq m} n_i T^*_{Z_i} X,$$

where Z_i are irreducible subvarieties of X and $n_i \in \mathbb{Z}_{>0}$. By the global index theorem (cf. Theorem 2.1) and Theorem 1.5, we have

$$\chi(X, \mathcal{P}) = CC(\mathcal{P}) \cdot T^*_X X = \sum_{1 \leq i \leq m} n_i T^*_{Z_i} X \cdot T^*_X \geq 0. \qquad \square$$

Notice that since $\pi_1(X)$ is finitely generated, having a large representation $\pi_1(X) \to \mathrm{GL}(r, K)$ for a field K of characteristic zero is equivalent to having a large representation $\pi_1(X) \to \mathrm{GL}(r, \mathbb{C})$. Therefore, in Theorem 1.5, we can assume either $\mathrm{char}(K) > 0$ or $K = \mathbb{C}$.

1.1. **Sketch of the proof.** We will use techniques from the linear Shafarevich conjecture in [DYK23, DY24, Eys04, EKPR12] to establish Theorem 1.5. As in [AW21], Arapura and the second named author proved Theorem 1.5 assuming ρ underlies a complex variation of Hodge structure ($\mathbb{C}$-VHS for short) with discrete monodromy. When ρ does not satisfy such property, in [DYK23] Yamanoi and the first named author showed that one can produce non-trivial multi-valued one-forms on X. Our approach involves an iterative utilization of both multi-valued one-forms and the period maps of $\mathbb{C}$-VHS to deduce the inequality (2).

Let us briefly explain our strategy. We first observe that if there is a d-valued one-form whose image in T^*X intersects $T^*_Z X$ at finitely many points, then the inequality (2) holds, since d times the left-hand side is equal to the number of intersection points counting multiplicity.

In general, we introduce a vanishing cycle functor Φ_η of a d-valued one-form η acting on the free abelian group generated by conormal varieties. We show that $d T^*_Z X \cdot T^*_X X = \Phi_\eta(T^*_Z X) \cdot T^*_X X$. Using Φ_η, we can reduce (2) to the same inequality with Z replaced by smaller subvarieties, as long as the restriction of η to the smooth locus of Z is non-trivial. Then the proof of Theorem 1.5 is divided into the following three cases.

Case (i): when $\mathrm{char}(K) > 0$. In this case, we can prove that the above reduction process continues until Z becomes points. When Z is a point, the inequality (2) is obvious.

Case (ii): when $K = \mathbb{C}$ and ρ is semisimple. In this case, when the above reduction process terminates, Z_{norm} underlies a $\mathbb{C}$-VHS with large monodromy representation and discrete monodromy group. As proved in [AW21], Z_{norm} then admits a finite morphism to the period domain of such $\mathbb{C}$-VHS. Then the inequality (2) can be deduced from the curvature property of the period domain.

Case (iii): when $K = \mathbb{C}$ and ρ is not semisimple. In this case, when the reduction process terminates, Z_{norm} admits a complex variation of mixed Hodge structure ($\mathbb{R}$-VMHS) such that its mixed period map $Z_{\mathrm{norm}}^{\mathrm{univ}} \to \mathscr{M}$ has discrete fibers. In this case, two extra difficulties occur compared to Case (ii):

- the monodromy group may not act on the mixed period domain discretely, and thus we cannot take the quotient of the period map by the monodromy group.
- The mixed period domain does not have the desired non-positive curvature in the pure case.

This is resolved by a technical result (cf. Proposition 5.3) which allows us to simultaneously explore the geometry of the mixed period map of the $\mathbb{R}$-VMHS and the period map of the $\mathbb{C}$-VHS corresponding to the semi-simplification of the $\mathbb{R}$-VMHS.

In [AW21], we obtain discreteness of the monodromy group from a deep theorem of Esnault-Groechenig ([EG18], see also [Esn23] for a survey). In this paper, the discreteness of monodromy groups are deduced from nonarchemidean Hodge theory, which is same as the approach in [BKT13].

In theory, we could bypass Case (ii) and directly prove the more general Case (iii). However, we decide to write the proof in separate cases to isolate the few technical arguments and to benefit the readers who would be satisfied understanding the proof up to Case (ii).

1.2. **Relation with the Shafarevich conjecture and hyperbolicity.** The Shafarevich conjecture predicts that the universal cover of a complex normal projective variety X is holomorphically convex. In particular, if X has large fundamental group, then its universal cover is conjectured to be Stein. Currently, this conjecture is proved for the following cases,

(1) when X is smooth and $\pi_1(X)$ admits a faithful linear representation over $\mathbb{C}$, by Eyssidieux et. al. in [EKPR12];
(2) when X is not necessarily smooth and $\pi_1(X)$ admits a faithful reductive linear representation over $\mathbb{C}$, by the first named author, Yamanoi and Katzarkov in [DYK23];
(3) when X is a normal surface and $\pi_1(X)$ admits an almost faithful linear representation over a field of positive characteristic in [DY24].

The proofs of the Shafarevich conjecture use nonabelian Hodge theories both in the archimedean setting by Simpson [Sim92] and non-archimedean setting by Gromov-Schoen [GS92].

In this paper, we do not use any result about holomorphic convexity or Steinness. Nevertheless, we use both archimedian and nonarchimedian nonabelian Hodge theories in the same way as in the proof of Shafarevich conjectures. These techiniques are robust tools in studying the Shafarevich conjecture, hyperbolicity of algebraic varieties (cf. e.g. [Yam10, CDY22, DY24]), and the Singer-Hopf conjecture in the linear case. Our main result Theorem 1.5 is certain positivity property of the cotangent bundle of X, which is known to be related to the hyperbolicity of X. For example, a projective manifold with ample cotangent bundle is Kobayashi hyperbolic.

Even though our result relies on X to have a large fundamental group, we hope our positivity (2) can be directly related to hyperbolicity. For example, does (2) always hold for irreducible subvarieties Z in a Kobayashi hyperbolic projective manifold X? In general, it is also interesting to find sufficient conditions on a representation of $\pi_1(X)$ such that (2) is always strict inequality. In [CDY22, DY24], it is proved that when the Zariski closure of a (generically) large representation $\varrho : \pi_1(X) \to \mathrm{GL}(r, K)$ is a semisimple algebraic group, then X is "almost" hyperbolic. This condition is sharp as one has to exclude the case of abelian varieties. Hence we conjecture that the inequality (2) is strict if the Zariski closure of $\varrho(\mathrm{Im}[\pi_1(Z_{\mathrm{norm}}) \to \pi_1(X)])$ is a semisimple algebraic group for every positive-dimensional subvariety Z of X. In this case, X is Kobayashi hyperbolic by [Yam10, CDY22, DY24].

Recently, various generalizations of the Singer-Hopf conjecture in the algebra-geometric setting are formulated, e.g., in the singular setting as in [Max23], and in the coherent setting as in [AMW23]. Here, we also want to propose Singer-Hopf type conjectures for quasi-projective varieties. For example, we conjecture that if the universal cover of a quasi-projective manifold X is a bounded symmetric space, then $\chi(X, \mathcal{P}) \geq 0$ for any perverse sheaf on X (with respect to an algebraic stratification).

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Notation and Convention.

- For any analytic/algebraic variety Z, we denote by Z_{norm} its normalization, by Z_{reg} the smooth locus of Z, and by $\widetilde{Z}^{\mathrm{univ}}$ its universal cover.
- For any algebraic variety X, any field K and any representation $\sigma : \pi_1(X) \to \mathrm{GL}(r, K)$, L_σ denotes the corresponding local system.
- Let $\sigma : \Gamma \to \mathrm{GL}(r, \mathbb{C})$ where Γ is a finitely generated group. We denote by $\sigma^{ss} : \Gamma \to \mathrm{GL}(r, \mathbb{C})$ its semisimplification.

2. Preliminaries

We first review some results about constructible functions, constructible sheaves, characteristic cycles and the global index formula. The recent survey article [MS22] gives a more detailed summery of these subjects.

2.1. **Constructible functions.** Let X be an analytic variety. A function $\gamma : X \to \mathbb{Z}$ is called constructible, if there exists a locally finite stratification of X into locally closed smooth subvarieties $X = \bigsqcup_{i\in I} X_i$, such that the restriction of γ to each X_i is constant. The Euler characteristic

$$\chi(\gamma) = \sum_{i\in I} \gamma(X_i) \cdot \chi(X_i)$$

is well-defined when the above stratification can be chosen to be finite and each stratum is homotopy equivalent to a finite CW-complex, e.g., when X is compact or when X is an algebraic variety and each stratum is a locally closed algebraic subvariety.

If $f : Y \to X$ is a holomorphic map between analytic varieties, then the pullback $f^*(\gamma) := \gamma \circ f$ is a constructible function on Y.

If $g : X \to Y$ is a proper holomorphic map between complex analytic varieties, then the pushforward of a constructible function $g_*(\gamma)$ defined by

$$g_*(\gamma)(y) = \chi\left(g^{-1}(Y), \gamma|_{g^{-1}(y)}\right)$$

is a constructible function. Moreover, we have $\chi(Y, g_*(\gamma)) = \chi(X, \gamma)$ when Y is compact or when g is a regular map of algebraic varieties and the constructibility is algebraic.

Given a constructible complex $\mathcal{F}$ defined over a field, we can define its stalkwise Euler characteristic function $\chi_{st}(\mathcal{F})$ by

$$\chi_{st}(\mathcal{F})(x) = \chi(\mathcal{F}_x).$$

2.2. **Constructible sheaves and characteristic cycles.** Let X be a complex manifold. Let $D_c^b(X, K)$ be the derived category of K-constructible complexes,and let $Perv(X)$ be its subcategory of perverse sheaves. We are interested in the Euler characteristics of perverse sheaves, e.g., $\mathbb{Q}_X[\dim X]$. The Euler characteristic of a constructible complex can be computed via characteristic cycles.

Given any irreducible analytic subvariety Z of X, its conormal variety T_Z^*X is defined to be the closure of the conormal bundle $T_{Z_{\mathrm{reg}}}^*X$ in T^*X. In particular, the conormal variety T_X^*X of X itself is equal to the zero section of T^*X. Every conormal variety T_Z^*X is conic and Lagrangian in T^*X, and conversely, every conic Lagrangian subvariety of T^*X is a conormal variety. Denote by $L(X)$ the abelian group of locally finite conic Lagrangian cycles on T^*X. The characteristic cycle is a group homomorphism

$$CC : K_0\big(D_c^b(X, K)\big) \to L(X),$$

where $K_0(D_c^b(X, K))$ is the Grothendieck group of $D_c^b(X, K)$. We will not review its precise definition (see e.g., [Dim04, Definition 4.3.19] and [KS90, Chapter IX]). Instead, we focus on the following two important properties.

Theorem 2.1 ([Kas85], [Dim04, Theorem 4.3.25])**.** *Let X be a complex manifold, and let $\mathcal{F}$ be a constructible complex on X with compact support. Then*

$$\chi(X, \mathcal{F}) = CC(\mathcal{F}) \cdot T_X^*X$$

where the right-hand side denotes the intersection number in T^*X.

Note that even though T^*X is not compact, the intersection number is well-defined because it can be defined as a zero cycle in the support of $\mathcal{F}$ in X (see [Ful98, Chapter 6]).

Proposition 2.2 ([Dim04, Corollary 5.2.24])**.** *For any perverse sheaf $\mathcal{P}$ on X, $CC(\mathcal{P})$ is effective. In other words,*

$$CC(\mathcal{P}) = \sum_{i\in I} n_i T^*_{Z_i} X,$$

where Z_i are irreducible subvarieties of X, $n_i \in \mathbb{Z}_{>0}$ and the sum is locally finite.

Given a constructible complex $\mathcal{F}$ on a complex manifold X, we can define its stalkwise Euler characteristic function $\chi_{st}(\mathcal{F}) : X \to \mathbb{Z}$ by

$$\chi_{st}(\mathcal{F})(x) = \chi(\mathcal{F}_x).$$

Denote the abelian group of $\mathbb{Z}$-valued constructible functions on X by $F(X)$. Then χ_{st} defines a group homomorphism

$$\chi_{st} : K_0\big(D^b_c(X, K)\big) \to F(X).$$

Moreover, the characteristic cycle $CC : K_0\big(D^b_c(X, K)\big) \to L(X)$ factors through χ_{st}. By abusing notations, we also use CC to denote induced group homomorphism

$$CC : F(X) \to L(X).$$

Given an irreducible subvariety Z of X, the local Euler obstruction function Eu_Z is a constructible function on Z (or on X with value 0 outside Z) uniquely characterized by the property that

$$CC(Eu_Z) = (-1)^{\dim Z} T^*_Z X.$$

It turns out that Eu_Z is an intrinsic invariant of Z, which does not depend on the embedding to a smooth variety (see e.g. [Dim04, Page 100-102] and [Mas20, Sections 3,4]). Moreover, as Z varies through all irreducible analytic subvarieties of X, Eu_Z form a basis of $F(X)$.

The above global index formula implies that

$$\chi(Eu_Z) = (-1)^{\dim Z} T^*_Z X \cdot T^*_X X. \tag{3}$$

The advantage of working with Eu_Z and $(-1)^{\dim Z}\chi(Eu_Z)$ instead of $T^*_Z X$ and $T^*_Z X \cdot T^*_X X$ is that the former does not need to involve an embedding.

Definition 2.3. A $\mathbb{Z}$-constructible function γ on an analytic variety X is called *CC-effective* if it is of the following form

$$\gamma = \sum_{i\in I} (-1)^{\dim Z_i} n_i Eu_{Z_i}, \tag{4}$$

where Z_i are irreducible subvarieties of X, $n_i \in \mathbb{Z}_{>0}$ and the sum is locally finite.

When X is smooth, γ is CC-effective if and only if $CC(\gamma)$ is an effective cycle in T^*X.

Lemma 2.4. *A $\mathbb{Z}$-constructible function γ on an analytic variety X is CC-effective if and only if for any $x \in X$ and a small open neighborhood $U_x \subset X$ of X, the restriction $\gamma|_{U_x}$ is CC-effective.*

Proof. Since Eu_Z form a basis of $F(X)$, for any constructible function γ there is a unique expression (4) with $n_i \in \mathbb{Z}$. Since the local Euler obstruction function is a local invariant, $Eu_Z|_{U_x}{=}Eu_{Z_i\cap U_x}$. Hence,

$$\gamma|_{U_x} = \sum_{i\in I,\ Z_i\cap U_x\neq 0} (-1)^{\dim Z_i} n_i Eu_{Z_i\cap U_x},$$

which is also the unique expression of $\gamma|_{U_x}$ in terms of local Euler obstruction functions. Therefore, γ is CC-effective if and only if all $\gamma|_{U_x}$ are CC-effective. □

The following proposition is proved in the algebraic setting in [AMSS22, Proposition 7.2 (2)]. Nevertheless, the argument also applies to the analytic setting.

Proposition 2.5 (Aluffi-Mihalcea-Schürmann-Su)**.** *Let $p : Z' \to Z$ be a finite morphism of irreducible analytic varieties. Then $p_*((-1)^{\dim Z'} Eu_{Z'})$ is CC-effective. More generally, if γ is a CC-effective constructible function on Z', then $p_*(\gamma)$ is also CC-effective.*

2.3. **A proper pushforward formula.** Given a constructible function γ on a complex manifold X, we have considered $CC(\gamma)$ as an analytic $\dim(X)$-cycle in T^*X. To study the functorial properties of the characteristic cycles, we also need to consider them as Borel-Moore homology classes in an appropriate subspace of T^*X.

Assume γ is constructible with respect to a Whitney stratification $\mathcal{S}$ of X. We define the *conormal space* of $\mathcal{S}$ to be

$$T^*_{\mathcal{S}}X = \bigcup_{S\in\mathcal{S}} T^*_{\overline{S}}X,$$

where $\overline{S}$ is the closure of the stratum S in X. Then $T^*_{\mathcal{S}}X$ is a locally finite union of conic Lagrangian subvarieties. The analytic cycle $CC(\gamma)$ is supported in $T^*_{\mathcal{S}}X$, and it represents a class in $H^{BM}_{2\dim X}(T^*_{\mathcal{S}}X, \mathbb{Z})$. For the rest of this section, we identify a characteristic cycle as the Borel-Moore homology class it represents.

Let $f : X \to Y$ be a proper holomorphic map between complex manifolds. Let γ be a constructible function on X. We review a formula to compute $CC(f_*(\gamma))$.

By the theorem in [GM88, Part I, Section 1.7], there exist Whitney stratifications $\mathcal{S}$ and $\mathcal{S}'$ of X and Y respectively, satisfying

(1) γ is constructible with respect to $\mathcal{S}$, and
(2) for any stratum $S \in \mathcal{S}$, there exists a stratum $S' \in \mathcal{S}'$ such that f induces a submersion $S \to S'$.

The map f induces

$$T^*X \xleftarrow{u_1} f^*T^*Y \xrightarrow{u_2} T^*Y \quad \text{and} \quad T^*_{\mathcal{S}}X \xleftarrow{u_1} u_1^{-1}T^*_{\mathcal{S}}X \xrightarrow{u_2} T^*_{\mathcal{S}'}Y.$$

Theorem 2.6 ([Sab85, Théorème 2.2], see also [MS22, Proposition10.3.46])**.** *Under the above notations,*

$$CC(f_*(\gamma)) = u_{2*}u_1^*(CC(\gamma)),$$

where $CC(\gamma) \in H^{BM}_{2\dim X}(T^*_{\mathcal{S}}X, \mathbb{Z})$, $u_1^*(CC(\gamma)) \in H^{BM}_{2\dim Y}(u_1^{-1}T^*_{\mathcal{S}}X, \mathbb{Z})$, and $u_{2*}u_1^*(CC(\gamma)) \in H^{BM}_{2\dim Y}(T^*_{\mathcal{S}'}Y, \mathbb{Z})$.

Here, u_{2*} is the proper pushforward of Borel-Moore homology, and u_1^* is defined via Poincaré duality as the pullback of local cohomology (see [MS22, Page 734] for more details).

2.4. **Multivalued one-forms.**

Definition 2.7 (Mutivalued one-form)**.** Let X be a complex manifold and let E be a holomorphic vector bundle on X. A *multivalued section* η on X is a formal sum $\sum_{i=1}^{m} n_i\Gamma_i$, where each $n_i \in \mathbb{Z}_{>0}$ and each Γ_i is an irreducible closed subvariety of E such that the natural projection $\Gamma_i \to X$ is a finite surjective morphism. Such multisection η is called *d-valued* for $d = \sum_{i=1}^{m} n_i \deg[\Gamma_i : X]$. We say that η is *reduced* if all Γ_i are distinct, and η is *irreducible* if $m = 1$ and $n_1 = 1$. We say that η is *trivial* if $m = 1$ and Γ_1 is the zero section of $E \to X$. Multivalued sections of T^*X will be called *mutivalued (holomorphic) one-forms*.

Let η be a mutivalued one-form on a complex manifold X. We sometimes denote by Γ_η instead of η to emphasize it is an analytic cycle in T^*X. By definition, there exists a largest Zariski open subset $X^\circ \subset X$ such that $\Gamma_\eta \cap T^*X^\circ \to X^\circ$ is étale. We call X° to be the *unbranching locus* of η and the complement $X \setminus X^\circ$ the *branching locus*. In this case, for any $x \in X^\circ$, there exist an open neighborhood of U of x and holomorphic one-forms $\eta_1, \dots, \eta_m$ on U such that $\eta|_U$ is equal to the union of $\eta_1, \dots, \eta_m$. Therefore, we say that η is *closed* if for any point $x \in X^\circ$, each of the above η_i is a closed one-form.

Remark 2.8. Let Y be a locally closed complex submanifold of X. Given any multivalued one-form η of X, using the pullback map $T^*X|_Y \to T^*Y$ we can define the restriction multivalued one-form $\eta|_Y$. Clearly, when η is closed, so is $\eta|_Y$.

Lemma 2.9. *Let X be a projective manifold. Every multivalued one-form η on X is closed.*

Proof. Without loss of generality, we can assume that η is irreducible. Since X is projective and the natural map $\Gamma_\eta \to X$ is finite, Γ_η is a projective variety. Let $Y \to \Gamma_\eta$ be a resolution of singularity, which is an isomorphism over the smooth locus of Γ_η. Let θ be the tautological holomorphic one form on T^*X, and let θ_Y be the pullback of θ via the composition $Y \to \Gamma_\eta \to T^*X$. Since Y is a projective manifold, by Hodge theory θ_Y must be closed. Over the unbranching locus of η, the composition $Y \to \Gamma_\eta \to X$ is étale, and the pushforward of θ_Y to X is equal to η. Thus, η is closed. □

By the work of [Eys04] and [CDY22], we can construct non-trivial multivalued one-forms from unbounded representations $\pi_1(X) \to \mathrm{GL}(r, K)$ where K is a non-archimedean local field.

Proposition 2.10. *Let X be a smooth projective variety and let $\rho : \pi_1(X) \to \mathrm{GL}(r, K)$ be a reductive representation where K is a non-archimedean local field. Then there exists a proper surjective morphism $s_\rho : X \to S_\rho$ to a normal projective variety with connected fibers and a closed multivalued one-form η_ρ such that for any irreducible closed subvariety $Z \subset X$, the following properties are equivalent:*

(1) $\rho(\mathrm{Im}[\pi_1(Z_{\mathrm{norm}}) \to \pi_1(X)])$ *is bounded;*
(2) $\rho(\mathrm{Im}[\pi_1(Z) \to \pi_1(X)])$ *is bounded;*
(3) $s_\rho(Z)$ *is a point;*
(4) the restriction $\eta_\rho|_{Z_{\mathrm{reg}}}$ *is trivial, where* Z_{reg} *is the smooth locus of* Z*.*

In particular, if $\rho(\pi_1(X))$ is unbounded, then η_ρ is non-trivial.

We will call the above map s_ρ the (Katzarkov-Eyssidieux) reduction map for ρ. The construction of the mutivalued holomorphic one-form η_ρ associated with ρ can be found in

[CDY22, Step 2 in the proof of Theorem H]. The closedness of η_ρ follows from Lemma 2.9. The equivalence of Item 1 and Item 3 is proved by Katzarkov [Kat97] and Eyssidieux [Eys04]. The equivalence of the first three items can be found in [CDY22, Theorem H]. Hence we only need to prove their equivalence to Item 4.

Proof of the equivalence of Items 3 and 4. To simplify the notation, we write η instead of η_ρ. By [CDY22, Definition 5.11], we can find a finite Galois cover $f : Y \to X$ of Galois group G (so-called *spectral covering*) from a normal projective variety such that f is étale outside the branching locus of $\Gamma_\eta \to X$, and $f^*\eta$ becomes single-valued, i.e. there exist sections $\{\omega_1, \ldots, \omega_m\} \subset H^0(Y, f^*\Omega^1_X)$ such that $f^*\eta = \{\omega_1, \ldots, \omega_m\}$. Let $a : Y \to A$ be the *partial Albanese map* associated with $\{\omega_1, \ldots, \omega_m\}$ (cf. [CDY22, Definition 5.19]), where A is an abelian variety. By [CDY22, Claim 5.15], $\{\omega_1, \ldots, \omega_m\}$ is invariant under G. Moreover, by [CDY22, Step 4 in the proof of Theorem H], G acts on A such that a is G-equivariant. Then by [CDY22, Proof of Theorem H], s_ρ is the Stein factorization of the quotient map $X \to A/G$, as in the commutative diagram,

$$
\begin{tikzcd}
Y \arrow[rr, "a"] \arrow[d, "f"] & & A \arrow[d, "\pi"] \\
X \arrow[r, "s_\rho"] & S_\rho \arrow[r] & A/G.
\end{tikzcd}
\tag{5}
$$

Let Z' be a connected component of $f^{-1}(Z)$. Then Z' and Z have the same image in A/G. Since s_ρ is the Stein factorization of $X \to A/G$,

$$a(Z') \text{ is a point} \Leftrightarrow \pi \circ a(Z') \text{ is a point} \Leftrightarrow s_\rho(Z) \text{ is a point}.$$

By [CDY22, Lemma 1.1], $a(Z')$ is a point if and only if $\omega_i|_{Z'_{\mathrm{reg}}} = 0$ for each i.

Clearly, $\omega_i|_{Z'_{\mathrm{reg}}} = 0$ for each i if and only if $\eta|_{Z_{\mathrm{reg}}}$ is trivial. Thus, the equivalence between Items 3 and 4 follows. □

2.5. **A factorization map.** In this subsection, we will review some constructions in the proof of the reductive Shafarevich conjecture in [DYK23]. We first apply Proposition 2.10 to construct a fibration which is essential in our proof. This construction allows us to factorize non-rigid representations into those underlying $\mathbb{C}$-VHS with discrete monodromy.

Definition 2.11 (Factorization map)**.** Let X be a smooth projective variety. We fix a positive integer $r > 0$. We define a *factorization map* $s_{\mathrm{fac},r} : X \to S_{\mathrm{fac},r}$ to be the *simultaneous Stein factorization* of the Katzarkov-Eyssidieux reductions $\{s_\tau : X \to S_\tau\}_\tau$, where $\tau : \pi_1(X) \to \mathrm{GL}(r, K)$ ranges over all semisimple representations with K a non-archimedean local field of characteristic 0. We refer the readers to [DYK23, Lemma 1.28] and [Car60, Lemma on page 7] for the precise definition of the simultaneous Stein factorization. In particular, $s_{\mathrm{fac},r} : X \to S_{\mathrm{fac},r}$ is a proper morphism to a normal projective variety with connected fibers such that

(1) all the above maps s_τ factor through $s_{\mathrm{fac},r}$;
(2) for any closed subvariety Z of X, $s_\tau(Z)$ is a point for all the above s_τ if and only if $s_{\mathrm{fac},r}(Z)$ is a point.

2.6. **Shafarevich morphism.** We will recall the definition of the Shafarevich morphism of a representation of the fundamental group of a projective variety.

Definition 2.12 (Shafarevich morphism)**.** Let X be a projective manifold.

(i) Let H be a normal subgroup of $\pi_1(X)$. The *Shafarevich morphism* of the pair (X, H) is a holomorphic map to a normal projective variety $\mathrm{sh}_H : X \to \mathrm{Sh}_H(X)$ with connected fibers such that for any subvariety Z of X, $\mathrm{Im}[\pi_1(Z_{\mathrm{norm}}) \to \pi_1(X)/H]$ is finite if and only if $\mathrm{sh}_H(Z)$ is a point.
(ii) The *Shafarevich morphism* of a linear presentations of $\varrho : \pi_1(X) \to \mathrm{GL}(r, K)$, denoted by $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$, is the Shafarevich morphism of the pair $(X, \ker\varrho)$.
(iii) Let M be a subset of the moduli space of representations $M_{\mathrm{B}}(\pi_1(X), \mathrm{GL}_r)(\mathbb{C})$. The *reductive Shafarevich morphism* of M, denoted by $\mathrm{sh}_M : X \to \mathrm{Sh}_M(X)$, is the Shafarevich morphism of the pair (X, H), where H is the intersection of kernels of all *semisimple* representations $\varrho : \pi_1(X) \to \mathrm{GL}(r, \mathbb{C})$ with $[\varrho] \in M$.

The Shafarevich morphism is unique if it exists. The existence of sh_M for various choices of M is proved in [Eys04, EKPR12, DYK23, DY24]. Note that for if $\varrho : \pi_1(X) \to \mathrm{GL}(r, K)$ is a large representation, then the Shafarevich morphism $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$ of ϱ is the identity map.

2.7. **Pure Hodge structures and period maps.** In this subsection we briefly review the definitions of $\mathbb{C}$-Hodge structures, pure period domains and period maps. We refer the readers to [CMSP17, SS22] for more details.

A polarized $\mathbb{C}$-Hodge structure (of weight m) is a triple $(V = \oplus_{p+q=m} V^{p,q}, S)$, where V is a $\mathbb{C}$-vector space together with a decomposition $V = \oplus_{p+q=m} V^{p,q}$, and S is the *polarization* that is a non-degenerate hermitian form on V such that the above decomposition is orthogonal with respect ot S and $(-1)^p S|_{V^{p,q}}$ is positive-definite. If (V, S) is endowed with a real structure such that we have the complex conjugate $\overline{F^{p,q}} = F^{q,p}$, then it is called a $\mathbb{R}$-Hodge structure.

The Hodge filtration is defined to be $F^p := \oplus_{i \geq p} V^{i,m-i}$. Fixing m and $\dim_{\mathbb{C}} F^p$, the set of all such filtration $F^\bullet$ is a complex flag manifold, which is denoted by $\mathscr{D}^\vee$. It is a closed submanifold of a product of Grassmannians, and is thus a projective manifold. The *period domain*, denoted by $\mathscr{D}$, is the subset of all complex polarized Hodge structures are charcterized by

(a) $F^p = F^p \cap (F^{p+1})^\perp \oplus F^{p+1}$.
(b) $(-1)^p S$ is positive definite over $F^p \cap (F^{p+1})^\perp$.

It is an open submanifold of $\mathscr{D}^\vee$. Since the groups $\mathrm{GL}(V)$ and $\mathrm{GL}(V, S)$ act transitively on $\mathscr{D}^\vee$ and $\mathscr{D}$ respectively, $\mathscr{D}^\vee$ and $\mathscr{D}$ are thus homogeneous spaces. We also use $F^\bullet$ to denote the $\mathbb{C}$-Hodge structure.

For any Hodge structure $F^\bullet \in \mathscr{D}^\vee$, the holomorphic tangent space $T_{F^\bullet}\mathscr{D}^\vee$ of $\mathscr{D}^\vee$ at $F^\bullet$ is isomorphic to

$$\mathrm{End}(V)/\{A \in \mathrm{End}(V) \mid A(F^p) \subset F^p \text{ for all } p\}.$$

For any $A \in \mathrm{End}(V)$, we denote by $[A]_F$ its image in $T_{F^\bullet}\mathscr{D}^\vee$. A tangent vector $[A]_{F^\bullet}$ in $T_{F^\bullet}\mathscr{D}^\vee$ is called *horizontal* if $A(F^p) \subset F^{p-1}$ for all p. All horizontal vectors form a vector subbundle of $T\mathscr{D}^\vee$, which we denote by $T^h\mathscr{D}^\vee$. A holomorphic map $f : \Omega \to \mathscr{D}^\vee$ from a complex manifold Ω is called *horizontal* if $df : T\Omega \to f^*T\mathscr{D}^\vee$ factors through $f^*T^h\mathscr{D}^\vee$.

A $\mathbb{C}$-*variation of Hodge structure* ($\mathbb{C}$-VHS for short) on a complex manifold X is a family of polarized $\mathbb{C}$-Hodge structures on X subject to a Griffiths transversality condition (see e.g., [Sim92, SS22] for more details). Given a $\mathbb{C}$-VHS with monodromy representation $\varrho : \pi_1(X) \to \mathrm{GL}(V, S)$, it induces a ϱ-equivariant horizontal holomorphic map $\phi : \widetilde{X}^{\mathrm{univ}} \to \mathscr{D}$, called the *period map*. The image $\varrho(\pi_1(X))$ is called the *monodromy group*.

2.8. **Mixed Hodge structures and mixed period maps.** We recall the definition of $\mathbb{R}$-mixed Hodge structures of weight length one and their mixed period maps. We refer the readers to [Pea00, Her99, Car87] for more details.

A graded polarized $\mathbb{R}$-mixed Hodge structure of length 1 is quadruple $(V_{\mathbb{R}}, W_\bullet, F^\bullet, S_i)$ consisting of

- A real finite dimensional vector space $V_{\mathbb{R}}$;
- an increased (weight) filtration $\{0\} = W_{-2} \subset W_{-1} \subset W_0 = V_{\mathbb{R}}$;
- a decreased (Hodge) filtration $F^\bullet$ of V, where $V := V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$;
- two non-degenerate hermitian form S_{-1} and S_0 on the graded quotients $\mathrm{Gr}^W_{-1} V_{\mathbb{R}}$ and $\mathrm{Gr}^W_0 V_{\mathbb{R}}$ of $V_{\mathbb{R}}$ respectively,

such that $\mathrm{Gr}^W_m V_{\mathbb{R}}$ carries a pure $\mathbb{R}$-Hodge structure of weight m polarized by S_m. Here the Hodge filtration $F^\bullet \mathrm{Gr}^W_m V$ is given by

$$F^p \mathrm{Gr}^W_m V := \frac{F^p \cap W_m \otimes \mathbb{C}}{F^p \cap W_{m-1} \otimes \mathbb{C}}.$$

We fix $(V_{\mathbb{R}}, W_\bullet, S_i)$. After fixing $\dim_{\mathbb{C}} F^p \mathrm{Gr}^W_m V$, the *mixed period domain* $\mathscr{M}$ is the set of polarized $\mathbb{R}$-mixed Hodge structures on $(V_{\mathbb{R}}, W_\bullet)$, i.e. the set of decreasing filtrations $F^\bullet$ such that $(V_{\mathbb{R}}, W_\bullet, F^\bullet, S_i)$ is a mixed Hodge structure.

Given an $\mathbb{R}$-mixed Hodge structure $(V_{\mathbb{R}}, W_\bullet, F^\bullet, S_i)$ in $\mathscr{M}$, since each $(\mathrm{Gr}^W_k V_{\mathbb{R}}, F^\bullet \mathrm{Gr}^W_k V, S_k)$ is classified by a pure period domain $\mathscr{D}_k$ for $k = -1, 0$, the graded quotient $\mathrm{Gr}^W_\bullet V_{\mathbb{R}}$ of $W_\bullet$ is then classified by a point of

$$\mathrm{Gr}^W \mathscr{M} := \mathscr{D}_{-1} \times \mathscr{D}_0.$$

Thus, we have a natural projection $\pi : \mathscr{M} \to \mathrm{Gr}^W \mathscr{M}$, which is a holomorphic map between complex manifolds.

Write $V_{k,\mathbb{R}} := \mathrm{Gr}^W_k V_{\mathbb{R}}$. Let $\mathrm{GL}(V_{\mathbb{R}})^W$ be the real subgroup of $\mathrm{GL}(V_{\mathbb{R}})$ preserving the weight filtration $W_\bullet$. Then we have a natural homomorphism

$$q : \mathrm{GL}^W(V_{\mathbb{R}}) \to \mathrm{GL}(V_{-1,\mathbb{R}}) \times \mathrm{GL}(V_{0,\mathbb{R}}).$$

By Section 2.7, the subgroup $\mathrm{GL}(V_{k,\mathbb{R}}, S_k)$ of $\mathrm{GL}(V_{k,\mathbb{R}})$ acts on $\mathscr{D}_k$ transitively. Let G be the inverse image of $\mathrm{GL}(V_{-1,\mathbb{R}}, S_{-1}) \times \mathrm{GL}(V_{0,\mathbb{R}}, S_0)$ under q, which is a real algebraic group. Then $G(\mathbb{R})$ acts on $\mathscr{M}$ and π is q-equivariant. Note that the kernel U of $G \to \mathrm{GL}(V_{-1,\mathbb{R}}, S_{-1}) \times \mathrm{GL}(V_{0,\mathbb{R}}, S_0)$ is a commutative real algebraic group isomorphic to a real vector space.

In the same vein as Section 2.7, we can define the horizontal bundle $T^h_{\mathscr{M}}$ on $\mathscr{M}$, which is a holomorphic subbundle of the tangent bundle of $\mathscr{M}$.

An $\mathbb{R}$-variation of mixed Hodge structure ($\mathbb{R}$-VMHS for short) on a complex manifold X is a family of $\mathbb{R}$-mixed Hodge structures subject to certain conditions (see [Pea00] for more details). In particular, an $\mathbb{R}$-VMHS determines a monodromy representation $\varrho : \pi_1(X) \to G(\mathbb{R})$ and a ϱ-equivariant holomorphic map $\phi : \widetilde{X}^{\mathrm{univ}} \to \mathscr{M}$ such that f is horizontal, i.e., $d\phi : T\widetilde{X} \to$

$f^*T^h\mathscr{M}$, and it satisfies the property that, $\pi\circ\phi : \widetilde{X}^{\mathrm{univ}} \to \mathrm{Gr}^W\mathscr{M}$ is a $q\circ\varrho$-equivariant horizontal holomorphic map, which defines an $\mathbb{R}$-VHS. Such ϕ is called the *mixed period map* of this $\mathbb{R}$-VMHS and $\varrho(\pi_1(X))$ is called the *monodromy group*.

We recollect the following standard fact about the mixed period domain (cf. [Car87, p. 218]).

Lemma 2.13. *Let $\mathscr{M}$ be as above. Then $\pi : \mathscr{M} \to \mathrm{Gr}^W\mathscr{M}$ is a holomorphic vector bundle with the fiber at a point $P \in \mathrm{Gr}^W\mathscr{M}$ being canonically isomorphic to* $\mathrm{Hom}(V_0, V_{-1})\otimes\mathbb{C}/F^0\mathrm{Hom}(V_0, V_{-1})\otimes\mathbb{C}$. *Here we denote by $V_i := \mathrm{Gr}^W_i V$, and* $\mathrm{Hom}(V_0, V_{-1})$ *is endowed with a natural Hodge structure of weight* -1 *induced from P. For the kernel U of the homomorphism $G \to \mathrm{GL}(V_{-1,\mathbb{R}}, S_{-1})\times\mathrm{GL}(V_{0,\mathbb{R}}, S_0)$, $U(\mathbb{C})$ acts on the fibers of π as a translation.* □

3. Nearby cycle functor of a multivalued one-form

In the first part of this section, we define a vanishing cycle functor of a multivalued one-form and prove some useful properties of this functor. In the second part, we prove Theorem 1.5 in the case when K is a field of positive characteristic.

3.1. **Definition of Φ_η.** Given an irreducible multivalued one-form η on a complex manifold X and an irreducible conic Lagrangian cycle Λ in T^*X, we define two irreducible cycles in $T^*X\times\mathbb{C}$: $\Lambda^\diamond = \Lambda\times\mathbb{C}$ and $\Gamma^\diamond_\eta$ is the cycle such that its restriction to $T^*X\times\{s\}$ is equal to $\Gamma_{s\eta}$. Since $\Gamma^\diamond_\eta$ is a multisection of $T^*X\times\mathbb{C}$ as a vector bundle over $X\times\mathbb{C}$, it follows that $\Lambda^\diamond\times_{X\times\mathbb{C}}\Gamma^\diamond_\eta$ is an $(m+1)$-cycle in $(T^*X\times\mathbb{C})\times_{X\times\mathbb{C}}(T^*X\times\mathbb{C})$, where $m = \dim X$. Denote the fiberwise addition map by

$$\mathfrak{S} : (T^*X\times\mathbb{C})\times_{X\times\mathbb{C}}(T^*X\times\mathbb{C}) \to T^*X\times\mathbb{C}.$$

Since the natural projection $\Gamma^\diamond_\eta \to X\times\mathbb{C}$ is a finite morphism, so is the restriction map

$$\mathfrak{S} : (T^*X\times\mathbb{C})\times_{X\times\mathbb{C}}\Gamma_\eta \to T^*X\times\mathbb{C}.$$

Thus, the restriction of $\mathfrak{S}$ to the support of $\Lambda^\diamond\times_{X\times\mathbb{C}}\Gamma^\diamond_\eta$ is also a finite morphism. The pushforward $\mathfrak{S}_*(\Lambda^\diamond\times_{X\times\mathbb{C}}\Gamma^\diamond_{s\eta})$ is a $(d+1)$-cycle in $T^*X\times\mathbb{C}$, which we also denote by $\Lambda^\diamond_\eta$.

Lemma 3.1. *As a subspace of $T^*X\times\mathbb{P}^1$, $\Lambda^\diamond_\eta$ is locally closed with respect to the analytic Zariski topology.*

Proof. If we consider $T^*X\times\mathbb{C}$ as a vector bundle over X, then it follows from definition that $\Lambda^\diamond_\eta$ is a conic cycle. Consider the fiberwise projective compactification

$$T^*X\times\mathbb{C} \subset \mathbb{P}\big((T^*X\times\mathbb{C})\oplus\mathbb{C}_X\big).$$

Since $\Lambda^\diamond_\eta$ is conic, it is locally closed in $\mathbb{P}\big((T^*X\times\mathbb{C})\oplus\mathbb{C}_X\big)$ with respect to the analytic Zariski topology.

Now, consider the fiberwise projective compactifications

$$T^*X \subset \mathbb{P}(T^*X\oplus\mathbb{C}_X), \quad \text{and} \quad X\times\mathbb{C} = \mathbb{C}_X \subset \mathbb{P}(\mathbb{C}_X\oplus\mathbb{C}_X) = X\times\mathbb{P}^1.$$

As partial compactifications of $T^*X\times\mathbb{C}$, $\mathbb{P}\big((T^*X\times\mathbb{C})\oplus\mathbb{C}_X\big)$ and $\mathbb{P}(T^*X\oplus\mathbb{C}_X)\times_X(X\times\mathbb{P}^1)$ are birational to each other. In fact, one can easily construct blowup and blowdown maps connecting them. Since $\Lambda^\diamond_\eta$ is locally closed in the first partial compactification, it is also a locally closed in the second. □

Taking closure in $T^*X\times\mathbb{P}^1$ and keeping the multiplicities, we have $\overline{\Lambda^\diamond_\eta}$, a $(d+1)$ cycle in $T^*X\times\mathbb{P}^1$. Let $\Phi_\eta\Lambda$ be the restriction of $\overline{\Lambda^\diamond_\eta}$ to $T^*X\times\{\infty\}$.

Remark 3.2. The $(m+1)$-cycle $\Lambda^\diamond\times_{X\times\mathbb{C}}\Gamma^\diamond_\eta$ may have higher multiplicities. In fact, it is defined as the scheme-theoretic intersection of $\Lambda^\diamond\times\Gamma^\diamond_\eta$ and the preimage of the diagonal of $(X\times\mathbb{C})\times(X\times\mathbb{C})$ under the map $(T^*X\times\mathbb{C})\times(T^*X\times\mathbb{C})\to(X\times\mathbb{C})\times(X\times\mathbb{C})$. It is straightforward to check that the intersection has expected dimension, but may have higher multiplicities (see [Ful98, Section 7.1]).

Remark 3.3. When η is a single-valued one-form, Γ_η is a section of T^*X, and $\Phi_\eta\Lambda$ is simply MacPherson's description of the deformation to normal cone (see [Ful98, Remark 5.1.1]).

Lemma 3.4. *Let Z be a closed complex submanifold of X, and denote T^*_ZX by Λ. Let η be a multivalued one-form on X, and let $\eta|_Z$ be the restriction of η to Z as in Remark 2.8. Then*

$$\Phi_\eta(\Lambda)=u^*\left(\Phi_{\eta|_Z}T^*_ZZ\right)\tag{6}$$

where $u: T^*X|_Z\to T^*Z$ is the pullback map and u^* is the flat pullback on analytic cycles.

Proof. Let $u_0: T^*X|_Z\times\mathbb{C}\to T^*Z\times\mathbb{C}$ and $u_1: T^*X|_Z\times\mathbb{P}^1\to T^*Z\times\mathbb{P}^1$ be the product of u and the identity maps on $\mathbb{C}$ and $\mathbb{P}^1$ respectively. Then it follows from definition that $\overline{\Lambda^\diamond_\eta}$ is contained in $T^*X|_Z\times\mathbb{P}^1$, and as analytic cycles on $T^*X|_Z\times\mathbb{C}$ and $T^*X|_Z\times\mathbb{P}^1$ respectively, we have

$$u_0^*\left(\Gamma^\diamond_{\eta|_Z}\right)=\Lambda^\diamond_\eta\quad\text{and}\quad u_1^*\left(\overline{\Gamma^\diamond_{\eta|_Z}}\right)=\overline{\Lambda^\diamond_\eta}.\tag{7}$$

By the flatness of u_1 and the definition of Φ_η, we have

$$u_1^*\left(\overline{\Gamma^\diamond_{\eta|_Z}}\cap T^*Z\times\{\infty\}\right)=\overline{\Lambda^\diamond_\eta}\cap T^*X|_Z\times\{\infty\}=\Phi_\eta(\Lambda).$$

On the other hand, if we let $\Lambda_Z=T^*_ZZ$, then by definition

$$\Phi_{\eta|_Z}(\Lambda_Z)=\overline{\Lambda^\diamond_{Z,\eta|_Z}}\cap T^*Z\times\{\infty\}\quad\text{and}\quad\overline{\Lambda^\diamond_{Z,\eta|_Z}}=\overline{\Gamma^\diamond_{\eta|_Z}}.$$

Thus, the desired equality (6) follows. □

Proposition 3.5. *Given an irreducible conic Lagrangian cycle $\Lambda=T^*_ZX$, $\Phi_\eta\Lambda$ is an effective conic cycle supported in $T^*X|_Z$. Moreover, if η is closed, then $\Phi_\eta\Lambda$ is also Lagrangian.*

Proof. By definition, $\Phi_\eta\Lambda$ is effective. Since the $\mathbb{C}^*$-action on $T^*X\times\mathbb{C}$ induced by the vector bundle structure (as in the proof of Lemma 3.1) extends to $T^*X\times\mathbb{P}^1$, the $\mathbb{C}^*$-action on $\Lambda^\diamond_\eta$ extends to the closure $\overline{\Lambda^\diamond_\eta}$. Thus, the restriction $\overline{\Lambda^\diamond_\eta}$ to $T^*X\times\{\infty\}$ is a conic cycle.

Since $\Phi_\eta\Lambda$ is the limit of $\Lambda^\diamond_\eta\cap(T^*X\times\{s\})$ as s approaches to ∞, and since the limit of Lagrangian cycles is also Lagrangian, it suffices to show that $\Lambda_{\eta,s}:=\Lambda^\diamond_\eta\cap(T^*X\times\{s\})$ is Lagrangian for any $s\in\mathbb{C}$. Since $\Lambda^\diamond_\eta$ is irreducible and of dimension $m+1$, $\Lambda_{\eta,s}$ is pure of dimension m. Here, we recall that m is the dimension of X.

By definition, identifying $T^*X\times\{s\}$ with T^*X, we have (not counting multiplicity)

$$\Lambda_{\eta,s}=\left\{(x,\zeta)\mid\zeta\in\Lambda|_x+s\cdot\Gamma_\eta|_x\right\},$$

where

$$\Lambda|_x+s\cdot\Gamma_\eta|_x=\left\{\alpha+s\beta\mid\alpha\in\Lambda\cap T^*_xX\text{ and }\beta\in\Gamma_\eta\cap T^*_xX\right\}.$$

Since the projection $\Gamma_\eta \to X$ is a finite morphism, over every point of Z, the fibers of the two projections $\Lambda = T^*_Z X \to Z$ and $\Lambda_{\eta,s} \to Z$ have the same dimension. Since Λ is irreducible and $\dim \Lambda = \dim \Lambda_{\eta,s} = m$, by counting dimensions, the restriction of $\Lambda_{\eta,s} \to Z$ to every irreducible component of $\Lambda_{\eta,s}$ is dominant. Thus, it suffices to show that $\Lambda_{\eta,s}$ is Lagrangian over a dense open subset of Z. By the first equation in (7), we have

$$\Lambda_{\eta,s}|_{Z_{\mathrm{reg}}} = u^*\left(\Gamma_{s\eta|_{Z_{\mathrm{reg}}}}\right)$$

where $u : T^*X|_{Z_{\mathrm{reg}}} \to T^*Z_{\mathrm{reg}}$ is the pullback map. Since locally, we can realize X as the product of Z_{reg} and another complex manifold, $u^*(\Gamma_{s\eta|_{Z_{\mathrm{reg}}}})$ is Lagrangian in T^*X if and only if $\Gamma_{s\eta|_{Z_{\mathrm{reg}}}}$ is Lagrangian in T^*Z_{reg}. Locally over the unbranching locus of $s\eta|_{Z_{\mathrm{reg}}}$, $\Gamma_{s\eta|_{Z_{\mathrm{reg}}}}$ is a union of closed holomorphic one-forms. It is a well-known fact that the image of a closed one-form in the cotangent bundle is a Lagrangian submanifold. Since η is a closed multivalued one-form, so is $s\eta|_{Z_{\mathrm{reg}}}$. Hence, over the unbraching locus, $\Gamma_{s\eta|_{Z_{\mathrm{reg}}}}$ is a Lagrangian submanifold of T^*Z_{reg}, and we have finished the proof. □

The following Proposition is analogous to the fact that the vanishing cycle of f is supported on the critical locus of f.

Proposition 3.6. *Let X be a complex manifold and let Z be a compact irreducible analytic subvariety of X. Denote by $\Lambda = T^*_Z X$. If η is a d-valued closed one-form on X, then*

$$\Phi_\eta(\Lambda) = n_0\Lambda + \sum_{1\le i\le m} n_i\, T^*_{Z_i}X \tag{8}$$

where n_0 is the multiplicity of the zero form in $\eta|_{Z_{\mathrm{reg}}}$ and all Z_i are proper closed subvarieties of Z. In particular, if $\eta|_{Z_{\mathrm{reg}}}$ is non-trivial, then $n_0 < d$.

Proof. It follows from definition that $\Phi_\eta \subset T^*X|_Z$. Thus, by Proposition 3.5, we have the presentation (8) with $n_i \ge 0$ for all i. The only remaining statement is that n_0 is equal to the multiplicity of the zero form in $\eta|_{Z_{\mathrm{reg}}}$. By Lemma 3.4, it suffices to prove the statement in the case when $Z = X$ and $\Lambda = T^*_X X$. Restricting to a small ball in the unbranching locus of η, we can also assume that η is the union of d single-valued one-forms η_j, $1 \le j \le d$. In this case,

$$\Phi_\eta(T^*_X X) = \sum_{1\le j\le d} \Phi_{\eta_i}(T^*_X X).$$

Obviously, if $\eta_j = 0$, then $\Phi_{\eta_j}(T^*_X X) = T^*_X X$. Otherwise, $\Phi_{\eta_j}(T^*_X X)$ is supported on a proper closed subset of X. Thus, n_0 is equal to the number of η_j which are zero one-forms. □

Remark 3.7. Assume that $x \in Z_{\mathrm{reg}}$ is in the unbranching locus of η. If near x, some branch of $\eta|_{Z_{\mathrm{reg}}}$ has nonempty degenerating locus, then it follows from the definition of Φ_η that $n_i > 0$ for some $0 \le i \le m$. Furthermore, if the degenerating locus has dimension strictly less than $\dim Z$, then $n_i > 0$ for some $1 \le i \le m$.

We have proved that Φ_η maps conic Lagrangian cycles to conic Lagrangian cycles. Hence it induces a group homomorphism $\Phi_\eta : L(X) \to L(X)$, which we call *the vanishing cycle functor* of η. The following proposition justifies this name.

Proposition 3.8 (Massey). *If f is a holomorphic function on a complex manifold X, then Φ_{df} is the total vanishing cycle functor. In other words, given a conic Lagrangian cycle Λ in T^*X,*

$$\Phi_{df}(\Lambda) = \sum_{t\in\mathbb{C}} \Phi_{f-t}(\Lambda) \tag{9}$$

where Φ_{f-t} is the standard vanishing cycle functor of the holomorphic function $f-t$. Note that the sum on the right-hand side is a locally finite sum, because restricting to a small ball in X, there are only finitely many t such that $\Phi_{f-t}(\Lambda) \neq 0$.

Proof. In this case, Γ_η is a section of T^*X. Then $\Phi_\eta\Lambda$ is the deformation to the normal cone (Remark 3.3), the equation (9) is equivalent to [Mas00, Theorem 2.10]. □

3.2. Some properties of Φ_η.

Proposition 3.9. *Suppose η is a d-valued closed one-form. For any $\Lambda \in L(X)$, we have*

$$d\,\Lambda \cdot T^*_X X = \Phi_\eta(\Lambda) \cdot T^*_X X$$

where $\cdot$ denotes the intersection number in T^*X.

Proof. Recall that $\Phi_\eta\Lambda$ is defined to be the restriction of $\overline{\Lambda^\circ_\eta}$ to $T^*X\times\{\infty\}$. Since the intersection number is invariant under rational equivalence,

$$\Phi_\eta(\Lambda)\cdot T^*_X X = \left(\overline{\Lambda^\circ_\eta} \cap (T^*X\times\{0\})\right)\cdot T^*_X X.$$

By definition, we have $\overline{\Lambda^\circ_\eta} \cap (T^*X\times\{0\}) = d\Lambda$ as n-cycles on T^*X. Therefore, the desired equality follows. □

Corollary 3.10. *Let X be a complex manifold and Z be a compact irreducible subvariety of X. If there exists a closed multivalued one-form η on X whose restriction to Z is non-trivial, then there exist proper closed subvarieties Z_i of Z and $\lambda_i \in \mathbb{Q}_{\geq 0}$ such that*

$$T^*_Z X\cdot T^*_X X = \sum_{1\leq i\leq m} \lambda_i\, T^*_{Z_i}X\cdot T^*_X X.$$

Proof. By Propositions 3.6 and 3.9, we have

$$dT^*_Z X\cdot T^*_X X = \Phi_\eta(\Lambda)\cdot T^*_X X = \Big(n_0 T^*_Z X + \sum_{1\leq i\leq m} n_i\, T^*_{Z_i}X\Big)\cdot T^*_X X.$$

By the assumption that the restriction of η to Z_{reg} is non-trivial, $n_0 < d$, and we have

$$T^*_Z\cdot T^*_X X = \sum_{1\leq i\leq m} \frac{n_i}{d-n_0}\, T^*_{Z_i}X\cdot T^*_X X. \qquad \square$$

The classical vanishing cycle functor is defined for constructible complexes. So we end the section with the following question.

Question 3.11. Let η be a multi-valued closed one-form on a complex manifold X. Is there a natural lifting of Φ_η to a functor of constructible complexes $\Phi_\eta : D^b_c(X,\mathbb{C}) \to D^b_c(X,\mathbb{C})$?

3.3. Proof of Theorem 1.5 in the case when $\mathrm{char}(K) > 0$.

Proposition 3.12. *Let X be a smooth projective variety and let $\rho : \pi_1(X) \to \mathrm{GL}(r, K)$ be a linear representation where K is a field of characteristic $p > 0$. If ρ is large, then there exists a closed mutivalued one-form η on X such that for any positive-dimensional closed subvariety Z of X, the restriction of η to $Z|_{\mathrm{reg}}$ is non-trivial.*

Proof. By [DY24, Theorem 2.7 & Corollary 2.10], there exists a semisimple representation $\tau : \pi_1(X) \to \mathrm{GL}(N, L)$ where L is a non-archimedean local field of characteristic p such that the Katzarkov-Eyssidieux reduction map s_τ is the Shafarevich morphism of ρ. Since ρ is large, its Shafarevich morphism is the identity map. Hence s_τ is the identity map. Let η be the associated closed multivalued one-form in Proposition 2.10. By the equivalence of item 3 and item 4 in Proposition 2.10, for any positive dimensional subvariety Z, the restriction $\eta|_{Z_{\mathrm{reg}}}$ is non-trivial. The proposition is proved. □

Proof of Theorem 1.5 assuming $\mathrm{char}(K) > 0$. In this case the axioms of Proposition 3.12 are satisfied, and hence we have a closed multivalued one-form η whose restriction to any positive dimensional subvariety is non-trivial. Iterating Corollary 3.10, we can express $T^*_Z X \cdot T^*_X X$ as a sum of finitely many $\lambda_i T^*_{Z_i} X \cdot T^*_X X$, where $\lambda_i > 0$ and Z_i is a point. If Z_i is a point, then $T^*_{Z_i} X \cdot T^*_X X = 1$. Thus, $T^*_Z X \cdot T^*_X X \geq 0$. □

4. Period maps of $\mathbb{C}$-VHS and positivity

4.1. Positivity from the period maps.

In this subsection, we prove the following generalization of [AW21, Theorem 1.9].

Proposition 4.1. *Let X be a projective manifold with a representation $\sigma : \pi_1(X) \to \mathrm{GL}(N, \mathbb{C})$ such that the associated local system L_σ underlies a $\mathbb{C}$-VHS. Assume that Z is an irreducible subvariety of X such that*

(1) the pullback of σ to $\pi_1(Z_{\mathrm{norm}})$ is large;
(2) $\Gamma \coloneqq \sigma(\mathrm{Im}[\pi_1(Z_{\mathrm{norm}}) \to \pi_1(X)])$ is a discrete subgroup of $\mathrm{GL}(N, \mathbb{C})$.

Then the intersection number $T^*_Z X \cdot T^*_X X \geq 0$. Equivalently, $(-1)^{\dim Z} \chi(Eu_Z) \geq 0$.

When Z happens to be normal, the proposition follows immediately from the arguments of [AW21]. However, some technical arguments are required to deal with the non-normal case.

First, we need to reduce to the case when Γ is torsion free.

Lemma 4.2. *Let X be a projective manifold, and $\pi : \widetilde{X} \to X$ be a finite covering map. For an irreducible subvariety Z of X, let $\widetilde{Z}$ be a connected component of $\pi^{-1}(Z)$. Then,*

$$T^*_{\widetilde{Z}} \widetilde{X} \cdot T^*_{\widetilde{X}} \widetilde{X} = d\, T^*_Z X \cdot T^*_X X$$

where d is the degree of the covering map $\pi|_{\widetilde{Z}} : \widetilde{Z} \to Z$.

Proof. Since $\pi|_{\widetilde{Z}} : \widetilde{Z} \to Z$ is a covering map of degree d, the pushforward of $Eu_{\widetilde{Z}}$ under $\pi|_{\widetilde{Z}}$ is equal to $d \cdot Eu_Z$. Hence

$$\chi\big(Eu_{\widetilde{Z}}\big) = \chi\big(\pi|_{\widetilde{Z}*}(Eu_{\widetilde{Z}})\big) = d \cdot \chi(Eu_Z).$$

By (3), the desired equation follows. □

By [Sel60, Lemma 8], $\sigma(\pi_1(X))$ has a torsion-free finite index subgroup. Replacing X by the induced finite cover, without loss of generality, we can assume that $\sigma(\pi_1(X))$ is torsion free. Then as a subgroup, Γ is also torsion free.

By the argument of [AW21], we know that there is a finite period map $Z_{\text{norm}} \to \Gamma\backslash D$. Since $\Gamma\backslash D$ has non-positive holomorphic bisectional curvature in the horizontal directions, we can deduce the fact that $(-1)^{\dim Z}\chi(Eu_{Z_{\text{norm}}}) \geq 0$. However, this is weaker than the inequality $(-1)^{\dim Z}\chi(Eu_Z) \geq 0$. To achieve the latter inequality, we introduce a new constructible function δ on Z_{norm} as follows.

Definition 4.3. Let Z be an algebraic variety and let $p : Z_{\text{norm}} \to Z$ be the normalization map. Given any point $x \in Z_{\text{norm}}$, we choose a small open neighborhood $U_x \subset Z_{\text{norm}}$ of x. The value of δ at x is defined to be the value of $(-1)^{\dim Z} Eu_{p(U_x)}$ at $p(x)$.

The function δ is constructible with respect to a stratification of the map $p : Z_{\text{norm}} \to Z$ (see [GM88, Chapter 1, 1.6 and 1.7]).

Example 4.4. Suppose that Z is a projective curve with two singular points: a cusp P and a node Q. Then the above defined function δ has value -1 on every point of Z_{norm} except at the preimage P' of P, where the value is -2. In this case, δ is not CC-effective on Z_{norm}, because

$$\delta = (-1)^{\dim Z} 1_{Z_{\text{norm}}} - 1_{P'},$$

where $1_{Z_{\text{norm}}} = Eu_{Z_{\text{norm}}}$, since Z_{norm} is smooth. Such Z could appear in Proposition 4.1, and in such case, we will see that the image of Z_{norm} in the period domain $\Gamma\backslash D$ must at least have a cusp singularity at the image of P'. So the pushforward of δ in $\Gamma\backslash D$ will be CC-effective.

Lemma 4.5. *Let $p : Z_{\text{norm}} \to Z$ and the constructible function δ be as in Definition 4.3. If $U \subset Z_{\text{norm}}$ is open, then $(p|_U)_*(\delta) = (-1)^{\dim Z} Eu_{p(U)}$. In particular, $p_*(\delta) = (-1)^{\dim Z} Eu_Z$ and $\chi(\delta) = (-1)^{\dim Z}\chi(Eu_Z)$.*

Proof. The lemma follows from the definition of δ and the fact that for a reducible analytic variety, its Euler obstruction function equals to the sum of the Euler obstruction of every irreducible components. ◻

Let $\pi : \widetilde{X} \to X$ be the covering map induced by $\ker(\sigma)$. Let $\pi_{Z_{\text{norm}}} : \widetilde{Z}_{\text{norm}} \to Z_{\text{norm}}$ be the covering map induced by $\ker[\pi_1(Z_{\text{norm}}) \to \pi_1(X) \xrightarrow{\sigma} \mathrm{GL}(N, \mathbb{C})]$. Then the map $Z_{\text{norm}} \to X$ lifts to a map $\widetilde{Z}_{\text{norm}} \to \widetilde{X}$. Let $\widetilde{Z}$ be the image of $\widetilde{Z}_{\text{norm}}$ in $\widetilde{X}$. Then we have the following commutative diagram:

$$\begin{tikzcd}
\widetilde{Z}_{\text{norm}} \arrow[r, "\tilde{p}"] \arrow[d, "\pi_{Z_{\text{norm}}}"] & \widetilde{Z} \arrow[r, "\tilde{\iota}"] \arrow[d, "\pi_Z"] & \widetilde{X} \arrow[d, "\pi"] \\
Z_{\text{norm}} \arrow[r, "p"] & Z \arrow[r, "\iota"] & X.
\end{tikzcd}$$

It follows from definition that $\widetilde{Z}_{\text{norm}}$ is a connected component of the fiber product $Z_{\text{norm}} \times_X \widetilde{X}$. Since $\iota \circ p : Z_{\text{norm}} \to X$ is a finite morphism, so are $\tilde{\iota} \circ \tilde{p} : \widetilde{Z}_{\text{norm}} \to \widetilde{X}$ and $\tilde{p} : \widetilde{Z}_{\text{norm}} \to \widetilde{Z}$ (of analytic varieties).

Remark 4.6. The map π_Z is a covering map over the Zariski open subset of Z where it is locally irreducible. For example, if Z is curve, everywhere smooth except at one nodal singular point, then $\widetilde{Z}$ is a covering space of either Z or Z_{norm} depending on whether the two compositions $\pi_1(Z_{\mathrm{norm}}) \to \pi_1(X) \xrightarrow{\sigma} \mathrm{GL}(N, \mathbb{C})$ and $\pi_1(Z) \to \pi_1(X) \xrightarrow{\sigma} \mathrm{GL}(N, \mathbb{C})$ have the same image or not.

Lemma 4.7. *Let $\tilde{\delta} = \pi^*_{Z_{\mathrm{norm}}}(\delta)$. Then $\tilde{p}_*(\tilde{\delta})$ is CC-effective.*

Proof. Since ι is a closed embedding, it suffices to show that $(\tilde{\iota} \circ \tilde{p})_*(\tilde{\delta})$ is CC-effective. By definition, near the image of $\tilde{\iota} \circ \tilde{p}$, $(\iota \circ \tilde{p})_*(\tilde{\delta})$ is equal to $\pi^*((\iota \circ p)_*\delta)$. Since π is a covering map, it suffices to show that $(\iota \circ p)_*\delta$ is CC-effective. By Lemma 4.5, $p_*\delta$ is CC-effective. Since ι is a closed embedding, $(\iota \circ p)_*\delta$ is also CC-effective. □

By the definition of $\widetilde{X}$, the pullback of L_σ to $\widetilde{X}$ becomes a trivial local system. Thus, the $\mathbb{C}$-VHS structure on L_σ induces a period map

$$\widetilde{\phi} : \widetilde{X} \to D.$$

Denote the pullback of $\widetilde{\phi}$ to $\widetilde{Z}$ by $\widetilde{\phi}_Z : \widetilde{Z} \to D$. Now, the deck transformation group of the normal covering map $\widetilde{Z}_{\mathrm{norm}} \to Z_{\mathrm{norm}}$ is equal to Γ, and Γ acts on D via the monodromy action of the $\mathbb{C}$-VHS L_σ. Thus, Γ acts equivariantly on the composition $\tilde{\phi} \circ \tilde{p} : \widetilde{Z}_{\mathrm{norm}} \to D$, and taking quotient by the Γ-action, we have the commutative diagram

$$\begin{tikzcd} \widetilde{Z}_{\mathrm{norm}} \arrow[r, "\tilde{p}"] \arrow[d] & \tilde{Z} \arrow[r, "\tilde{\phi}_Z"] & D \arrow[d, "\pi_D"] \\ Z_{\mathrm{norm}} \arrow[rr, "\phi"] & & \Gamma\backslash D. \end{tikzcd} \tag{10}$$

Since Γ is torsion free and discrete, we know that $\Gamma\backslash D$ is a complex manifold.

Since the pullback of σ to $\pi_1(Z_{\mathrm{norm}})$ is large, by [AW21, Proof of Theorem 1.9], we have the following.

Lemma 4.8. *The period map $\phi : Z_{\mathrm{norm}} \to \Gamma\backslash D$ is a finite morphism.* □

Proposition 4.9. *Let δ be the constructible function as in Definition 4.3. Then the pushforward $\phi_*(\delta)$ is CC-effective.*

Proof. The proof will be based on the commutative diagram (10).

Let $\tilde{\delta}$ be the pullback of δ to the covering space $\widetilde{Z}_{\mathrm{norm}}$. Notice that ignoring $\widetilde{Z}$, (10) is a Cartesian square. Thus, it suffices to show that $(\tilde{\phi}_Z \circ \tilde{p})_*(\tilde{\delta})$ is CC-effective. By Lemma 4.7, $\tilde{p}_*(\tilde{\delta})$ is CC-effective. Since $\tilde{\phi}_Z$ is a finite morphism, by Proposition 2.5, $(\tilde{\phi}_Z \circ \tilde{p})_*(\tilde{\delta})$ is CC-effective. □

Remark 4.10. The horizontal subbundle $T^{-1,1}D$ defined in section 2.7 is preserved by the Γ-action, and hence descends to a subbundle $T^{-1,1}(\Gamma\backslash D)$. A subvariety Y of $\Gamma\backslash D$ is called horizontal if the image of TY_{reg} in $T(\Gamma\backslash D)|_{Y_{\mathrm{reg}}}$ is contained in $T^{-1,1}(\Gamma\backslash D)|_{Y_{\mathrm{reg}}}$. The image of a period map is always horizontal. It is proved in [AW21, Corollary 5.4] that the subvariety of a horizontal variety is also horizontal.

The following proposition was proved using the curvature property of the period domain.

Proposition 4.11. [AW21, Proposition 6.1] *Let Y be a horizontal compact irreducible analytic subvariety of $\Gamma\backslash D$. Then*

$$T_Y^*\Gamma\backslash D \cdot T_{\Gamma\backslash D}^*\Gamma\backslash D \geq 0$$

as intersection number in $T^*\Gamma\backslash D$. Equivalently, $(-1)^{\dim Y}\chi(Eu_Y) \geq 0$.

Proof of Proposition 4.1. By Lemma 4.5,

$$(-1)^{\dim Z}\chi(Eu_Z) = \chi(\delta). \tag{11}$$

Since $\phi : Z_{\mathrm{norm}} \to \Gamma\backslash D$ is a finite morphism,

$$\chi(\delta) = \chi(\phi_*(\delta)). \tag{12}$$

Moreover, the support of $\phi_*(\delta)$ is compact. By Proposition 4.9, we know that

$$\phi_*(\delta) = \sum_{1\leq i\leq m} (-1)^{\dim Y_i} n_i Eu_{Y_i} \tag{13}$$

for irreducible compact subvarieties Y_i of $\Gamma\backslash D$ and $n_i > 0$. By Remark 4.10, $\phi(Z_{\mathrm{norm}})$ is horizontal. Since all Y_i are contained in $\phi(Z_{\mathrm{norm}})$, they are also horizontal. Thus, by Proposition 4.11, $(-1)^{\dim Y_i}\chi(Eu_{Y_i}) \geq 0$ for all i. Therefore, by (13), $\chi(\phi_*(\delta)) \geq 0$. By (11) and (12), $(-1)^{\dim Z}\chi(Eu_Z) \geq 0$. □

4.2. **Proof of Theorem 1.5 in the case when $K = \mathbb{C}$ and ρ is semisimple.** Let $\rho : \pi_1(X) \to \mathrm{GL}(r, \mathbb{C})$ be a large and semisimple representation. The proof of the reductive Shafarevich conjecture as in [DYK23] gives us the desired multivalued one-forms and $\mathbb{C}$-VHS as in the following proposition.

Proposition 4.12. *Let X be a projective manifold and let $\rho : \pi_1(X) \to \mathrm{GL}(r, \mathbb{C})$ be a large and semisimple representation. Then there exists a semisimple representation $\sigma : \pi_1(X) \to \mathrm{GL}(N, \mathbb{C})$ on X underlying a $\mathbb{C}$-VHS, and a closed multivalued one form η on X such that for any irreducible subvariety Z of X, one of the following two statements holds.*

(1) The restriction of η to Z is non-trivial.
(2) The pullback of σ to the normalization Z_{norm} of Z is large and has discrete monodromy.

Proof. Let $s_{\mathrm{fac},r} : X \to S_{\mathrm{fac},r}$ be the fibration defined in Definition 2.11. By [DYK23, Proposition 3.13], there exist a semisimple representation $\sigma : \pi_1(X) \to \mathrm{GL}(N, \mathbb{C})$ underlying a $\mathbb{C}$-VHS, such that for any closed subvariety Z with $s_{\mathrm{fac},r}(Z)$ being a point, the following properties hold:

(1) let $p : Z_{\mathrm{norm}} \to Z$ be the normalization map. Then the image of $p^*\sigma : \pi_1(Z_{\mathrm{norm}}) \to \mathrm{GL}(N, \mathbb{C})$ is a discrete subgroup.
(2) For each semisimple representation $\tau : \pi_1(X) \to \mathrm{GL}(r, \mathbb{C})$, $p^*\tau$ is conjugate to a direct factor of $p^*\sigma : \pi_1(Z_{\mathrm{norm}}) \to \mathrm{GL}(N, \mathbb{C})$.

Since ρ is large, $p^*\rho : \pi_1(Z_{\mathrm{norm}}) \to \mathrm{GL}(N, \mathbb{C})$ is also large. Item 2 implies that $p^*\sigma : \pi_1(Z_{\mathrm{norm}}) \to \mathrm{GL}(N, \mathbb{C})$ is large.

By [DYK23, Theorem 1.28], there exist semisimple representations $\{\rho_i : \pi_1(X) \to \mathrm{GL}(r, K_i)\}_{i=1,\ldots,k}$ where each K_i is a non-archimedean local field of characteristic zero such that the Stein factorization of $(s_{\rho_1}, \ldots, s_{\rho_k}) : X \to S_{\rho_1} \times \cdots \times S_{\rho_k}$ coincides with $s_{\mathrm{fac},r} : X \to S_{\mathrm{fac},r}$.

Recall in Proposition 2.10, each ρ_i gives rise to a multivalued one-form η_i on X such that the properties in Proposition 2.10 are equivalent. Let η be the union of $\eta_1, \dots, \eta_k$.

Suppose that Z is a closed subvariety of X such that $\eta|_{Z_{\mathrm{reg}}}$ is trivial. Then we need to show that $s_{\mathrm{fac},r}(Z)$ is not a point. In fact, since $\eta|_{Z_{\mathrm{reg}}}$ is trivial, each $\eta_i|_{Z_{\mathrm{reg}}}$ is trivial. By Proposition 2.10, $s_{\rho_i}(Z)$ is a point for each i. By the property of the simultaneous Stein factorization, $s_{\mathrm{fac},r}(Z)$ is also a point. So the above property (1) holds, that is, the pullback of σ to Z_{norm} is large and has discrete monodromy. The proposition is proved. □

Therefore, given any conic Lagrangian cycle $\Lambda \subset T^*X$, we can apply Proposition 4.12 and Corollary 3.10 to iterate the vanishing cycle functor Φ_η to achieve the following.

Corollary 4.13. *Let X be a smooth projective variety and let $\rho : \pi_1(X) \to \mathrm{GL}(r, \mathbb{C})$ be a semisimple and large representation. Let $\sigma : \pi_1(X) \to \mathrm{GL}(N, \mathbb{C})$ be the representation constructed in Proposition 4.12, and let L_σ be the associated local system. Then, given any conic Lagrangian cycle $T^*_Z X$, there exist subvarieties Z_i and rational numbers $\mu_i > 0$ with $1 \le i \le m$ satisfying the following properties.*

(1) As intersection numbers in T^*X, we have

$$T^*_Z X \cdot T^*_X X = \sum_{1\le i\le m} \mu_i T^*_{Z_i} X \cdot T^*_X X.$$

(2) For each i, the pullback of L_σ to the normalization $Z_{i,\mathrm{norm}}$ of Z_i is large and has discrete monodromy.

Remark 4.14. If $\eta = df$ for some holomorphic function f on X, then the support of $\Phi_\eta(T^*_Z X)$ in X is contained in the critical locus of $f|_Z$ in the stratified sense. In particular, f has constant value on each irreducible component of the support of $\Phi_\eta(T^*_Z X)$. Consequently, $\Phi_\eta \circ \Phi_\eta(T^*_Z X) = \Phi_\eta(T^*_Z X)$. However, when η is a closed multivalued one-form on X, the above identity may not hold even up to a scalar. In fact, over a small ball in the unbranching locus X°, η can be considered as a union of closed one-forms $\eta_1, \dots, \eta_l$ (possibly with multiplicity), and locally we can assume $\eta_i = df_i$. Then locally, $\Phi_\eta = \sum_{1\le i\le l} \Phi_{df_i}$. Even though $\Phi_{df_i} \circ \Phi_{df_i}(T^*_Z X) = \Phi_{df_i}(T^*_Z X)$, we have no control about the terms $\Phi_{df_i} \circ \Phi_{df_j}(T^*_Z X)$.

Proof of Theorem 1.5 in the case when $K = \mathbb{C}$ and ρ is semisimple. By Corollary 4.13 (1), it is sufficient to show that $T^*_{Z_i} X \cdot T^*_X X \ge 0$, where Z_i satisfies the conditions in Corollary 4.13 (2). By Proposition 4.1, the above inequality holds. □

5. Mixed period maps of $\mathbb{R}$-VMHS and positivity

5.1. **Positivity from the mixed period maps.** In this subsection, we prove the following analogue of Proposition 4.1 for mixed period maps.

Proposition 5.1. *Let X be a projective manifold with two representations $\{\sigma_i : \pi_1(X) \to \mathrm{GL}(N_i, \mathbb{C})\}_{i=1,2}$ such that*

(1) σ_1 is semisimple and L_{σ_1} underlies a $\mathbb{C}$-VHS;
(2) L_{σ_2} underlies a $\mathbb{R}$-VMHS of weight $-1, 0$.

Let Z be an irreducible subvariety of X, and let $p : Z_{\mathrm{norm}} \to Z$ be its normalization. Let $\Gamma_1 := \sigma_1(\mathrm{Im}[\pi_1(Z_{\mathrm{norm}} \to \pi_1(X)])$ and $\Gamma_2^{ss} := \sigma_2^{ss}(\mathrm{Im}[\pi_1(Z_{\mathrm{norm}}) \to \pi_1(X)])$, where

$\sigma_2^{ss} : \pi_1(X) \to \mathrm{GL}(N_2, \mathbb{C})$ is the semisimplification of σ_2, which underlies a $\mathbb{C}$-VHS $L_{\sigma_2^{ss}}$. Assume that

(i) the product of the period map of $p^*\sigma_1$ and the mixed period map of $p^*\sigma_2$, denote by $\widetilde{Z}^{\mathrm{univ}}_{\mathrm{norm}} \to \mathscr{D}_1 \times \mathscr{M}$, has discrete fibers. Here $\widetilde{Z}^{\mathrm{univ}}_{\mathrm{norm}}$ is the universal cover of Z_{norm}, $\mathscr{D}_1$ is the period domain of L_{σ_1}, and $\mathscr{M}$ is the mixed period domain of L_{σ_2}.
(ii) The monodromy group Γ_1 (resp. Γ_2^{ss}) acts on $\mathscr{D}_1$ (resp. $\mathscr{D}_2$) respectively. Here $\mathscr{D}_2$ is the period domain of the $\mathbb{C}$-VHS $L_{\sigma_2^{ss}}$.

Then, the intersection number $T^*_Z X \cdot T^*_X X \geq 0$. Equivalently, $(-1)^{\dim Z}\chi(Eu_Z) \geq 0$.

Using the same finite covering trick as in the beginning of the previous section, we can reduce to the case when Γ_1 and Γ_2^{ss} are torsion free. We will work under this assumption for the remaining of this subsection.

Since the image of $\sigma_2 : \pi_1(X) \to \mathrm{GL}(N_2, \mathbb{R})$ may not be discrete, we will need a technical proposition to deduce desired positivity. Before stating the proposition, we need a lemma.

Lemma 5.2. *Let $0 \to N \to M \to M' \to 0$ be a short exact sequence of holomorphic vector bundles on a complex manifold X. Assume that M' is trivial of rank r. Consider N as a complex submanifold of M, and denote by $\iota : N \to M$ the inclusion map. Let C be an equi-dimensional subvariety of M, and $[C] \in H^{BM}_{2\dim C}(C, \mathbb{Z})$ be the fundamental class of C. Then $\iota^*[C] \in H^{BM}_{2\dim C-2r}(C \cap N, \mathbb{Z})$ can be represented by an effective analytic $(\dim C - r)$-cycle in $C \cap N$, where the pullback class $\iota^*[C]$ is defined as the Borel-Moore homology analogue of* [Ful98, Chapter 8].

Proof. Since M' is a trivial bundle, the normal bundle of N in M is also trivial. Using the deformation to normal cone [Ful98, Chapter 5], we can replace M by a trivial vector bundle over N and C a cone of the new vector bundle M. Now, we can take a general global section Γ of M, and the image of $\Gamma \cap C$ in N is contained in $N \cap C$ and it represents the class $\iota^*[C]$. □

Proposition 5.3. *Let $\pi_B : E \to B$ be a holomorphic vector bundle over a complex ball B. Suppose that there is a $\mathbb{Z}^k$-action on E by fiberwise translations, i.e., there are sections $s_1, \ldots, s_k$ of the vector bundle, and $(m_1, \ldots, m_k) \in \mathbb{Z}^k$ acts on E by fiberwise addition with $m_1 s_1 + \cdots + m_k s_k$. Let $\pi_X : \widetilde{X} \to X$ be a $\mathbb{Z}^k$-covering map of complex manifolds. Denote the support of the constructible function γ by* $\mathrm{Supp}(\gamma)$. *Suppose we have a commutative diagram*

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\tilde{f}"] \arrow[d, "\pi_X"] & E \arrow[d, "\pi_B"] \\ X \arrow[r, "f"] & B \end{tikzcd} \tag{14}$$

where $\tilde{f}$ is a $\mathbb{Z}^k$-equivariant holomorphic map. If γ is a constructible function on X satisfying

(1) for any $e \in E$, $\pi_X^{-1}\,\mathrm{Supp}(\gamma) \cap \tilde{f}^{-1}(e)$ is a discrete set,
(2) $f|_{\mathrm{Supp}(\gamma)} : \mathrm{Supp}(\gamma) \to B$ is proper,
(3) γ is CC-effective,

then $f_*(\gamma)$ is also CC-effective.

Proof. Since being CC-effective is a local property, by possibly shrinking B, we can assume that γ is constructible with respect to a finite Whitney stratification of X. Furthermore, without loss of generality, we can assume that $\gamma = (-1)^{\dim Z} Eu_Z$ for some irreducible closed analytic subvariety Z of X. Then $CC(\gamma) = T^*_Z X$, which we denote by Λ. We will construct some an effective conic Lagrangian cycle on B and show that it is equal to $CC(f_*(\gamma))$.

Consider the maps of vector bundles on $\widetilde{X}$:

$$\pi_X^* f^* T^* B = \tilde{f}^* \pi_B^* T^* B \to \tilde{f}^* T^* E \to T^* \widetilde{X},$$

induced by $\tilde{f}$ and π_B. Since $\tilde{f}$ is $\mathbb{Z}^k$-equivariant, there is a natural $\mathbb{Z}^k$-action on $\tilde{f}^* T^* E$. There are also obvious $\mathbb{Z}^k$-actions on $\pi_X^* f^* T^* B$ and $T^* \widetilde{X}$, and moreover, the above two maps are $\mathbb{Z}^k$ equivariant. Taking quotient of the $\mathbb{Z}^k$-action, we have a commutative diagram

$$\begin{tikzcd}
\tilde{f}^* \pi_B^* T^* B \arrow[r, "\tilde{\iota}"] \arrow[d] & \tilde{f}^* T^* E \arrow[r, "\tilde{\nu}_1"] \arrow[d, "\pi_E"] & T^* \widetilde{X} \arrow[d, "\pi_T"] \\
f^* T^* B \arrow[r, "\iota"] & \tilde{f}^* T^* E / \mathbb{Z}^k \arrow[r, "\nu_1"] & T^* X
\end{tikzcd}$$

where the first row consists of vector bundles on $\widetilde{X}$, the second row consists of vector bundles on X, all horizontal maps are vector bundle maps and vertical maps are covering maps.

Claim 5.4. *The set-theoretic preimage $\nu_1^{-1}(\Lambda)$ is an analytic subset of dimension at most* $\dim E$.

Proof of the claim. Let $\widetilde{\Lambda} = \pi_T^{-1}(\Lambda)$. Then $\widetilde{\Lambda} = T^*_{\pi_X^{-1} Z} \widetilde{X}$ is a conic Lagrangian (not necessarily connected) subvariety of $T^* \widetilde{X}$.

Consider the maps induced by $\tilde{f} : \widetilde{X} \to E$:

$$T^* \widetilde{X} \xleftarrow{\tilde{\nu}_1} \tilde{f}^* T^* E \xrightarrow{\tilde{\nu}_2} T^* E.$$

It follows from [Kas77, Proposition 4.9] that there is a locally finite conic Lagrangian cycle Λ' in $T^* E$ such that $\tilde{\nu}_1^{-1}(\widetilde{\Lambda}) \subset \nu_2^{-1}(\Lambda')$. The property (1) of the proposition implies that the preimage of any point under $\tilde{\nu}_2$ is also a discrete set. Thus,

$$\dim \tilde{\nu}_1^{-1}(\widetilde{\Lambda}) \leq \dim \nu_2^{-1}(\Lambda') \leq \dim \Lambda' = \dim E,$$

where $\dim E$ is the dimension of the total space of E. Since $\tilde{\nu}_1^{-1}(\widetilde{\Lambda})$ is a covering space of $\nu_1^{-1}(\Lambda)$, it follows that $\dim \nu_1^{-1}(\Lambda) \leq \dim E$. □

Consider the following maps induced by $f : X \to B$:

$$T^* X \xleftarrow{u_1} f^* T^* B \xrightarrow{u_2} T^* B.$$

Notice that the composition $\tilde{\nu}_1 \circ \tilde{\iota} : (\tilde{f} \circ \pi_B)^* T^* B \to T^* \widetilde{X}$ is equal to the natural pullback map of cotangent bundle induced by $\tilde{f} \circ \pi_B$. Thus, the quotient map $\nu_1 \circ \iota : f^* T^* B \to T^* X$ is equal to the natural pullback map u_1.

Notice that on E, we have a short exact sequence of holomorphic vector bundles

$$0 \to \pi_B^* T^* B \to T^* E \to \pi_B^* E^\vee \to 0,$$

where $E^\vee$ is the dual vector bundle of E. Taking the pullback to $\widetilde{X}$, we have

$$0 \to \tilde{f}^* \pi_B^* T^* B \to \tilde{f}^* T^* E \to \tilde{f}^* E^\vee \to 0.$$

Taking the quotient by the $\mathbb{Z}^k$-action, we have a short exact sequence of vector bundles on X:

$$0 \to f^*T^*B \to \tilde{f}^*T^*E/\mathbb{Z}^k \to f^*E^\vee \to 0.$$

Now, take the scheme-theoretic preimage $v_1^{-1}(\Lambda)$, which, by Claim 5.4, can be regarded as an effective analytic $\dim(E)$-cycle in $\tilde{f}^*T^*E/\mathbb{Z}^k$. By the above argument and Lemma 5.2, there exists an effective analytic $\dim(B)$-cycle Λ'' on $\iota^{-1}v_1^{-1}(\Lambda)$, representing the class $\iota^*[v_1^{-1}(\Lambda)]$. Since f is a proper map, so is u_2, and hence $u_{2\star}(\Lambda'')$ is a well-defined $\dim(B)$-analytic cycle in T^*B.

Claim 5.5. *The analytic* $\dim(B)$*-cycle* $u_{2\star}(\Lambda'')$ *is equal to* $CC(f_*(\gamma))$*. In particular,* $u_{2\star}(\Lambda'')$ *does not depend on the choice of* Λ'' *we made when applying Lemma 5.2.*

Proof of the claim. Let us give the precise Borel-Moore homology groups in which each cycle/class is defined. As in section 2.3, there exist Whitney stratifications $\mathcal{S}$ and $\mathcal{S}'$ of X and B respectively such that γ is constructible with respect to $\mathcal{S}$ and f is a stratified map with respect to $\mathcal{S}$ and $\mathcal{S}'$. By possibly shrinking B, we can assume that both $\mathcal{S}$ and $\mathcal{S}'$ are finite stratifications.

Now, we can regard $CC(\gamma) = T^*_Z X = \Lambda$ as an element in $H^{BM}_{2\dim X}(T^*_{\mathcal{S}}X, \mathbb{Z})$. Then the scheme-theoretic preimage $v_1^{-1}(\Lambda)$, considered as an analytic $\dim(E)$-cycle, represents the element $v_1^*(\Lambda) \in H^{BM}_{2\dim E}(v_1^{-1}(T^*_{\mathcal{S}}X), \mathbb{Z})$. Since $v_1 \circ \iota = u_1$, we have

$$\iota^* v_1^*(\Lambda) = u_1^*(\Lambda) = [\Lambda''] \in H^{BM}_{2\dim B}(u_1^{-1}(T^*_{\mathcal{S}}X), \mathbb{Z}).$$

Then,

$$u_{2*}u_1^*(\Lambda) = [u_{2\star}(\Lambda'')] \in H^{BM}_{2\dim B}(T^*_{\mathcal{S}'}B, \mathbb{Z}). \tag{15}$$

By section 2.3, $u_{2\star}(\Lambda'')$ and $CC(f_*(\gamma))$ represent the same homology classes in $H^{BM}_{2\dim B}(T^*_{\mathcal{S}'}B, \mathbb{Z})$. Since $T^*_{\mathcal{S}'}B$ is the union of finitely many $\dim(B)$-dimensional irreducible varieties, $H^{BM}_{2\dim B}(T^*_{\mathcal{S}'}B, \mathbb{Z})$ is generated by their fundamental classes, and hence the equality (15) implies that $u_{2\star}(\Lambda'') = CC(f_*(\gamma))$ as analytic $\dim(B)$-cycles. □

By the construction, $u_{2\star}(\Lambda'')$ is evidently effective. Thus, $f_*(\gamma)$ is CC-effective. The proposition is proved. □

Proof of Proposition 5.1. Under the assumptions of Proposition 5.1, we can construct the following commutative diagram analogous to diagram (10),

$$\begin{tikzcd}
\widetilde{Z}_{\mathrm{norm}} \arrow[r, "\tilde{p}"] \arrow[d] & \widetilde{Z} \arrow[r, "\iota"] \arrow[d] & \widetilde{X} \arrow[r, "\tilde{\phi}"] \arrow[d] & \mathscr{D}_1 \times \mathscr{M} \arrow[d, "\pi_M"] \\
\widetilde{Z}^{ss}_{\mathrm{norm}} \arrow[r, "\tilde{p}^{ss}"] \arrow[d] & \widetilde{Z}^{ss} \arrow[r, "\iota^{ss}"] & \widetilde{X}^{ss} \arrow[r, "\tilde{\phi}^{ss}"] & \mathscr{D}_1 \times \mathscr{D}_2 \arrow[d, "\pi_\Gamma"] \\
Z_{\mathrm{norm}} \arrow[rrr, "\phi"] & & & \Gamma\backslash\mathscr{D}_1 \times \mathscr{D}_2
\end{tikzcd} \tag{16}$$

where

- Γ is the image of the composition

$$\pi_1(Z_{\mathrm{norm}}) \xrightarrow{\tau} \pi_1(X) \xrightarrow{\sigma_1 \times \sigma_2^{ss}} \mathrm{GL}(N_1, \mathbb{C}) \times \mathrm{GL}(N_2, \mathbb{C}),$$

- $\widetilde{X}$ and $\widetilde{X}^{ss}$ are the covering spaces of X induced by $\ker[\sigma_1 \times \sigma_2]$ and $\ker[\sigma_1 \times \sigma_2^{ss}]$ respectively;
- $\widetilde{Z}_{\text{norm}}$ and $\widetilde{Z}_{\text{norm}}^{ss}$ are the covering spaces of Z_{norm} induced by $\ker[(\sigma_1 \times \sigma_2) \circ \tau]$ and $\ker[(\sigma_1 \times \sigma_2^{ss}) \circ \tau]$ respectively;
- choosing a lifting $\widetilde{Z}_{\text{norm}}^{ss} \to \widetilde{X}^{ss}$ of $Z_{\text{norm}} \to X$, let $\widetilde{Z}_{\text{norm}}^{ss} \xrightarrow{\tilde{p}^{ss}} \widetilde{Z}^{ss} \xrightarrow{\iota^{ss}} \widetilde{X}^{ss}$ be the factorization such that $\tilde{p}^{ss}$ is surjective and ι^{ss} is injective;
- choosing a lifting $\widetilde{Z}_{\text{norm}} \to \widetilde{X}$ of $\widetilde{Z}_{\text{norm}}^{ss} \to \widetilde{X}^{ss}$, let $\widetilde{Z}_{\text{norm}} \xrightarrow{\tilde{p}} \widetilde{Z} \xrightarrow{\iota} \widetilde{X}$ be the factorization such that $\tilde{p}^{ss}$ is surjective and ι^{ss} is injective;
- $\tilde{\phi}$, $\tilde{\phi}^{ss}$ and ϕ are the period maps.

Notice that all vertical maps are covering maps except π_M. Moreover, $\phi : Z_{\text{norm}} \to \Gamma\backslash\mathscr{D}_1 \times \mathscr{D}_2$, $\tilde{\phi} \circ \iota \circ \tilde{p} : \widetilde{Z}_{\text{norm}} \to \mathscr{D}_1 \times \mathscr{M}$ and $\tilde{\phi}^{ss} \circ \iota^{ss} \circ \tilde{p}^{ss} : \widetilde{Z}_{\text{norm}}^{ss} \to \mathscr{D}_1 \times \mathscr{M}$ are all proper holomorphic maps. By definition, Γ is a subgroup of $\Gamma_1 \times \Gamma_2$. Since both Γ_1 and Γ_2 are discrete and torsion-free, so is Γ.

Claim 5.6. *Let δ be the constructible function on Z_{norm} as defined in Definition 4.3. Under the above notations, $\phi_*(\delta)$ is CC-effective.*

Proof. Let $\tilde{\delta}$ be the pullback of δ to $\widetilde{Z}_{\text{norm}}^{ss}$. As an analog of Lemma 4.7, $\tilde{p}_*^{ss}(\tilde{\delta})$ is CC-effective. Since ι^{ss} is a closed embedding, $(\iota^{ss} \circ \tilde{p}^{ss})_*(\tilde{\delta})$ is also CC-effective.

Let $\widetilde{Z}_{\text{norm}}^{\text{univ}} \to \mathscr{D}_1 \times \mathscr{M}$ be the product of the period map of $p^*\sigma_1$ and the mixed period map of $p^*\sigma_2$. By the assumptions in Item (i) it has discrete fibers. Note that it factors through the proper map $\tilde{\phi} \circ \iota \circ \tilde{p} : \widetilde{Z}_{\text{norm}} \to \mathscr{D}_1 \times \mathscr{M}$ is proper and the étale cover $\widetilde{Z}_{\text{norm}}^{\text{univ}} \to \widetilde{Z}_{\text{norm}}$. It implies that $\tilde{\phi} \circ \iota \circ \tilde{p} : \widetilde{Z}_{\text{norm}} \to \mathscr{D}_1 \times \mathscr{M}$ is a finite morphism. Since $\tilde{\phi}$ is surjective, $\iota \circ \tilde{p} : \widetilde{Z} \to \mathscr{D}_1 \times \mathscr{M}$ is also a finite morphism.

By definition, the deck transfomration group of the normal covering map $\widetilde{Z}_{\text{norm}}^{ss} \to Z_{\text{norm}}$ is equal to Γ. Thus, the bottom rectangle of (16) is Cartesian. Since Z_{norm} is projective, ϕ is proper, and hence $\tilde{\phi}^{ss} \circ \iota^{ss} \circ \tilde{p}^{ss} : \widetilde{Z}_{\text{norm}}^{ss} \to \mathscr{D}_1 \times \mathscr{D}_2$ is also proper. Since $\tilde{p}^{ss}$ is surjective, $\tilde{\phi}^{ss} \circ \iota^{ss} : \widetilde{Z}^{ss} \to \mathscr{D}_1 \times \mathscr{D}_2$ is also proper.

Denote the kernel of the natural map $\mathrm{Im}(\sigma_1 \times \sigma_2) \to \mathrm{Im}(\sigma_1 \times \sigma_2^{ss})$ by Γ_0. Then the monodromy action of Γ_0 on $\mathscr{D}_1 \times \mathscr{D}_2$ is trivial. By definition, Γ_0 is also equal to the deck transformation group of the normal covering map $\widetilde{X} \to \widetilde{X}^{ss}$. Since the period maps are compatible with the monodromy actions, the map $\tilde{\phi}$ is Γ_0-equivariant. As well-known facts for mixed period domain and mixed period maps (see e.g. [Her99, Section 2]), the map $\mathscr{M} \to \mathscr{D}_2$, and hence $\pi_M : \mathscr{D}_1 \times \mathscr{M} \to \mathscr{D}_1 \times \mathscr{D}_2$, are vector bundles. Moreover, Γ_0 acts on the fibers of π_M by linear translations.

Therefore, the constructible function $(\iota^{ss} \circ \tilde{p}^{ss})_*(\tilde{\delta})$ satisfies the axiom of Proposition 5.3 with respect to the most upper-right square of (16) restricted to a small ball in $\mathscr{D}_1 \times \mathscr{D}_2$. Since CC-effectiveness is a local property, Proposition 5.3 implies that

$$\tilde{\phi}_*^{ss}(\iota^{ss} \circ \tilde{p}^{ss})_*(\tilde{\delta}) = \tilde{\phi}_*^{ss}\iota_*^{ss}\tilde{p}_*^{ss}(\tilde{\delta})$$

is CC-effective. Since the bottom rectangle of (16) is Cartesian, $\pi_\Gamma^*(\phi_*(\delta))$ is equal to $\tilde{\phi}_*^{ss}\iota_*^{ss}\tilde{p}_*^{ss}(\tilde{\delta})$. Thus, $\pi_\Gamma^*(\phi_*(\delta))$ is CC-effective. Since π_Γ is a covering map, we can conclude that $\phi_*(\delta)$ is CC-effective. □

We can follow the same arguments as in the proof of Proposition 4.1. By Claim 5.6 and Proposition 4.11, we know that $\chi(\phi_*(\delta)) \geq 0$. The equations (11) and (12) also apply here, and hence we have

$$(-1)^{\dim Z}\chi(Eu_Z) = \chi(\delta) = \chi(\phi_*(\delta)) \geq 0. \qquad \square$$

5.2. **Techniques from the linear Shafarevich conjecture.** In this subsection, we show the mixed analogue of Proposition 4.12, using the techniques in the proof of the linear Shafarevich conjecture in [EKPR12].

Proposition 5.7. *Let X be a projective manifold and let $\rho : \pi_1(X) \to \mathrm{GL}(r, \mathbb{C})$ be a large representation. Then there exists*

- *a semisimple representation $\sigma_1 : \pi_1(X) \to \mathrm{GL}(N_1, \mathbb{C})$ underlying a $\mathbb{C}$-VHS;*
- *a representation $\sigma_2 : \pi_1(X) \to \mathrm{GL}(N_2, \mathbb{C})$ such that the associated local system L_{σ_2} underlies an $\mathbb{R}$-VMHS of weight $-1, 0$;*
- *and a multivalued closed holomorphic one form η on X*

such that for any irreducible subvariety Z of X, when $\eta|_{Z_{\mathrm{reg}}}$ is trivial,

(i) *let $p : Z_{\mathrm{norm}} \to Z$ be the normalization. Then $p^*\sigma_1 : \pi_1(Z_{\mathrm{norm}}) \to \mathrm{GL}(N_1, \mathbb{C})$ and $p^*\sigma_2^{ss} : \pi_1(Z_{\mathrm{norm}}) \to \mathrm{GL}(N_2, \mathbb{C})$ both have discrete images.*
(ii) *Letting $\phi_1 : \widetilde{Z}_{\mathrm{norm}}^{\mathrm{univ}} \to \mathscr{D}_1$ be the period map of L_{σ_1} and $\phi_2 : \widetilde{Z}_{\mathrm{norm}}^{\mathrm{univ}} \to \mathscr{M}$ be the mixed period map of L_{σ_2}, then*

$$(\phi_1, \phi_2) : \widetilde{Z}_{\mathrm{norm}}^{\mathrm{univ}} \to \mathscr{D}_1 \times \mathscr{M} \tag{17}$$

has discrete fibers.

Proof. Let $s_{\mathrm{fac},r} : X \to S_{\mathrm{fac},r}$ be the factorization map defined in Definition 2.11. By the same arguments in the proof of Proposition 4.12, there exists a closed mutivalued one-form η on X such that for any closed subvariety Z, $\eta|_{Z_{\mathrm{reg}}}$ is trivial if and only if $s_{\mathrm{fac},r}(Z)$ is a point. From now on, we will assume that $s_{\mathrm{fac},r}(Z)$ is a point.

In [DYK23, Proposition 3.13], the authors constructed

- a semisimple representation $\sigma_1 : \pi_1(X) \to \mathrm{GL}(N_1, \mathbb{C})$ underlying a $\mathbb{C}$-VHS;
- semisimple representations $\{\rho_i : \pi_1(X) \to \mathrm{GL}(r, k)\}_{i=1,\ldots,n}$ where k is a number field,

with the following properties.

(1) The image of $p^*\rho_i : \pi_1(Z_{\mathrm{norm}}) \to \mathrm{GL}(r, k)$ is contained in $\mathrm{GL}(r, O_k)$, where O_k is the ring of integer of k. Moreover, the direct sum

$$\bigoplus_{i=1}^{k} \bigoplus_{w \in \mathrm{Ar}(k)} p^*\rho_{i,w} : \pi_1(Z_{\mathrm{norm}}) \to \prod \mathrm{GL}(r, \mathbb{C})$$

is conjugate to $p^*\sigma_1$, where $\mathrm{Ar}(k)$ is the set of archimedean places of k and $\rho_{i,w} := w\rho_i$ for each $w \in \mathrm{Ar}(k)$. In particular, $p^*\sigma_1 : \pi_1(Z_{\mathrm{norm}}) \to \mathrm{GL}(N_1, \mathbb{C})$ is a $\mathbb{C}$-VHS with discrete monodromy.
(2) Each geometric connected component of M contains some $[\varrho_{i,w}]$.
(3) For any semisimple representation $\tau : \pi_1(X) \to \mathrm{GL}(r, \mathbb{C})$ such that $[\tau]$ is in the same geometric component of $[\varrho_{i,w}]$, $p^*\tau$ is conjugate to $p^*\rho_{i,w}$.

The representation $\sigma_2 : \pi_1(X) \to \mathrm{GL}(N_2, \mathbb{R})$ underlying $\mathbb{R}$-VMHS of weight $-1, 0$ is constructed in [EKPR12, Lemma 5.4], and let us briefly recall it. We define M^{VHS} to be the set of semisimple representations $\pi_1(X) \to \mathrm{GL}(r, \mathbb{C})$ underlying a $\mathbb{C}$-VHS. Define $\mathcal{T}_M^{\mathrm{VHS}}$ to be the set of the tensor product $V_1 \otimes \cdots \otimes V_n$ with each V_i in M^{VHS}. In [EKPR12, Lemma 5.4], σ_2 is defined to be the monodromy representation of $\sum_{i=1}^{\ell}(\mathbb{D}_1(\mathbb{V}_i) + \overline{\mathbb{D}_1(\mathbb{V}_i)})$ where $\{\mathbb{V}_i\}_{i=1,\ldots,\ell}$ are certain objects in $\mathcal{T}_M^{\mathrm{VHS}}$. Here $\mathbb{D}_1(\mathbb{V}_i)$ is the 1-step $\mathbb{C}$-MVHS (hence of weight length 1) constructed in [EKPR12, Definition 2.11] whose graded part is a direct sum of $\mathbb{V}_i$ (cf. also [ES11, Theorem 3.15]), and $\overline{\mathbb{D}_1(\mathbb{V}_i)}$ a $\mathbb{C}$-VMHS which is the conjugate of $\mathbb{D}_1(\mathbb{V}_i)$. Hence L_{σ_2} is an $\mathbb{R}$-VMHS of weight length 1.

By the definition of $\mathcal{T}^{\mathrm{VHS}}$, we know that there exists $\{\varrho_n : \pi_1(X) \to \mathrm{GL}(r, \mathbb{C})\}_{n=1,\ldots,m}$ underlying $\mathbb{C}$-VHS such that $V_j = L_{\varrho_1} \otimes \cdots \otimes L_{\varrho_m}$. By Item 2, there exists ϱ_{j_n, w_n} such that $[\varrho_{j_n, w_n}]$ is in the same geometric component of $[\varrho_n]$. For any $w \in \mathrm{Ar}(k)$, by Simpson's ubiquity [Sim92, Theorem 3], we know that there exists a semisimple representation $\varrho_{j_n,w}^{\mathrm{vhs}}$: $\pi_1(X) \to \mathrm{GL}(r, \mathbb{C})$ underlying a $\mathbb{C}$-VHS such that $[\varrho_{j_n,w}^{\mathrm{vhs}}]$ is in the same geometric connected component of $[\varrho_{j_n,w}]$ for each $w \in \mathrm{Ar}(k)$. Let L_n be the local system associated to the direct sum representation

$$\bigoplus_{w \in \mathrm{Ar}(k), w \neq w_n} \varrho_{j_n,w}^{\mathrm{vhs}} \oplus \varrho_n.$$

Then L_n is a $\mathbb{C}$-VHS. Moreover, by Item 3, p^*L_n is isomorphic to the local system corresponding to the representation

$$\bigoplus_{w \in \mathrm{Ar}(k)} p^* \varrho_{j_n,w} : \pi_1(Z_{\mathrm{norm}}) \to \prod \mathrm{GL}(r, \mathbb{C}).$$

By Item 1, we know that p^*L_n has discrete monodromy. Note that $\mathbb{V}_j$ is a direct factor of $L_1 \otimes \cdots \otimes L_m$. We can replace each $\mathbb{V}_j$ by $L_1 \otimes \cdots \otimes L_m$ and [EKPR12, Lemma 5.4] will still holds. In this case, $p^*\sigma_2^{ss}$ has discrete image by our construction. Proposition 5.7.(i) is proved.

Let $M := M_{\mathrm{B}}(\pi_1(X), \mathrm{GL}_r)(\mathbb{C})$. We define $\widetilde{H_M^0}$ to be the intersection of the kernels of all semisimple representations $\pi_1(X) \to \mathrm{GL}(r, \mathbb{C})$. Let $\widetilde{H_M^1} \subset \widetilde{H_M^0}$ be the intersection of $\widetilde{H_M^0}$ and the kernels of the monodromy representation of $\mathbb{D}_1(L_\sigma)$ with $\sigma \in \mathcal{T}_M^{\mathrm{VHS}}$. Let $\pi_X : \widetilde{X}^{\mathrm{univ}} \to X$ be the universal covering map. Denote by $\widetilde{X_M^i} := \widetilde{X}^{\mathrm{univ}}/\widetilde{H_M^i}$ for $i = 0, 1$ and let $\pi_i : \widetilde{X_M^i} \to X$ be the covering map. In [DYK23, Lemma 3.30], it is proved that

Claim 5.8. *Each connected component of the fiber of the holomorphic map*

$$(s_{\mathrm{fac},r} \circ \pi_0, \tilde{\varphi}_1) : \widetilde{X_M^0} \to S_{\mathrm{fac},r} \times \mathscr{D}_1 \tag{18}$$

is compact. Here $\tilde{\varphi}_1 : \widetilde{X_M^0} \to \mathscr{D}_1$ is the period map of L_{σ_1}.

By the generalized Stein factorization discussed in Section 2.5, the set $\widetilde{S}_M$ of connected components of fibres of can be endowed with the structure of a normal complex space such that (18) factors through a proper holomorphic fibration $\Psi_0 : \widetilde{X_M^0} \to \widetilde{S_M^0}$. The Galois group $\pi_1(X)/\widetilde{H_M^0}$ induces a properly discontinuous action on $\widetilde{S_M^0}$ such that Ψ_0 is equivariant. By [DYK23, Proposition 3.13], the *reductive Shafarevich morphism* $\mathrm{Sh}_M : X \to \mathrm{Sh}_M(X)$ of M is the quotient of Ψ_0 by this action $\pi_1(X)/\widetilde{H_M^0}$. Namely, for any subvariety Z of Z, $\mathrm{Sh}_M(Z)$ is a point if and only if $\tau(\mathrm{Im}[\pi_1(Z_{\mathrm{norm}}) \to \pi_1(X)])$ is finite for any semisimple representation $\tau : \pi_1(X) \to \mathrm{GL}(r, \mathbb{C})$.

Note that the mixed period map $\varphi_2 : \widetilde{X}^{\mathrm{univ}} \to \mathscr{M}$ of L_{σ_2} factors through $\tilde{\varphi}_2 : \widetilde{X^1_M} \to \mathscr{M}$. It is proved in [EKPR12, Lemma 5.5] that each connected component of the fiber of the holomorphic map

$$(\tilde{\varphi}_2, \mathrm{sh}_M \circ \pi_1) : \widetilde{X^1_M} \to \mathscr{M} \times \mathrm{Sh}_M(X) \tag{19}$$

is compact. By the generalized Stein factorization again, the set $\widetilde{S^1_M}$ of connected components of fibres of can be endowed with the structure of a normal complex space such that (19) factors through a proper holomorphic fibration $\Psi_1 : \widetilde{X^1_M} \to \widetilde{S^1_M}$. The Galois group $\pi_1(X)/\widetilde{H^1_M}$ induces a properly discontinuous action on $\widetilde{S^1_M}$ such that Ψ_1 is equivariant. Let $\mathrm{sh}^1_M : X \to \mathrm{Sh}^1_M(X)$ be the quotient of Ψ_1 by this action $\pi_1(X)/\widetilde{H^1_M}$. Let H be the intersection of the kernels of all linear representations $\rho : \pi_1(X) \to \mathrm{GL}(r, \mathbb{C})$. A crucial fact proven in [EKPR12, Proposition 3.10 & p.1549] is that, $\mathrm{sh}^1_M : X \to \mathrm{Sh}^1_M(X)$ is the Shafarevich morphism $\mathrm{sh}_H : X \to \mathrm{Sh}_H(X)$ of (X, H).

Claim 5.9. $\mathrm{sh}_H : X \to \mathrm{Sh}_H(X)$ *is the identity map.*

Proof. Note that $H \subset \ker \rho$. Hence there is an injection

$$\pi_1(X)/\ker \rho \to \pi_1(X)/H.$$

Since $\rho : \pi_1(X) \to \mathrm{GL}(r, \mathbb{C})$ is a large representation, for any closed subvariety $Z \subset X$, the image $\pi_1(Z_{\mathrm{norm}}) \to \pi_1(X)/\ker \rho$ is an infinite group. Hence $\mathrm{Im}[\pi_1(Z_{\mathrm{norm}}) \to \pi_1(X)/H]$ is also infinite. Therefore, the fibers of $\mathrm{sh}_H : X \to \mathrm{Sh}_H(X)$ are zero dimensional. Since sh_H has connected fibers, and $\mathrm{Sh}_H(X)$ is normal, it follows that sh_H is the identity map. The claim is proved. □

Therefore, $\mathrm{sh}^1_M : X \to \mathrm{Sh}^1_M(X)$ is the identity map. By our construction of sh^1_M and sh^0_M, we conclude that

$$(s_{\mathrm{fac},r} \circ \pi_X, \varphi_1, \varphi_2) : \widetilde{X}^{\mathrm{univ}} \to S_{\mathrm{fac},r} \times \mathscr{D}_1 \times \mathscr{M} \tag{20}$$

has discrete fibers, where $\varphi_1 : \widetilde{X}^{\mathrm{univ}} \to \mathscr{M}$ is the period map of L_{σ_1}. Since $s_{\mathrm{fac},r}(Z)$ is a point, it follows that

$$(\phi_1, \phi_2) : \widetilde{Z}^{\mathrm{univ}}_{\mathrm{norm}} \to \mathscr{D}_1 \times \mathscr{M}$$

has discrete fibers. Proposition 5.7.(ii) is proved. We complete the proof of the proposition. □

5.3. **Proof of Theorem 1.5 in the case when $K = \mathbb{C}$ and ρ is linear.** After proving Propositions 5.1 and 5.7, we are in the same senario as in the semisimple case. So we only give a sketch.

Proof of Theorem 1.5 in the case when $K = \mathbb{C}$ and ρ is not necessarily semisimple. We apply Proposition 5.7 to construct a semisimple representation $\sigma_1 : \pi_1(X) \to \mathrm{GL}(N_1, \mathbb{C})$ underlying a $\mathbb{C}$-VHS, a representation $\sigma_2 : \pi_1(X) \to \mathrm{GL}(N_2, \mathbb{R})$ underlying a $\mathbb{R}$-VMHS of weight $-1, 0$, and a multivalued one form η on X such that they satisfies the properties therein. We use the vanishing cycle functor Φ_η and apply Corollary 3.10 repeatedly so that we reduce the proof to the case where $\eta|_Z$ is trivial. Then the properties in Proposition 5.7 are fulfilled. We then apply Proposition 5.1 to conclude that $T^*_Z X \cdot T^*_X X \geq 0$. $\square$

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