A FLATNESS CRITERION FOR PSEUDO-EFFECTIVE SHEAVES ON COMPACT KÄHLER SPACES

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ABSTRACT. In this paper, we prove that if $\mathcal{E}$ is a pseudo-effective sheaf with vanishing first Chern class on a klt compact Kähler space X, then, after passing to a finite quasi-étale cover, the reflexive pullback of $\mathcal{E}$ is locally free and flat. This extends the flatness criterion of Höring–Peternell, originally established for projective varieties, to the Kähler setting.

The proof relies on two main ingredients, both of which are new even in the projective case. The first is a flatness theorem for stable sheaves: we show that a slope-stable pseudo-effective sheaf with vanishing first Chern class is Hermitian flat. This is obtained by combining Hermitian–Einstein theory with the subharmonicity properties of direct image sheaves. The second is a singular Kähler analogue of Simpson's flatness theorem for extensions of locally free Hermitian flat sheaves.

1. INTRODUCTION

In complex geometry, a fundamental theme is to extract a *flat part* from "semipositive" curvature conditions in order to understand the global structure of the underlying spaces. A guiding principle, going back to the celebrated work of Demailly–Peternell–Schneider [DPS94], is that "semipositively curved" vector bundles with vanishing first Chern class should be flat. This principle is one of the key ingredients in the structure theorem for compact Kähler manifolds with nef tangent bundle.

There are several ways to formulate semipositivity. In the projective setting, positivity in the sense of Viehweg is one of the most useful notions of semipositivity, especially in the classification theory of projective varieties. For sheaves on normal compact Kähler spaces, the notion relevant to our purposes is *pseudo-effectivity* (see Definition 2.2 and Remark 2.3). This is a weak positivity property which arises naturally in complex geometry, particularly in the study of structure theorems for varieties with non-negative curvature; see, for example, [CH19, EIM23, GKP21, GKP22, LPT18, Mat22, MW25b, Wan22], among many others. Also, in recent years, the Minimal Model Program in the Kähler setting has also seen substantial progress [Das, DH25, DHP24, DO24, Fuj22,

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GP25, HP16, HP18]. These developments make it natural and important to extend the principle of Demailly–Peternell–Schneider to pseudo-effective sheaves on singular Kähler spaces, and such an extension is expected to have various applications (see [Mat, MQ25, MWWZ]).

The purpose of this paper is to establish two flatness criteria for pseudo-effective sheaves on compact Kähler spaces with mild singularities; see Theorem 1.1 and Theorem 1.2 below.

Theorem 1.1. *Let X be a compact Kähler space with potentially klt singularities (i.e. there exists an effective $\mathbb{Q}$-divisor Δ such that (X, Δ) is a klt pair). Let $\mathcal{E}$ be a pseudo-effective sheaf on X whose first Chern class vanishes. Then the following assertions hold*:

(i) *There exists a finite quasi-étale cover $\nu\colon X' \to X$ such that the reflexive pullback $\nu^{[*]}\mathcal{E} := (\nu^*\mathcal{E})^{**}$ is a numerically flat locally free sheaf on X'.*

(ii) *The restriction of the reflexive hull $\mathcal{E}^{**}$ to the smooth locus X_{reg} is a flat locally free sheaf on X_{reg}.*

We refer to Definition 2.2 for the definition of pseudo-effective sheaves, and to Section 2.5 for the definitions of numerical flatness and Hermitian flatness. Theorem 1.1 extends the flatness criterion of Höring–Peternell [HP19], originally proved in the projective case, to the Kähler setting. In the previous work [HP19], the variety X is assumed to be smooth in codimension 2, whereas our result does not require this assumption (see Remark 2.3 for a detailed comparison).

We briefly explain the proof of Theorem 1.1 (i). One of the main difficulties is to prove the local freeness of $\mathcal{E}^{**}$. First, by recent work [Bra21, GKKP11, DHP24, Fuj22], we may take a maximal quasi-étale cover $\nu\colon X' \to X$ such that every flat locally free sheaf on the smooth locus X'_{reg} extends to a flat locally free sheaf on X' (see Theorem 2.9). The reflexive pullback $\nu^{[*]}\mathcal{E}$ is again pseudo-effective and satisfies $c_1(\nu^{[*]}\mathcal{E}) = 0$. By Proposition 2.7, the sheaf $\nu^{[*]}\mathcal{E}$ is slope-semistable. We may therefore consider the Jordan–Hölder filtration of $\nu^{[*]}\mathcal{E}$:

$$0 =: \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_k := \nu^{[*]}\mathcal{E}.$$

By proving a regularity property for this filtration in Proposition 3.1, we reduce the problem to showing that each graded quotient is Hermitian flat. This is precisely the content of Theorem 1.2 below.

For the proof of Theorem 1.1 (ii), it remains to descend the flat connection on $\nu^{[*]}\mathcal{E}$ to X_{reg}. This descent is not automatic in our singular setting and requires a variant of Simpson's theorem on extensions of flat bundles [Sim92] (see Subsection 2.5). To prove this variant, we use the explicit construction of flat connections in [Den21].

The next theorem gives a flatness criterion for pseudo-effective sheaves on normal compact Kähler spaces, without assuming klt singularities.

Theorem 1.2. *Let X be a normal compact Kähler space, and let $\mathcal{E}$ be a pseudo-effective sheaf on X whose first Chern class vanishes. If $\mathcal{E}$ is slope-stable with respect to some Kähler class ω, then the restriction of the reflexive hull $\mathcal{E}^{**}$ to the smooth locus X_{reg} is Hermitian flat.*

Before explaining the idea of the proof of Theorem 1.2, we recall two standard approaches to proving flatness. The first is to deduce flatness from the stability of all reflexive symmetric powers $\mathrm{Sym}^{[m]}\,\mathcal{E}$ rather than merely polystability, as in [HP19, CCP]. However, in our situation, the sheaves $\mathrm{Sym}^{[m]}\,\mathcal{E}$ need not be stable. The second approach is to prove the vanishing of the second Chern class and then apply a flatness criterion, as in [BS94, CCM21, LT18, GKP16b]; see also the recent developments in the Kähler setting [FO25, GP25, GP26]. However, it is not clear how to prove the vanishing of the second Chern class in the present setting, even when X has terminal singularities.

In the proof of Theorem 1.2, we avoid using the second Chern class. Instead, we develop an idea from [CCP], namely using the theory of Hermitian–Einstein metrics together with the subharmonicity of direct image sheaves established in [CCP]. This approach works well in the singular setting and even in the non-Kähler setting. The proof proceeds roughly as follows. By the definition of pseudo-effective sheaves, there exist singular metrics $\{h_\varepsilon\}_{\varepsilon>0}$ on $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ whose curvature is bounded below by $-\varepsilon p^*\omega_X$. Here ω_X is a Kähler form on X, and $p\colon \mathbb{P}(\mathcal{E}) \to X$ is the natural projection with hyperplane bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. Since $\mathcal{E}$ is not locally free in general, we have to treat $\mathbb{P}(\mathcal{E})$ with some care. To this end, we establish a characterization of pseudo-effective sheaves (see Theorem 4.5), which is highly nontrivial and is one of the main contributions of this paper.

In this introduction, we keep the following discussion at a heuristic level. A key point is to study the singularities of a limit metric h on $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ obtained from the metrics $\{h_\varepsilon\}_{\varepsilon>0}$. If the singular locus of h is not dominant over X, then the positivity of direct image sheaves implies that the L^2-metric H on $\mathcal{E}$ has semipositive curvature, using the condition $c_1(\mathcal{E}) = 0$. Hence, by Raufi's result [Rau15], the sheaf $\mathcal{E}$ is Hermitian flat and locally free on X_{reg}.

It remains to exclude the opposite case, where the singular locus of h is dominant over X. We do so by constructing a destabilizing subsheaf of $\mathcal{E}$. For simplicity, assume that $\det \mathcal{E}$ is a trivial invertible sheaf. We consider the natural inclusion

$$
\begin{aligned}
\mathcal{S} :=& p_*\big(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(K_{\mathbb{P}(\mathcal{E})/X} + \mathcal{O}_{\mathbb{P}(\mathcal{E})}(r+1)) \otimes \mathcal{I}(h^{r+1})\big)\\
\subset& p_*\big(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(K_{\mathbb{P}(\mathcal{E})/X} + \mathcal{O}_{\mathbb{P}(\mathcal{E})}(r+1))\big) = \mathcal{E},
\end{aligned}
$$

where $\mathcal{I}(\bullet)$ is the multiplier ideal sheaf and r is the rank of $\mathcal{E}$. The L^2-metric on $\mathcal{S}$ is positively curved, and hence its slope is semipositive. Therefore, we expect that $\mathcal{S}$, or a closely related direct image sheaf, gives a destabilizing subsheaf of $\mathcal{E}$, contradicting the stability of $\mathcal{E}$. The difficulty is that it is not clear a priori that $\mathcal{S}$ is nonzero and properly contained in $\mathcal{E}$.

To overcome this difficulty, we take a smooth metric g on $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ arising from Hermitian–Einstein metrics, as in [CGNPPW], and consider the subsheaf

$$\mathcal{S}_\delta := p_*\big(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(K_{\mathbb{P}(\mathcal{E})/X}) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(r+1) \otimes \mathcal{I}((h^{1-\delta} \cdot g^{\delta})^{r+1})\big).$$

By analyzing the singularities of the metric $h^{1-\delta} \cdot g^{\delta}$ on $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ in Proposition 3.2, we find some $0 < \delta < 1$ such that $\mathcal{S}_\delta$ is nonzero and properly contained in $\mathcal{E}$. Moreover, although the curvature of g is not necessarily semipositive, the slope of $\mathcal{S}_\delta$ is still non-negative by the subharmonicity of direct image sheaves [CCP]. This gives a destabilizing subsheaf, contradicting the stability of $\mathcal{E}$.

We emphasize that the argument explained above is new even in the projective setting. In the projective setting, the flatness criteria of this type, as used for instance in [CH19], have played an important role in the study of singular analogues of the Beauville–Bogomolov–Yau decomposition. Our flatness criterion may provide a more direct proof of such decomposition theorems in the Kähler setting. Also, one promising application of our flatness criterion is to extend the Beauville–Bogomolov–Yau decomposition for klt pairs established in [MW25b] to compact Kähler spaces. This is carried out in forthcoming work, which will appear soon.

Finally, in Section 5, we apply the techniques developed above to study numerically flat vector bundles on compact complex manifolds which are not necessarily Kähler. In particular, we extend an important characterization of numerically flat vector bundles on compact Kähler manifolds due to Demailly–Peternell–Schneider [DPS94, Theorem 1.18] (see also Theorem 1.2) to the non-Kähler setting. This gives a positive answer to the question raised in [DPS94, Remark 1.21].

Theorem 1.3. *Let X be a compact complex manifold and let E be a locally free sheaf on X. Assume that E is numerically flat, namely $c_1(E) = 0$ and $\mathcal{O}_{\mathbb{P}(E)}(1)$ is nef. Then E admits a filtration by holomorphic subbundles*

$$\{0\} = E_0 \subset E_1 \subset \cdots \subset E_k = E \tag{1.3.1}$$

such that each graded quotient E_i/E_{i-1} is Hermitian flat. In particular, all Chern classes $c_k(E)$ vanish.

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2. Tools and Preparations

In this section, we summarize some basic material and preliminary results. Throughout this paper, let X be a normal analytic variety of dimension n (i.e. an irreducible and reduced complex analytic space with at worst normal singularities), and let $\mathcal{E}$ be a torsion-free coherent sheaf on X of rank r. Set $X_0 := X_{\mathrm{reg}} \cap X_{\mathcal{E}}$, where X_{reg} is the smooth locus of X and $X_{\mathcal{E}}$ is the maximal locus on which $\mathcal{E}$ is locally free. We use the terms *locally free sheaf* and *vector bundle* interchangeably, and we reserve the term *vector bundle* for occasions when we want to emphasize the bundle viewpoint.

2.1. Notation and conventions.

For a positive integer $m \in \mathbb{Z}_+$, we define

- the reflexive tensor power $\mathcal{E}^{[\otimes m]} := (\mathcal{E}^{\otimes m})^{**}$;
- the reflexive symmetric power $\mathrm{Sym}^{[m]}\mathcal{E} := (\mathrm{Sym}^m \mathcal{E})^{**}$;

where $\mathcal{E}^* := \mathrm{Hom}(\mathcal{E}, \mathcal{O}_X)$ is the dual sheaf. The reflexive top exterior power $\det \mathcal{E} := \wedge^{[r]}\mathcal{E}$ is called the *determinant sheaf* of $\mathcal{E}$. Further, for a torsion-free sheaf $\mathcal{F}$ on X, we define the reflexive tensor product $\mathcal{E}[\otimes]\mathcal{F} := (\mathcal{E} \otimes \mathcal{F})^{**}$.

Let $f\colon Y \to X$ be a morphism between normal analytic varieties. We define the reflexive pullback $f^{[*]}\mathcal{E} := (f^*\mathcal{E})^{**}$. In this paper, we often say that a property (P) holds *over* a subset $U \subset X$ if it holds on the inverse image $f^{-1}(U)$. For example, we say that the sheaf $\mathcal{G}$ is locally free *over* X_0 if it is locally free on the inverse image $f^{-1}(X_0)$.

2.2. Singular Hermitian metrics and pseudo-effective sheaves.

We first recall the notion of psh (resp. quasi-psh) functions on a normal analytic variety X. See [Dem85] and [BG13, Section 4.6] for details.

A function $\varphi\colon X \to [-\infty, +\infty[$ is said to be *plurisubharmonic* (psh for short) if there exists an open covering $X = \bigcup_i U_i$ and embeddings $U_i \hookrightarrow \mathbb{C}^N$ such that $\varphi|_{U_i}$ is the restriction of a psh function defined on an open subset of $\mathbb{C}^N$. A function $\varphi\colon X \to [-\infty, +\infty[$ is said to be *quasi-psh* if it can be locally written as the sum of a psh function and a smooth function. Similarly, a $(1,1)$-form ω_X on X is said to be a *Kähler form* on X if $\omega_X|_{U_i}$ can be written as the restriction of a Kähler form defined on an open subset of $\mathbb{C}^N$. A Kähler form ω_X in our sense always admits local potentials (i.e. locally there exists a smooth function f such that $\omega_X = \sqrt{-1}\partial\bar{\partial} f$). Psh functions on normal varieties satisfy the Hartogs-type extension property, which will often be used in this paper without explicit mention.

Theorem 2.1 ([GR56, Satz 4])**.** *Let X be a normal analytic variety and let $Z \subset X$ be a Zariski closed subset of codimension $\geqslant 2$. Then, any psh function φ on $X \setminus Z$ uniquely extends to a psh function on X.*

Using quasi-psh functions, we review singular Hermitian metrics and pseudo-effectivity for torsion-free sheaves. Below, we assume familiarity with the basic theory of singular Hermitian metrics on vector bundles (see [Rau15, HPS18, Pau18, PT18]). We recommend [HPS18, Sections 16–19] for a detailed exposition.

A singular Hermitian metric h on a torsion free coherent sheaf $\mathcal{E}$ is a (possibly singular) Hermitian metric on the vector bundle $\mathcal{E}|_{X_0}$, where $X_0 := X_{\mathrm{reg}} \cap X_{\mathcal{E}}$. In particular, outside a subset of measure zero in X, the metric $h(x)$ defines a finite Hermitian inner product on $\mathcal{E}_x$. For a smooth $(1,1)$-form θ on X with local potential, we write

$$\sqrt{-1}\Theta_h \geqslant \theta \otimes \mathrm{id} \ \text{ on } X$$

if, for any local section $e \in H^0(U, \mathcal{E}^*)$ on an open set $U \subset X$, the function $\log |e|_{h^*} - f$ is psh on $U \cap X_0$, where f is a local potential of θ and h^* is the induced metric on the dual sheaf $\mathcal{E}^*$. Note that a singular Hermitian metric is only defined on X_0, even if it is smooth. Nevertheless, the function $\log |e|_{h^*}$ extends to a quasi-psh function across $X \setminus X_0$ by Theorem 2.1.

The following definition extends the notion of pseudo-effectivity for vector bundles to torsion-free sheaves.

Definition 2.2 ([Mat23, Definition 2.1])**.** Let X be a normal compact Kähler space and let ω_X be a Kähler form on X. A torsion-free sheaf $\mathcal{E}$ on X is said to be *pseudo-effective* if, for every $m \in \mathbb{Z}_+$, there exists a singular Hermitian metric h_m on the m-th symmetric power $\mathrm{Sym}^m \mathcal{E}|_{X_0}$ such that

$$\sqrt{-1}\Theta_{h_m} \geqslant -\omega_X \otimes \mathrm{id}\,.$$

Remark 2.3. Suppose that X is a projective variety with an ample line bundle A.

(1) We compare our definition of pseudo-effective sheaves with Viehweg-type positivity in the projective setting. By [Mat23, Proposition 2.4] (see also [Pau18, Theorem 2.21]), the pseudo-effectivity of $\mathcal{E}$ is equivalent to the following condition: for every $a \in \mathbb{Z}_+$, there exists $b \in \mathbb{Z}_+$ such that the reflexive hull $\mathrm{Sym}^{[ab]} \mathcal{E} \otimes (bA)$ is globally generated at a general point of X.

(2) In [HP19, Theorem 2.1], Höring–Peternell proved a flatness result for $\mathcal{E}$ when $\mathcal{E}$ is pseudo-effective in the sense of [HP19, Definition 2.1] and $\mathrm{Sym}^{[m]} \mathcal{E}$ is stable for every m. Here, pseudo-effectivity in the sense of [HP19, Definition 2.1] means that for every $c \in \mathbb{Z}_+$ there exist integers $i, j \in \mathbb{Z}_+$ such that $i > cj$ and

$$H^0\big(X, \mathrm{Sym}^{[i]} \mathcal{E} \otimes A^{\otimes j}\big) \neq 0.$$

This definition is weaker than ours, even when X is smooth and $\mathcal{E}$ is locally free. Our Definition 2.2 implies that the non-nef locus of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is not dominant over X.

The following propositions will be frequently used in this paper.

Proposition 2.4. *Let X be a normal analytic space and let*

$$\mathcal{E} \to \mathcal{F}$$

be a generically surjective morphism between torsion-free sheaves $\mathcal{E}$ and $\mathcal{F}$. If a Hermitian metric h on $\mathcal{E}$ satisfies

$$\sqrt{-1}\Theta_h \geqslant \theta \otimes \mathrm{id} \ \text{ on } X,$$

then the induced metric h_q on $\mathcal{F}$ satisfies the same condition. In particular, if $\mathcal{E}$ is pseudo-effective, then so is $\mathcal{F}$.

Proof. A key point is that $\mathcal{E} \to \mathcal{F}$ is only generically surjective, and not necessarily surjective everywhere. The proof follows from the argument of [PT18, Lemma 2.4.3]. $\square$

Proposition 2.5. *Let $\mathcal{E}$ be a torsion-free sheaf on a normal compact Kähler space X. Let $\Gamma^a(\mathcal{E})$ be the reflexive hull of $\Gamma^a(\mathcal{E}|_{X_0})$, where $\Gamma^a(\mathcal{E}|_{X_0})$ is the locally free sheaf associated with a tensor representation of highest weight $a = (a_1, \cdots, a_r)$ with $a_1 \geqslant \cdots \geqslant a_r \geqslant 0$. Assume that $\mathcal{E}$ is pseudo-effective. Then $\Gamma^a(\mathcal{E})$ is pseudo-effective. In particular, the reflexive exterior power $\Lambda^{[m]}\mathcal{E}$ is pseudo-effective. Moreover, if $c_1(\mathcal{E}) = 0$, then $\mathcal{E}^*$ is also pseudo-effective.*

Proof. The proof follows from the argument of [DPS94, Proposition 1.14], and thus, we briefly explain only the main strategy. Take singular Hermitian metrics $\{h_m\}$ on $\mathrm{Sym}^m \mathcal{E}|_{X_0}$ such that $\sqrt{-1}\Theta_{h_m} \geqslant -\omega_X \otimes \mathrm{id}$. After replacing X with X_0, we may assume that $\mathcal{E}$ is locally free.

Recall that $\mathrm{Sym}^m(\Gamma^a(\mathcal{E}))$ is a direct summand of $\Gamma^b(\mathcal{E})$ with $|b| = m|a|$, and that $\Gamma^b(\mathcal{E})$ is a direct summand of $\mathrm{Sym}^{b_1}(\mathcal{E}) \otimes \cdots \otimes \mathrm{Sym}^{b_r}(\mathcal{E})$. On the other hand, the product metric $h_{b_1} \otimes \cdots \otimes h_{b_r}$ satisfies

$$\sqrt{-1}\Theta_{h_{b_1} \otimes \cdots \otimes h_{b_r}} \geqslant -r\, \omega_X \otimes \mathrm{id}\,.$$

Note that, although h_m is singular, the tensor product metric can be defined by approximating h_m locally by smooth metrics via convolution. By Proposition 2.4, the induced quotient metric g_m on $\mathrm{Sym}^m(\Gamma^a(\mathcal{E}))$ satisfies the same inequality. This shows that $\Gamma^a(\mathcal{E})$ is pseudo-effective.

For the last assertion, assume that $c_1(\mathcal{E}) = 0$. Then $\Gamma^a(\mathcal{E}) \otimes (\det \mathcal{E})^*$ is pseudo-effective on X by (see Proposition 4.7). In particular, taking $a = (1, 0, \ldots, 0)$, we obtain that $\mathcal{E}^*$ is pseudo-effective. $\square$

2.3. **Stabilities of torsion-free sheaves.** The determinant sheaf $\det \mathcal{E}$ is reflexive, but it is not necessarily invertible when X has singularities. Nevertheless, we can define the first Chern class $c_1(\mathcal{E})$ as a functional on the classes of d-closed forms as follows: Take a modification $\pi \colon \tilde{X} \to X$ such that $\tilde{X}$ is smooth and $(\pi^*\mathcal{E})/\operatorname{tor}$ is a locally free sheaf, where tor is the maximal torsion subsheaf of $\pi^*\mathcal{E}$. Then, for a d-closed $(n-1, n-1)$ form γ, the first Chern class $c_1(\mathcal{E})$ is defined by

$$\int_X c_1(\mathcal{E}) \wedge \gamma := \int_{\tilde{X}} c_1\big((\pi^*\mathcal{E})/\operatorname{tor}\big) \wedge \pi^*\gamma$$

This definition is independent of the choice of $\pi \colon \tilde{X} \to X$. As in the smooth case, the slope (and hence stability) is defined as follows:

Definition 2.6. Let X be a normal compact Kähler space and let ω_X be a Kähler form on X. Let $\mathcal{E}$ be a torsion-free sheaf on X, and let $\pi\colon \tilde{X} \to X$ be a desingularisation of X such that $(\pi^*\mathcal{E})/\operatorname{tor}$ is a locally free sheaf. Then, the slope of $\mathcal{E}$ with respect to ω_X is defined by

$$\begin{aligned}\mu_{\omega_X}(\mathcal{E}) : &= \frac{1}{\operatorname{rank}\mathcal{E}} \int_X c_1(\mathcal{E}) \wedge \omega_X^{n-1} \\ &:= \frac{1}{\operatorname{rank}\mathcal{E}} \int_{\tilde{X}} c_1\big((\pi^*\mathcal{E})/\operatorname{tor}\big) \wedge \pi^*\omega_X^{n-1}.\end{aligned}$$

The right-hand side is independent of the choice of desingularisation, since

$$\int_E \pi^*\omega_X^{n-1} = 0$$

for any π-exceptional divisor E for dimensional reasons.

As a consequence of Propositions 2.4 and 2.5, we have the following:

Proposition 2.7. *Let X be a normal compact Kähler space and let $\mathcal{E}$ be a pseudo-effective sheaf on X. Let ω_X be a Kähler form on X. If $c_1(\mathcal{E}) = 0$, then $\mathcal{E}$ is semistable with respect to ω_X.*

Proof. Let $\mathcal{S} \subset \mathcal{E}$ be a saturated subsheaf, and set $\mathcal{Q} := \mathcal{E}/\mathcal{S}$. Then $\mathcal{Q}$ is pseudo-effective by Proposition 2.4, and so is $\det\mathcal{Q}$ by Proposition 2.5. This shows that the slope of $\mathcal{S}$ is semipositive. □

2.4. **Maximal quasi-étale covers in the Kähler setting.** Following [Gue25, Section 3.2] and [FO25, Subsection 2.B], we summarize some results on maximally quasi-étale covers in the analytic setting. Note that [Gue25, Corollary 3.6] and [FO25, Theorem 2.5] are formulated for klt complex analytic varieties, but the same argument applies to potentially klt singularities.

Definition 2.8. Let X be a normal analytic variety. A maximally quasi-étale cover of X is a quasi-étale Galois cover $X' \to X$ such that the naturally induced morphism

$$\iota_*\colon \hat{\pi}_1(X'_{\mathrm{reg}}) \longrightarrow \hat{\pi}_1(X')$$

of étale fundamental groups is an isomorphism.

Theorem 2.9 ([GKP16a, Theorem 1.5], [Gue25, Cor 3.6], [FO25, Theorem 2.5])**.** *Let X be a potentially klt compact analytic variety. Then, we have*:

(1) *There exists a maximally quasi-étale cover $\nu\colon X' \to X$ of X.*

(2) *Suppose that a normal analytic variety X is maximally quasi-étale* (*i.e. X itself satisfies the above property*)*. Then, any linear representation $\rho_0\colon \pi_1(X_{\mathrm{reg}}) \to \mathrm{GL}(r,\mathbb{C})$ factors through $\pi_1(X)$:*

$$\begin{tikzcd} \pi_1(X_{\mathrm{reg}}) \arrow[r, two heads, "i_*"'] \arrow[rr, bend left, "\rho_0"] & \pi_1(X) \arrow[r, "\rho"'] & \mathrm{GL}(r,\mathbb{C}). \end{tikzcd}$$

Proof. As explained in [Gue25, Corollary 3.6] and [FO25, Subsection 2.B], the first conclusion follows from the finiteness of the local fundamental group of X_{reg} thanks to the arguments in [Bra21, Xu14], where the existence of a plt blowup plays a crucial role via [BCHM10]. This has recently been established using developments of the Minimal Model Program for projective morphisms in the analytic setting [DHP24, Fuj22].

The second conclusion follows from Malcev's theorem, as in [GKP16a]. We briefly explain the proof for the reader's convenience. Note that $\pi_1(X_{\mathrm{reg}})$ is finitely generated. Except for this point, the statement is purely group-theoretic, so we set

$$G := \pi_1(X_{\mathrm{reg}}), \quad H := \pi_1(X), \quad i_*\colon G \to H.$$

Let $\widehat{G}$ (resp. $\widehat{H}$) be the profinite completion of G (resp. H). Consider the commutative diagram

$$\begin{tikzcd} G \arrow[r, "i_*"] \arrow[d, "p_G"'] & H \arrow[d, "p_H"] \\ \widehat{G} \arrow[r, "\widehat{i_*}"] & \widehat{H}. \end{tikzcd}$$

Since X is maximally quasi-étale, the bottom arrow $\widehat{i_*}$ is an isomorphism.

It is enough to show that $\operatorname{Ker}(i_*) \subset \operatorname{Ker}(\rho_0)$. Let $\gamma \in \operatorname{Ker}(i_*)$ and assume for contradiction that $\rho_0(\gamma) \neq 1$. Set $\Gamma := \rho_0(G) \subset \mathrm{GL}(r, \mathbb{C})$. Since Γ is a finitely generated linear group, Malcev's theorem shows that Γ is residually finite. Therefore, there exists a homomorphism $\psi\colon \Gamma \to F$ to some finite group F such that $\psi(\rho_0(\gamma)) \neq 1$. Then $\operatorname{Ker}(\psi \circ \rho_0) \triangleleft G$ is a normal subgroup of G of finite index and does not contain γ. This means that $p_G(\gamma) \neq 1$, which contradicts the commutative diagram. $\square$

2.5. **Numerically flat locally free sheaves.** We first recall three related notions of flatness.

Definition 2.10. Let X be a normal analytic variety and let $\mathcal{E}$ be a locally free sheaf on X.

(1) The sheaf $\mathcal{E}$ is said to be *flat* if it is constructed by a linear representation $\pi_1(X) \to \mathrm{GL}(r, \mathbb{C})$. If X is smooth, this is equivalent to saying that $\mathcal{E}$ admits a holomorphic connection with vanishing curvature.
(2) The sheaf $\mathcal{E}$ is said to be *Hermitian flat* if it admits a smooth Hermitian metric with vanishing Chern curvature.
(3) Suppose that X is compact. The sheaf $\mathcal{E}$ is said to be *numerically flat* if both $\mathcal{E}$ and $\mathcal{E}^*$ are nef. Equivalently, $\mathcal{E}$ is nef and $c_1(\mathcal{E}) = 0$.

Note that these notions are defined only for locally free sheaves. We assume that X is compact when discussing numerical flatness, while (1) and (2) are also defined for non-compact varieties. Recall that $\mathcal{E}$ is *numerically effective* (nef for short) if $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is a nef line bundle on $\mathbb{P}(\mathcal{E})$. Demailly–Peternell–Schneider in [DPS94] proved that $\mathcal{E}$ is nef if and only if, for every $m \in \mathbb{Z}_+$, the m-th symmetric power $\mathrm{Sym}^m \mathcal{E}$ admits a smooth Hermitian metric h_m such that

$$\sqrt{-1}\Theta_{h_m} \geqslant -\omega_X \otimes \mathrm{id}\,.$$

Moreover, they established a flatness criterion for numerically flat locally free sheaves:

Theorem 2.11 ([DPS94, Theorem 1.18]). *Let X be a compact Kähler manifold and let $\mathcal{E}$ be a locally free sheaf on X. If $\mathcal{E}$ is numerically flat, then $\mathcal{E}$ admits a filtration of holomorphic subbundles*

$$\{0\} = E_0 \subset E_1 \subset \cdots \subset E_k = \mathcal{E} \tag{2.11.1}$$

such that each graded quotient E_i/E_{i-1} is Hermitian flat.

Combining Theorem 2.11 with Simpson's result [Sim92] (see also [Den21] for an explicit construction), we obtain the following:

Corollary 2.12. *Let X be a compact Kähler manifold and let $\mathcal{E}$ be a locally free sheaf on X. If $\mathcal{E}$ is numerically flat, then $\mathcal{E}$ admits a holomorphic flat connection which is compatible with the filtration (2.11.1).*

Therefore, Theorem 1.1 can be viewed as a generalization of Theorem 2.11 and Corollary 2.12 to the singular setting: the sheaf $\mathcal{E}$ need not be locally free, the metrics h_m may be singular, and the base space X itself may also be singular.

3. Technical results

In this section, we prepare several auxiliary propositions. The first one, although technical, generalizes [DPS94, Proposition 1.16] to compact Kähler spaces.

Proposition 3.1. *Let X be a compact Kähler space with potentially klt singularities. Let $\mathcal{E}$ be a pseudo-effective reflexive sheaf on X with vanishing first Chern class. Consider an exact sequence*

$$0 \to \mathcal{S} \to \mathcal{E} \to \mathcal{Q} \to 0 \quad \text{on } X,$$

where $\mathcal{S}$ is reflexive and $\mathcal{Q}$ is torsion-free with vanishing first Chern class. Then, there exists a Zariski closed subset $Z \subset X$ of codimension $\geqslant 3$ such that $\mathcal{Q}$ is reflexive on $X \setminus Z$.

Proof. Set $X_0 := X_{\mathrm{reg}} \cap X_{\mathcal{E}}$. By [IMZ26, Lemma 2.9], the quotient $\mathcal{Q}$ is locally free, and the sequence in the proposition is an exact sequence of vector bundles on X_0. However, this is not enough, since $\operatorname{codim}(X \setminus X_0) \geqslant 3$ does not hold in general. The proof is divided into four steps.

Step 1. Consider the reflexive sheaf

$$\mathcal{F} := \Lambda^{[s]}\mathcal{E}[\otimes](\det \mathcal{Q})^*,$$

where s is the rank of $\mathcal{Q}$. Let $\tau \in H^0(X, \mathcal{F}^*)$ be the natural section obtained from the inclusion $0 \to \mathcal{O}_X \to \mathcal{F}^*$ induced by the exact sequence in the proposition. Since

$\mathcal{E}$ is pseudo-effective and $c_1(\mathcal{Q}) = 0$, there exist singular Hermitian metrics $\{h_m\}$ on $\operatorname{Sym}^m \mathcal{F}$ such that

$$\sqrt{-1}\Theta_{h_m}(\operatorname{Sym}^m \mathcal{F}) \geqslant -\omega_X \otimes \mathrm{id} \quad \text{ on } X,$$

where ω_X is a fixed Kähler form on X (see Proposition 4.7). The function $\log|\tau^m|_{h_m^*}$ satisfies

$$\sqrt{-1}\partial\bar{\partial}\Big(\frac{1}{m}\log|\tau^m|_{h_m^*}\Big) \geqslant \frac{1}{m}\omega_X$$

on X_0, and hence extends to a quasi-psh function on X with the same curvature inequality, since $\operatorname{codim}(X \setminus X_0) \geqslant 2$ and X is normal. Thus, after passing to a suitable normalization and a subsequence, we may assume that the functions $(1/m)\log|\tau^m|_{h_m^*}$ converge to a psh function on X, which is constant since X is compact.

Step 2. Since X has potentially klt singularities, we can take a Zariski closed subset $Z_1 \subset X$ of codimension $\geqslant 3$ such that $X \setminus Z_1$ has only quotient singularities. Thus, for a sufficiently small open set $U \subset X \setminus Z_1$, we can take a finite quasi-étale cover $\nu\colon V \to U$ such that V is smooth. Since V is smooth and $\nu^{[*]}\mathcal{E}$ is reflexive, there exists a Zariski closed subset $Z \subset U$ of codimension $\geqslant 3$ such that the reflexive pullback

$$\mathcal{E}_{\text{orb}} := \nu^{[*]}\mathcal{E}$$

is locally free on $U \setminus Z$. Although Z is defined only locally on U, the non-reflexive locus of $\mathcal{Q}$ is a globally defined Zariski closed subset of X. Thus, for our purposes, it suffices to prove that $\mathcal{Q}$ is reflexive on $U \setminus Z$.

Step 3. Let $\mathcal{S}_{\text{sat}}$ be the saturation of $\nu^{[*]}\mathcal{S} \subset \nu^{[*]}\mathcal{E}$, and consider

$$0 \to \mathcal{S}_{\text{sat}} \to \mathcal{E}_{\text{orb}} = \nu^{[*]}\mathcal{E} \to \mathcal{Q}_{\text{tf}} := \mathcal{E}_{\text{orb}}/\mathcal{S}_{\text{sat}} \to 0 \quad \text{ on } V. \tag{3.1.1}$$

Note that $\mathcal{Q}_{\text{tf}}$ is torsion-free on V. As in Step 1, we obtain the natural section $\tau_{\text{orb}} \in H^0(V, \mathcal{G}^*)$ induced by (3.1.1), where

$$\mathcal{G} := \Lambda^{[s]}\mathcal{E}_{\text{orb}}[\otimes](\det \mathcal{Q}_{\text{tf}})^*.$$

In this step, we prove that τ_{orb} is nowhere vanishing over $U \setminus Z$. Note that if τ_{orb} is nowhere vanishing over $U \setminus Z$, then by [DPS94, Lemma 1.20], both $\mathcal{S}_{\text{sat}}$ and $\mathcal{Q}_{\text{tf}}$ are locally free over $U \setminus Z$.

Consider singular Hermitian metrics $\{h_{\text{orb,m}}\}$ on $\operatorname{Sym}^m \mathcal{G}$ defined by pulling back $\{h_m\}$, namely $h_{\text{orb,m}} := \nu^* h_m$. More precisely, the metric h_m is a priori defined only on $X_{\text{reg}} \cap X_{\operatorname{Sym}^m \mathcal{F}}$. Hence the pullback $h_{\text{orb,m}}$ is defined on $V_1 := \nu^{-1}(X_{\text{reg}} \cap X_{\operatorname{Sym}^m \mathcal{F}})$ via $\nu^*\big(\operatorname{Sym}^m \mathcal{G}\big) = \operatorname{Sym}^m \mathcal{F}$. By $\operatorname{codim}(V \setminus V_1) \geqslant 2$, the metric $h_{\text{orb,m}}$ extends to a singular Hermitian metric on $\operatorname{Sym}^m \mathcal{G}$. By construction, we have

$$\tau_{\text{orb}} = \nu^*\tau \quad \text{and} \quad \log|\tau_{\text{orb}}^m|_{h_{\text{orb,m}}^*} = \nu^*\big(\log|\tau^m|_{h_m^*}\big) \quad \text{ on } V_1.$$

The function $\log|\tau_{\text{orb}}^m|_{h_{\text{orb,m}}^*}$ is quasi-psh on V_1, and hence extends across $V \setminus V_1$. On the other hand, the function $\log|\tau^m|_{h_m^*}$ also extends to X. Therefore, we obtain

$$\log|\tau_{\text{orb}}^m|_{h_{\text{orb,m}}^*} = \nu^*\big(\log|\tau^m|_{h_m^*}\big) \quad \text{ on } V. \tag{3.1.2}$$

Assume for contradiction that $\tau_{\mathrm{orb}}(p)=0$ for some point $p\in\nu^{-1}(U\setminus Z)$. To consider Lelong numbers on smooth varieties, we take a resolution of singularities $\pi\colon\tilde{X}\to X$ and a smooth variety $\tilde{V}$ fitting into the commutative diagram

$$\begin{tikzcd}
\tilde{V} \arrow[r, "\tilde{\nu}"] \arrow[d, "\tilde{\pi}"] & \pi^{-1}(U) \arrow[r, hook] & \tilde{X} \arrow[d, "\pi"] \\
V \arrow[r, "\nu"] & U \arrow[r, hook] & X.
\end{tikzcd}$$

Take a point $\tilde{p}\in\tilde{\pi}^{-1}(p)$. Since $\mathcal{G}$ is locally free around p, the Lelong number

$$\frac{1}{m}\nu\big(\tilde{\pi}^*(\log|\tau_{\mathrm{orb}}^m|_{h^*_{\mathrm{orb,m}}}),\tilde{p}\big)=\frac{1}{m}\nu\big(\log|\tilde{\pi}^*\tau_{\mathrm{orb}}^m|_{\tilde{\pi}^*h^*_{\mathrm{orb,m}}},\tilde{p}\big)$$

is bounded from below uniformly in m. Indeed, writing τ_{orb}^m locally as

$$\tau_{\mathrm{orb}}^m=\sum_I\tau_I e_I,$$

each holomorphic function τ_I vanishes to order $\geqslant m$ at p by $\tau_{\mathrm{orb}}(p)=0$. Here $\{e_i\}_{i=1}^r$ is a local frame of $\mathcal{G}$, I is a multi-index of degree m, and $e_I:=\prod_{i\in I}e_i$. Moreover, since $\log|u|_{h^*_{\mathrm{orb,m}}}$ is quasi-psh for any local section u, the inner product $\langle e_I,e_J\rangle_{h^*_{\mathrm{orb,m}}}$ is locally bounded from above. Consequently, there exists a constant $C>0$ such that

$$|\tau_{\mathrm{orb}}^m|_{h^*_{\mathrm{orb,m}}}\leqslant C\sum_I|\tau_I|.$$

This implies that the Lelong number of $(1/m)\log|\tau_{\mathrm{orb}}^m|_{h^*_{\mathrm{orb,m}}}$ at p is at least 1.

By [Fav99, Theorem 2], the Lelong number

$$\frac{1}{m}\nu\big(\pi^*\log|\tau^m|_{h^*_m},\tilde{\nu}(\tilde{p})\big)$$

is also bounded from below uniformly in m. However, the function $(1/m)\pi^*\log|\tau^m|_{h^*_m}$ converges to a constant by (3.1.2), which is a contradiction.

Step 4. After taking a Galois closure, we may assume that $\nu\colon V\to U$ is a Galois cover with Galois group G. By Step 3, both $\mathcal{S}_{\mathrm{sat}}$ and $\mathcal{Q}_{\mathrm{tf}}$ are locally free over $U\setminus Z$. In particular, $\mathcal{S}_{\mathrm{sat}}$, $\mathcal{E}_{\mathrm{orb}}$, and $\mathcal{Q}_{\mathrm{tf}}$ are reflexive over $U\setminus Z$. Hence, their G-invariant pushforwards $(\nu_*(\bullet))^G$ are also reflexive over $U\setminus Z$ by [GKKP11, Lemma A.3]. Thus, since $\mathcal{S}_{\mathrm{sat}}$, $\mathcal{E}_{\mathrm{orb}}$, and $\mathcal{Q}_{\mathrm{tf}}$ are the pullbacks of $\mathcal{S}$, $\mathcal{E}$, and $\mathcal{Q}$ over $U_{\mathrm{reg}}\cap U_{\mathcal{E}}$, we obtain

$$(\nu_*(\mathcal{S}_{\mathrm{sat}}))^G=\mathcal{S}^{**}=\mathcal{S},\quad(\nu_*(\mathcal{E}_{\mathrm{orb}}))^G=\mathcal{E}^{**}=\mathcal{E},\quad(\nu_*(\mathcal{Q}_{\mathrm{tf}}))^G=\mathcal{Q}^{**}.$$

on $U\setminus Z$. Together with the exactness of the functor $(\nu_*(\bullet))^G$ (see [GKKP11, Lemma A.3]), we deduce

$$0\to\mathcal{S}\to\mathcal{E}\to\mathcal{Q}^{**}\to 0\quad\text{ on }U\setminus Z.$$

This shows that $\mathcal{Q}\cong\mathcal{Q}^{**}$ on $U\setminus Z$. This completes the proof.

$\square$

The next proposition analyzes the singularities of the limit metric on $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ induced by Hermitian metrics on $\mathcal{E}$.

Proposition 3.2. *Let $\{H_\varepsilon\}$ be a sequence of smooth Hermitian metrics on $\mathbb{C}^{n+1}$, and let h_ε be the induced metrics on $\mathcal{O}_{\mathbb{P}^n}(1)$ via the quotient map $\mathbb{P}^n \times \mathbb{C}^{n+1} \to \mathcal{O}_{\mathbb{P}^n}(1)$.*

(1) After applying a unitary change of coordinates on $\mathbb{C}^{n+1}$, rescaling H_ε, and passing to a subsequence, the metrics $\{h_\varepsilon\}$ converge to a singular metric h on $\mathcal{O}_{\mathbb{P}^n}(1)$ whose weight can be written as

$$\log \sum_{i=0}^{n} \lambda_i |\zeta_i|^2,$$

where $[\zeta_0 \colon \cdots \colon \zeta_n]$ are the homogeneous coordinate on $\mathbb{P}^n$ and $\{\lambda_i\}$ are real numbers with $\lambda_0 \geqslant \cdots \geqslant \lambda_n \geqslant 0$.

(2) Suppose that the limit metric h is singular (i.e. $\lambda_n = 0$). Let g be a smooth metric on $\mathcal{O}_{\mathbb{P}^n}(1)$. Then, there exists $0 < \delta < 1$ such that the space of L^2-integrable sections

$$H^0\big(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(n+2) \otimes \mathcal{I}\big((g^{(1-\delta)} \cdot h^{\delta})^{(n+2)}\big)\big)$$

is nonzero and is a proper subspace of $H^0(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes \mathcal{O}(n+2))$.

Remark 3.3. (1) The second conclusion does not hold for an arbitrary singular metric h on $\mathcal{O}_{\mathbb{P}^n}(1)$. A key point in the proposition is that h arises as a limit of metrics induced by Hermitian metrics on $\mathbb{C}^{n+1}$. Indeed, in the one-dimensional case (i.e. $n = 1$), consider the singular metric h on $\mathcal{O}_{\mathbb{P}^1}(1)$ such that

$$\sqrt{-1}\Theta_h = \frac{1}{3}([p] + [q] + [r]),$$

where $p, q, r \in \mathbb{P}^1$ are distinct points. Then, in the case $\delta = 1$, the line bundle

$$K_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(n+2) \otimes \mathcal{I}\big((g^{(1-1)} \cdot h^1)^3\big) \cong \mathcal{O}_{\mathbb{P}^1}(-2)$$

has no nonzero sections. Moreover, for any $1 > \delta > 0$, the space of L^2-integrable sections is not a proper subspace of the ambient space, by $\mathcal{I}\big((g^{(1-\delta)} \cdot h^{\delta})^3\big) = \mathcal{O}_X$.

(2) The proposition concerns limit metrics arising from the family $\{H_\varepsilon\}$ in the definition of weakly positively curved sheaves. From this viewpoint, it is natural to study limit metrics appearing in the definition of pseudo-effective sheaves. Although this is not used in this paper, we formulate the following question.

Question 3.4. *Let $\{H_m\}$ be smooth Hermitian metrics on the m-th symmetric power $\mathrm{Sym}^m(\mathbb{C}^{n+1})$, and let $\{h_m\}$ be the metrics on $\mathcal{O}_{\mathbb{P}^n}(m)$ induced by the surjective morphism*

$$\mathbb{P}^n \times \mathrm{Sym}^m \mathbb{C}^{n+1} \to \mathcal{O}_{\mathbb{P}^n}(m).$$

Can we describe the limit metric h obtained from $\{h_m^{1/m}\}$, or the singularities of h?

Proof of Proposition 3.2. (1) Let G be the standard Hermitian metric on $\mathbb{C}^{n+1}$ and let $(e_0, \ldots, e_n)$ be the standard basis. Then, there exists a unitary transformation

$U_\varepsilon \in U(n+1)$ such that $(U_\varepsilon(e_0), \ldots, U_\varepsilon(e_n))$ is an orthogonal basis with respect to H_ε. Therefore, we can write

$$G = \sum_{i=0}^{n} e_i^* \otimes \overline{e_i^*} = \sum_{i=0}^{n} U_\varepsilon(e_i)^* \otimes \overline{U_\varepsilon(e_i)^*}, \quad H_\varepsilon = \sum_{i=0}^{n} \lambda_{i,\varepsilon}\, U_\varepsilon(e_i)^* \otimes \overline{U_\varepsilon(e_i)^*},$$

where $\lambda_{0,\varepsilon} \geqslant \cdots \geqslant \lambda_{n,\varepsilon} > 0$ are the eigenvalues of H_ε with respect to G. Replacing H_ε with $(1/\lambda_{0,\varepsilon}) \cdot H_\varepsilon$, we may assume that $\lambda_{0,\varepsilon} = 1$.

Let h_ε be the induced metric on $\mathcal{O}_{\mathbb{P}^n}(1)$. After passing to a subsequence, we may assume that $\lambda_{i,\varepsilon} \to \lambda_i \in [0,1]$ for all i, and that $U_\varepsilon \to U$, since the unitary group is compact. Then h_ε converges (in the sense of weights in L^1_{loc}) to a singular metric on $\mathcal{O}_{\mathbb{P}^n}(1)$, which we denote by h. More precisely, for $\zeta = (\zeta_0, \ldots, \zeta_n) \in \mathbb{C}^{n+1} \setminus \{0\}$, let $[\,\zeta\,] \in \mathbb{P}^n$ be the corresponding point. If we set $\eta_\varepsilon := U_\varepsilon^{-1}\zeta$, then the weight function of h_ε is given by

$$\varphi_\varepsilon([\,\zeta\,]) = \log\Big(\sum_{i=0}^{n} \lambda_{i,\varepsilon}\, |\eta_{\varepsilon,i}|^2\Big) = \log\Big(\sum_{i=0}^{n} \lambda_{i,\varepsilon}\, |(U_\varepsilon^{-1}\zeta)_i|^2\Big).$$

Hence φ_ε converges to

$$\varphi([\,\zeta\,]) = \log\Big(\sum_{i=0}^{n} \lambda_i\, |(U^{-1}\zeta)_i|^2\Big),$$

which defines the limit metric h on $\mathcal{O}_{\mathbb{P}^n}(1)$. Finally, after changing homogeneous coordinates by the unitary transformation U, the weight of h can be written as $\log \sum_{i=0}^{n} \lambda_i |\zeta_i|^2$.

(2) For the second conclusion, after applying a unitary change of coordinates, we may assume that G is the standard Hermitian inner product, that g is induced by G, and that h can be written as in (1). Fix a basis $(e_0, \ldots, e_n)$ of

$$\mathbb{C}^{n+1} \cong H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \cong H^0(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(n+2)),$$

which induces the homogeneous coordinates in (1).

Define a singular metric H_δ on $\mathcal{O}_{\mathbb{P}^n}(n+2)$ by

$$H_\delta := (g^{(1-\delta)} \cdot h^{\delta})^{(n+2)}.$$

By assumption, there exists an integer $0 < p < n$ such that

$$\lambda_0 \geqslant \cdots \geqslant \lambda_p > \lambda_{p+1} = \cdots = \lambda_n = 0.$$

A straightforward computation shows that, for any $0 \leqslant k \leqslant n$, we have

$$\|e_0\|^2_{H_\delta} = \int_{\mathbb{C}^n} \frac{1}{(1 + \sum_{i=1}^{n} |z_i|^2)^{(1-\delta)(n+2)} \cdot (\lambda_0 + \sum_{i=1}^{p} \lambda_i |z_i|^2)^{\delta(n+2)}}\, dV$$

$$\|e_k\|^2_{H_\delta} = \int_{\mathbb{C}^n} \frac{|z_k|^2}{(1 + \sum_{i=1}^{n} |z_i|^2)^{(1-\delta)(n+2)} \cdot (\lambda_0 + \sum_{i=1}^{p} \lambda_i |z_i|^2)^{\delta(n+2)}}\, dV,$$

where dV denotes the Lebesgue measure on $\mathbb{C}^n$.

From these expressions, we obtain the following integrability conditions:

- $\|e_k\|^2_{H_\delta} < \infty$ for $0 \leqslant k \leqslant p$ if and only if $(1-\delta)(n+2) > (n-p)$.

- $\|e_k\|^2_{H_\delta} < \infty$ for $p+1 \leqslant k \leqslant n$ if and only if $(1-\delta)(n+2) > (n-p+1)$.

Choose $0 < \delta < 1$ such that $n-p+1 \geqslant (1-\delta)(n+2) > n-p$. Then, the space

$$H^0(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(n+2) \otimes \mathcal{I}\big((g^{(1-\delta)} \cdot h^\delta)^{(n+2)}\big)\big)$$

is nonzero, since it contains the subspace spanned by $\{e_i\}_{i=0}^p$, and it is properly contained in $H^0(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(n+2))$ since it does not contain $e_{p+1}, \dots, e_n$.

Checking these integrability conditions is straightforward. For the reader's convenience, we verify the second one. (The first one can be proved in the same way.) It is enough to check integrability as $|z| \to \infty$. Since $\lambda_0, \dots, \lambda_p$ are positive constants, they do not affect integrability, and we may assume $\lambda_i = 1$ for simplicity. Set

$$a := (n+2)(1-\delta), \quad b := (n+2)\delta, \quad q := n-p.$$

Let us consider the integrability of

$$I := \int_{\mathbb{C}^n} \frac{|z_{p+1}|^2}{\big(1+\sum_{i=1}^n |z_i|^2\big)^a \big(1+\sum_{i=1}^p |z_i|^2\big)^b}\, dV.$$

Write $z = (z', z'') \in \mathbb{C}^p \times \mathbb{C}^q$ with

$$z' = (z_1, \dots, z_p), \quad z'' = (z_{p+1}, \dots, z_n),$$

and set $r := \|z'\|$, $s := \|z''\|$ so that $\|z\|^2 = r^2 + s^2$. Using polar coordinates on $\mathbb{C}^p \cong \mathbb{R}^{2p}$ and $\mathbb{C}^q \cong \mathbb{R}^{2q}$, we write

$$z' = r\,\xi, \quad (r \in [0,\infty),\ \xi \in S^{2p-1}), \quad z'' = s\,\omega, \quad (s \in [0,\infty),\ \omega \in S^{2q-1}).$$

The Euclidean volume forms can be written as

$$dV_{\mathbb{C}^p} = r^{2p-1}\, dr\, d\sigma_p(\xi), \quad dV_{\mathbb{C}^q} = s^{2q-1}\, ds\, d\sigma_q(\omega),$$

where $d\sigma_p$ and $d\sigma_q$ denote the standard sphere measures on S^{2p-1} and S^{2q-1}, respectively.

Since $|z_{p+1}|^2$ depends only on z'', by writing $z_{p+1} = s\omega_1$ and using Fubini's theorem, we obtain

$$\begin{aligned} I &= \int_0^\infty \int_0^\infty \int_{S^{2p-1}} \int_{S^{2q-1}} \frac{s^2|\omega_1|^2}{(1+r^2+s^2)^a(1+r^2)^b}\, r^{2p-1} s^{2q-1}\, d\sigma_q(\omega)\, d\sigma_p(\xi)\, ds\, dr \\ &= C \int_0^\infty \int_0^\infty \frac{r^{2p-1} s^{2q+1}}{(1+r^2+s^2)^a(1+r^2)^b}\, ds\, dr, \end{aligned}$$

where $C := \big(\int_{S^{2p-1}} d\sigma_p\big)\big(\int_{S^{2q-1}} |\omega_1|^2\, d\sigma_q\big) > 0$. Thus it suffices to consider

$$\begin{aligned} I' &:= \int_0^\infty \int_0^\infty \frac{r^{2p-1} s^{2q+1}}{(1+r^2+s^2)^a(1+r^2)^b}\, ds\, dr \\ &= \int_0^\infty \frac{r^{2p-1}}{(1+r^2)^b}\, dr \Big(\int_0^\infty \frac{s^{2q+1}}{(1+r^2+s^2)^a}\, ds \Big). \end{aligned}$$

Changing variables $s = \sqrt{1+r^2}\, t$, we have

$$1 + r^2 + s^2 = (1+r^2)(1+t^2), \quad ds = \sqrt{1+r^2}\, dt, \quad s^{2q+1} = (1+r^2)^{q+\frac{1}{2}} t^{2q+1}.$$

Hence, we obtain

$$\int_0^\infty \frac{s^{2q+1}}{(1+r^2+s^2)^a}\,ds = (1+r^2)^{q+1-a}\int_0^\infty \frac{t^{2q+1}}{(1+t^2)^a}\,dt.$$

Therefore, we have

$$\begin{aligned} I' &= \Big(\int_0^\infty \frac{t^{2q+1}}{(1+t^2)^a}\,dt\Big)\cdot\Big(\int_0^\infty r^{2p-1}(1+r^2)^{q+1-a-b}\,dr\Big) \\ &= \Big(\int_0^\infty \frac{t^{2q+1}}{(1+t^2)^a}\,dt\Big)\cdot\Big(\int_0^\infty r^{2p-1}(1+r^2)^{q+1-(n+2)}\,dr\Big), \end{aligned}$$

where we used $a+b=n+2$.

Now the first integral converges if and only if $a>q+1$, i.e. $a>n-p+1$, since its integrand behaves like $t^{2q+1-2a}$ as $t\to\infty$. The second integral always converges, since its integrand behaves like $r^{2p-1+2(q+1-(n+2))}=r^{2(p+q-(n+1))-1}$ as $r\to\infty$. Thus $I<\infty$ holds if and only if $a>n-p+1$, which is exactly the second condition. $\square$

4. Proofs of the main results

4.1. **Characterization of pseudo-effective sheaves.** In this subsection, we study a torsion-free sheaf $\mathcal{E}$ on a normal compact analytic variety X. We then establish Theorem 4.2, which gives a characterization of the pseudo-effectivity of $\mathcal{E}$ in terms of the hyperplane bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. As an application, we give a criterion for the condition $c_1(\det\mathcal{E})=0$ in Proposition 4.7.

We first fix the notation used throughout this subsection.

Setup 4.1. Let $\mathcal{E}$ be a torsion-free sheaf on a normal compact analytic variety X. Let $\pi_{\mathcal{E}}\colon \mathbb{P}(\mathcal{E})\to X$ denote the normalization of the irreducible component of the projectivization $\mathbf{Proj}(\bigoplus_{m=0}^\infty \operatorname{Sym}^m\mathcal{E})$ of the graded sheaf $\bigoplus_{m=0}^\infty \operatorname{Sym}^m\mathcal{E}$ that dominates X. Let $\pi\colon P\to\mathbb{P}(\mathcal{E})$ be a desingularization of $\mathbb{P}(\mathcal{E})$ fitting into the commutative diagram:

(4.1.1)

$$\begin{tikzcd} \mathbb{P}(\mathcal{E}) \arrow[d, "\pi_{\mathcal{E}}"'] & P \arrow[l, "\pi"'] \arrow[dl, bend left, "p"] \\ X. & \end{tikzcd}$$

Set $X_0 := X_{\mathrm{reg}}\cap X_{\mathcal{E}}$ and $P_0 := p^{-1}(X_0)$. Note that $\operatorname{codim}(X\setminus X_0)\geqslant 2$. We may assume that $\pi\colon P\to\mathbb{P}(\mathcal{E})$ is an isomorphism over X_0 so that $p\colon P_0\to X_0$ can be identified with the projective space bundle $\mathbb{P}(\mathcal{E}|_{X_0})\to X_0$ associated with the locally free sheaf $\mathcal{E}|_{X_0}$. We use the following notation:

- $L:=\pi^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, where $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is the pullback of the hyperplane bundle on $\mathbf{Proj}(\bigoplus_{m=0}^\infty \operatorname{Sym}^m\mathcal{E})$ to the normalization of the main component;
- ω_P is a Kähler form on P;
- ω_X is a Kähler form on X;

- Λ is an effective p-exceptional divisor such that
$$p_*\mathcal{O}_P(m(L+\Lambda)) = \mathrm{Sym}^{[m]}\,\mathcal{E}.$$
See [Nak04, §III.5, Lemma 5.10, pp. 107–108] for the existence of such a divisor Λ.

After replacing P by a further blow-up, we may assume that
$$P \setminus P_0 \text{ is divisorial and the support of } \Lambda \text{ coincides with } P \setminus P_0.$$

Then, we have the following theorem.

Theorem 4.2. *In the notation of Setup 4.1, the sheaf $\mathcal{E}$ is pseudo-effective (in the sense of Definition 2.2) if and only if the line bundle $L+\Lambda$ admits singular metrics $\{h_\varepsilon\}_{\varepsilon>0}$ satisfying the following conditions:*
(1) *$\sqrt{-1}\Theta_{h_\varepsilon} + \varepsilon p^*\omega_X \geqslant \delta_\varepsilon \omega_P$ holds on P for some $\delta_\varepsilon > 0$;*
(2) *$\{x \in P \,|\, \nu(h_\varepsilon, x) > 0\}$ does not dominate X, where $\nu(h_\varepsilon, x)$ denotes the Lelong number of the local weight of h_ε.*

By Remark 4.4, the line bundle L does not admit the desired metrics $\{h_\varepsilon\}_{\varepsilon>0}$ without adding Λ. The "if" part follows from the positivity of direct image sheaves and a similar argument is given in the proof of Theorem 1.2. Therefore, we focus on proving the "only if" part, which depends on the following two lemmas.

Lemma 4.3. *In the notation of Setup 4.1, assume that $\mathcal{E}$ is pseudo-effective. Then, the line bundle $L|_{P_0}$ admits singular Hermitian metrics $\{h_\varepsilon\}_{\varepsilon>0}$ satisfying the same conditions in Theorem 4.2, with P and X replaced by P_0 and X_0, respectively.*

Proof. We first show that there exists a singular metric h on L such that h is smooth on P_0 and satisfies
$$\sqrt{-1}\Theta_h + C\, p^*\omega_X \geqslant a\,\omega_P$$
for some constants $C \gg 1$ and $1 \gg a > 0$. The hyperplane bundle associated with $\mathbf{Proj}(\bigoplus_{m=0}^\infty \mathrm{S}^m\mathcal{E})$ is relatively ample with respect to $\mathbf{Proj}(\bigoplus_{m=0}^\infty \mathrm{S}^m\mathcal{E}) \to X$, and the structure sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{E})}$ is relatively ample with respect to the normalization $\mathbb{P}(\mathcal{E}) \to \mathbf{Proj}(\bigoplus_{m=0}^\infty \mathrm{S}^m\mathcal{E})$. This shows that the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is relatively $\pi_{\mathcal{E}}$-ample. Hence, there exists a smooth metric g_1 on $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ such that $\sqrt{-1}\Theta_{g_1} + C\,\pi_{\mathcal{E}}^*\omega_X > 0$ for some constant $C > 0$. Let E be an effective π-exceptional divisor such that $-E$ is relatively π-ample. Then, the line bundle $\mathcal{O}_P(E)$ admits a smooth metric g_2 such that
$$\begin{aligned}&\sqrt{-1}\Theta_{\pi^*g_1} + Cp^*\omega_X - \delta\sqrt{-1}\Theta_{g_2}(E)\\ =&\pi^*\big(\sqrt{-1}\Theta_{g_1} + C\pi_{\mathcal{E}}^*\omega_X\big) - \delta\sqrt{-1}\Theta_{g_2}(E) > 0\end{aligned}$$
for $1 \gg \delta > 0$. We define a singular metric h on $L = (L-\delta E) + \delta E$ by
$$h := (\pi^*g_1)\cdot g_2^{-\delta}\cdot g_{[\delta E]},$$
where $g_{[\delta E]}$ denotes the singular metric associated with the effective divisor δE. Then h satisfies the required properties.

Next, we construct singular metrics $\{g_\varepsilon\}_{\varepsilon>0}$ on $L|_{P_0}$ satisfying the following properties:

- $\sqrt{-1}\Theta_{g_\varepsilon} + \varepsilon p^*\omega_X \geqslant 0$ on P_0;
- $\{x \in P_0 \mid \nu(g_\varepsilon, x) > 0\}$ does not dominate X_0.

By the definition of pseudo-effective sheaves, there exist singular Hermitian metrics $\{h_m\}_{m\in\mathbb{Z}_+}$ on $\mathrm{S}^{[m]}\mathcal{E}$ such that $\sqrt{-1}\Theta_{h_m} \geqslant -\omega_X \otimes \mathrm{id}$. Note that $p\colon P_0 \to X_0$ is the projective space bundle $\mathbb{P}(\mathcal{E}|_{X_0}) \to X_0$ of the locally free sheaf $\mathcal{E}|_{X_0}$ Hence, the pullbacks p^*h_m define singular Hermitian metrics on $p^*\operatorname{Sym}^m\mathcal{E}|_{P_0}$ such that $\sqrt{-1}\Theta_{p^*h_m} \geqslant -p^*\omega_X \otimes \mathrm{id}$ on P_0. Let H_m be the singular Hermitian metric on $mL|_{P_0}$ induced by the natural surjective morphism

$$p^*\operatorname{Sym}^m\mathcal{E}|_{P_0} \to mL|_{P_0}.$$

By construction, we have $\sqrt{-1}\Theta_{H_m} \geqslant -p^*\omega_X$, and $\{x \in P_0 \mid \nu(H_m, x) > 0\}$ does not dominate X_0. Choosing m such that $\varepsilon > 1/m$, the metrics $H_m^{1/m}$ on $L|_{P_0}$ give the desired family $\{g_\varepsilon\}_{\varepsilon>0}$.

Finally, we define singular metrics on $L|_{P_0}$ by

$$h_\varepsilon := g_{\varepsilon/2}^{1-\delta} \cdot h^\delta.$$

From the curvature properties of h and g_ε, we obtain

$$\sqrt{-1}\Theta_{h_\varepsilon} \geqslant -\left(\frac{\varepsilon}{2} + \delta C\right) p^*\omega_X + \delta a\omega_P.$$

Setting $\delta := \varepsilon/2C$, we conclude that h_ε satisfies the required properties. $\square$

Remark 4.4. In general, a metric h_ε on $L|_{P_0}$ *cannot* be extended to a singular Hermitian metric on L. For example, let $\mathcal{E}$ be the ideal sheaf of a submanifold $Z \subset X$ of codimension $\geqslant 2$. Then $\mathcal{E}$ is pseudo-effective in the sense of Definition 2.2. We may take P to be the blow-up of X along Z with exceptional divisor E, so that $L = -E$. Then, the line bundle $L = -E$ does not admit singular Hermitian metrics h_ε satisfying $\sqrt{-1}\Theta_{h_\varepsilon} \geqslant -\varepsilon\omega_P$ for all $\varepsilon > 0$. Nevertheless, since we may take $\Lambda := E$, the corresponding metrics on $L + \Lambda = \mathcal{O}_P$ can extend.

As noted above, extending a metric h_ε defined only on P_0 to P is a delicate problem, since $P \setminus P_0$ can be divisorial. Nevertheless, by considering $L + \Lambda$ instead of L, we can extend h_ε to a metric on $L + \Lambda$ using the identification $L|_{P_0} \cong (L + \Lambda)|_{P_0}$, which is a key step in the proof of Theorem 4.2.

Theorem 4.5. *Let h be a singular metric on $L|_{P_0}$ such that*

$$\sqrt{-1}\Theta_h + C\,p^*\omega_X \geqslant a\omega_P \quad \text{on } P_0,$$

for some constants $C \gg 1$ and $0 < a \ll 1$. Then, when h is regarded as a metric on $(L+\Lambda)|_{P_0}$ via the identification $L|_{P_0} \cong (L+\Lambda)|_{P_0}$ (for which we use the same notation h), the weight of h is locally bounded above near $P \setminus P_0$. In particular, the metric h extends across $P \setminus P_0$ to a singular metric on $L + \Lambda$ (which we denote by the same symbol h).

More precisely, this means that the weight of the metric $h \cdot h_\Lambda$ on $(L+\Lambda)|_{P_0}$ is locally bounded above, and hence extends to a singular metric on $L+\Lambda$, where h_Λ is the singular metric on Λ associated with the effective divisor Λ.

As a direct consequence, the metric h_ε constructed in Lemma 4.3 extends to a singular metric on $L + \Lambda$ satisfying the desired properties as in Theorem 4.2, completing the proof of Theorem 4.2.

Proof. We regard the metric h on $L|_{P_0}$ as a metric on $(L+\Lambda)|_{P_0}$ via the identification $L|_{P_0} \cong (L+\Lambda)|_{P_0}$. It suffices to show that the local weight φ of h is bounded from above in a neighborhood of $P \setminus P_0$. The problem is local on X, so we may assume that X is Stein. Then, after replacing h by $h\, e^{-p^*\psi}$, where ψ is a local potential of $C\omega_X$, we may assume that $\sqrt{-1}\Theta_h$ is a Kähler current by the curvature assumption.

For $m \in \mathbb{Z}_{>0}$, we consider the L^2-space

$$H^0_{L^2}(P_0, m(L+\Lambda))_{h^m,\omega_P}$$

of holomorphic sections σ on P_0 with finite L^2-norm $\|\sigma\|_{h^m,\omega_P} < \infty$. Let B_m denote the Bergman kernel metric of this Hilbert space, whose local weight b_m is defined by

$$b_m(p) := \frac{1}{m} \sup\Big\{ \log|\sigma|(p) \;\Big|\; \|\sigma\|_{h^m,\omega_P} \leqslant 1 \Big\} \qquad \text{for } p \in P_0.$$

Then, for every relatively compact set $K \Subset X$, there exists a constant $C = C(K)$, independent of m, such that

$$\varphi(p) \leqslant b_m(p) + \frac{C}{m} \qquad \text{for } p \in P_0 \cap p^{-1}(K). \tag{4.5.1}$$

This can be verified by the standard L^2-methods for $\bar\partial$-equations (see, for example, [Dem10, (13.21) Theorem, p. 138]). We briefly explain the proof for the reader's convenience.

Fix a point $p_0 \in P_0 \cap p^{-1}(K)$. Take an open ball $B \Subset P$ centered at p_0 and a hypersurface $H \subset X$ such that $X \setminus X_0 \subset H$ and $p_0 \notin p^{-1}(H)$. Note that $P \setminus p^{-1}(H)$ is a weakly pseudoconvex Kähler manifold and that $B \setminus p^{-1}(H)$ is Stein. This allows us to solve $\bar\partial$-equations on $P \setminus p^{-1}(H)$ and to apply the L^2-extension theorem on $B \setminus p^{-1}(H)$. If $\varphi(p_0) = -\infty$, the inequality is obvious, so we may assume $\varphi(p_0) > -\infty$. For a given $\alpha \in \mathbb{C}$, by the L^2-extension theorem, there exists a holomorphic function f on $B \setminus p^{-1}(H)$ such that

$$f(p_0) = \alpha \quad \text{ and } \quad \int_{B\setminus p^{-1}(H)} |f|^2 e^{-m\varphi}\, dV_{\omega_P} \leqslant C_1 |\alpha|^2 e^{-m\varphi(p_0)}.$$

Fix a relatively compact subset $K_P \Subset B$ and consider a cut-off function θ supported in B and equal to 1 on K_P. Since $\sqrt{-1}\Theta_h$ is a Kähler current, the standard L^2-methods for $\bar\partial$-equations enable us to solve $\overline{\partial} g = \overline{\partial}(\theta f)$ on $P \setminus p^{-1}(H)$ with the L^2-estimate

$$\int_{P\setminus p^{-1}(H)} |g|^2_{h^m}\, e^{-2n\theta \log|z - z(p_0)|}\, dV_{\omega_P} \leqslant C_2 \int_{B\setminus p^{-1}(H)} |f|^2 e^{-m\varphi}\, dV_{\omega_P},$$

where z is a coordinate on B and $n := \dim P$. Then, the section $\sigma := \theta f - g$ satisfies

$$\sigma(p_0) = f(p_0) = \alpha \quad \text{and} \quad \|\sigma\|_{h^m} \leqslant C_3 |\alpha|^2 e^{-m\varphi(p_0)}. \tag{4.5.2}$$

Here, on the left-hand side, we identify σ with a holomorphic function via a fixed local frame. The section σ is defined on $P \setminus p^{-1}(H)$. On the other hand, since h is defined even near every point of $p^{-1}(H) \cap P_0$, every L^2-section on $P \setminus p^{-1}(H)$ extends to P_0, giving the isomorphism

$$H^0_{L^2}(P_0, m(L + \Lambda))_{h^m, \omega_P} \cong H^0_{L^2}(P \setminus p^{-1}(H), m(L + \Lambda))_{h^m, \omega_P}.$$

This shows that σ can be regarded as a section on P_0. We can choose α so that the right-hand side in (4.5.2) equals 1, and thus we obtain inequality (4.5.1).

Henceforth, to simplify notation, we assume $m = 1$ and set $L_\Lambda := L + \Lambda$. We will show that the weight $b := b_1$ of the Bergman kernel is locally bounded from above near $P \setminus P_0$. The crucial point is that every section of L_Λ on P_0 extends holomorphically to P, since $p_*(L_\Lambda)$ is reflexive by the choice of Λ.

Fix a trivialization of L_Λ on an open ball $B \subset P$. Then, for each point $p \in B_0 := B \cap P_0$, we consider the evaluation map

$$\mathrm{ev}_p \colon H^0_{L^2}(P_0, L_\Lambda)_{h, \omega_P} \longrightarrow H^0(B_0, L_\Lambda) \cong H^0(B_0, \mathcal{O}_B) \longrightarrow \mathbb{C}.$$

By taking duals and the operator norm, we can interpret the weight b as the composition

$$b = b_1 \colon\ B_0 \xrightarrow{\mathrm{ev}^*} \left(H^0_{L^2}(P_0, L_\Lambda)_{h, \omega_P}\right)^* \xrightarrow{\|\cdot\|_{\mathrm{op}}} \mathbb{R},$$

where the dual space is equipped with the strong topology. By Montel's theorem, the map ev^* is continuous. Hence, it is enough to extend ev^* continuously to B; once this is done, the weight function $b = b_1$ is bounded on compact subsets of B.

The restriction map

$$H^0(P, L_\Lambda) \longrightarrow H^0(P_0, L_\Lambda)$$

is an isomorphism of Fréchet spaces with the topology induced by local sup-norms. Indeed, this map is clearly continuous, injective, and linear. Moreover, it is surjective, since we have

$$H^0(P, L_\Lambda) \cong H^0(X, p_*(L_\Lambda)) \cong H^0(X_0, p_*(L_\Lambda)) \cong H^0(P_0, L_\Lambda)$$

by reflexivity. Hence, the map is a homeomorphism by the open mapping theorem. This induces a continuous map

$$H^0(P, L_\Lambda)^* \cong H^0(P_0, L_\Lambda)^* \to \left(H^0_{L^2}(P_0, L_\Lambda)_{h, \omega_P}\right)^*.$$

On the other hand, there exists a continuous map

$$B \to H^0(B, L_\Lambda)^* \to H^0(P, L_\Lambda)^*.$$

The composition of these maps extends ev^* continuously to B, proving the desired local boundedness. $\square$

The following theorem is an almost direct consequence of the above arguments. For completeness, we briefly sketch the proof.

Theorem 4.6. *Let G be a singular Hermitian metric on $\operatorname{Sym}^m \mathcal{E}$ such that*

$$\sqrt{-1}\Theta_G \geqslant \theta \otimes \mathrm{id},$$

where θ is a smooth $(1,1)$-form on X which locally admits potentials and $m \in \mathbb{Z}_+$ is a fixed integer. Consider the singular metric g on $mL|_{P_0}$ induced by G and the natural surjective morphism

$$p^* \operatorname{Sym}^m \mathcal{E}|_{P_0} \to mL|_{P_0}.$$

Then, after replacing Λ by a sufficiently large multiple, the metric g extends to a singular metric on $m(L+\Lambda)$ (which we denote by the same symbol g) via the identification $L|_{P_0} \cong (L+\Lambda)|_{P_0}$, and the extension satisfies $\sqrt{-1}\Theta_g \geqslant p^*\theta$.

Proof. As explained in the proof of Lemma 4.3, there exists a singular metric h on L (not only on $L|_{P_0}$) such that $\sqrt{-1}\Theta_h = [\delta E]$ modulo smooth forms and

$$\sqrt{-1}\Theta_h + C\, p^*\omega_X \geqslant a\, \omega_P$$

for some constants $C \gg 1$ and $1 \gg a > 0$, where E is an effective π-exceptional divisor. Then, the metric $g \cdot h^\delta$ on P_0 satisfies the assumptions needed to apply the argument of Theorem4.5. Thus, its weight is locally bounded above on P when $g \cdot h^\delta$ is regarded as a metric on $(L+\Lambda)|_{P_0}$. Precisely, this means that $g \cdot h^\delta \cdot h_\Lambda$ extends to a metric on $(L+\Lambda)$. Hence $g \cdot h_{2\Lambda}$ also extends, since $\operatorname{Supp}(E) \subset \operatorname{Supp}(\Lambda)$ and $\sqrt{-1}\Theta_h = [\delta E]$ modulo smooth forms. □

We now give a characterization of the condition $c_1(\mathcal{E}) = 0$. Although this characterization may look obvious at first glance, the proof relies on Theorem 4.6 and [CGNPPW, Theorem 1.1], and hence involves a subtle point.

Proposition 4.7. *Let $\mathcal{E}$ be a torsion-free sheaf of rank 1 on X. Let $p\colon P \to X$ be a modification such that $M := (p^*\mathcal{E})/\operatorname{tor}$ is an invertible sheaf. Then, the following conditions are equivalent:*

1. *$c_1(M)$ is represented by a (not necessarily effective) p-exceptional divisor.*
2. *$\mathcal{E}$ admits a singular metric h such that $\sqrt{-1}\Theta_h = 0$ holds on the Zariski open set $X_0 = X_{\mathrm{reg}} \cap X_{\mathcal{E}}$.*

If, in addition, the sheaf $\mathcal{E}$ is pseudo-effective, then the above conditions are equivalent to the following condition:

3. *$\int_X c_1(\mathcal{E}) \wedge \omega^{n-1} = 0$ holds for some Kähler form ω on X.*

Proof. We apply Setup 4.1 in the case where $\mathrm{rk}\mathcal{E} = 1$. We may assume that the morphism $p\colon P \to X$ in (4.1.1) is a modification such that it is an isomorphism over X_0 and $M := (p^*\mathcal{E})/\operatorname{tor}$ is an invertible sheaf. Since p is an isomorphism over X_0, there exists a (not necessarily effective) p-exceptional divisor F such that

$$M \cong L + F.$$

Indeed, the natural morphisms

$$p^*p_*\big(\mathcal{O}_P(M)\big) \to p^*p_{[*]}\big(\mathcal{O}_P(M)\big) \quad \text{and} \quad p^*p_*\big(\mathcal{O}_P(M)\big) \to \mathcal{O}_P(M)$$

are isomorphisms on P_0, and hence both their kernels and cokernels are torsion sheaves supported on a p-exceptional divisor. This implies that the difference $p^{[*]}p_{[*]}\big(\mathcal{O}_P(M)\big) - M$ is represented by a p-exceptional divisor. The same argument applies to $\mathcal{O}_P(L)$. The claimed isomorphism follows from

$$p_{[*]}\big(\mathcal{O}_P(M)\big) \cong \mathcal{E}^{**} \cong p_{[*]}\big(\mathcal{O}_P(L)\big).$$

Assume (2). Then $L|_{P_0}$ admits a smooth metric h with $\sqrt{-1}\Theta_h = 0$. By Theorem 4.6, this metric on $L|_{P_0}$ extends to a possibly singular metric on $L+\Lambda$ via $L|_{P_0} \cong (L+\Lambda)|_{P_0}$ (which we denote by the same symbol h). Moreover, the curvature $\sqrt{-1}\Theta_h$ is supported on the p-exceptional locus. Therefore, the first Chern class of $L+\Lambda$, and hence that of M, is represented by a p-exceptional divisor, which yields (1).

Conversely, assume (1). Since the support of Λ coincides with the p-exceptional locus, we see that $c_1(L+m\Lambda)$ is represented by an effective p-exceptional divisor E for $m \gg 1$. Therefore, there exists a singular metric g on $L+m\Lambda$ whose curvature current is the integration current $[E]$. The metric $h := p_*g$ on $\mathcal{E}|_{X_0}$ defined by pushforward satisfies the desired property.

Clearly (1) implies (3). Assume (3). Since $\mathcal{E}$ is pseudo-effective, the above argument shows that $L+m\Lambda$ admits a singular metric h whose curvature current is semipositive for $m \gg 1$. Then, we see from (3) that $\sqrt{-1}\Theta_h = 0$ on P_0, which yields (1). □

4.2. **Proof of Theorem 1.2.** We begin the proof of Theorem 1.2.

Proof of Theorem 1.2. We assume that $\mathcal{E}$ is pseudo-effective and has vanishing first Chern class. We first show that $\mathcal{E}$ is weakly positively curved in the following sense:

Lemma 4.8. *The sheaf $\mathcal{E}$ admits singular metrics $\{H_\varepsilon\}$ such that $\sqrt{-1}\Theta_{H_\varepsilon} \geqslant -\varepsilon\omega_X \otimes \mathrm{id}$ and each H_ε is smooth over a non-empty Zariski open subset of X. Moreover, let $\{h_\varepsilon\}$ be the singular metrics on $L+\Lambda$ obtained by extending the induced metrics on $L|_{P_0}$ as in Theorem 4.6. Then $\{h_\varepsilon\}$ satisfy the following properties:*

1. *$\sqrt{-1}\Theta_{h_\varepsilon} + \varepsilon p^*\omega_X \geqslant \delta_\varepsilon\,\omega_P$ holds on P for some $\delta_\varepsilon > 0$;*
2. *h_ε is induced by H_ε and the surjective morphism $p^*\mathcal{E}|_{P_0} \to L|_{P_0}$;*
3. *h_ε is smooth over a non-empty Zariski open subset of P.*

Proof. In the notation of Setup 4.1, by [MWWZ, Lemmas 2.3 and 2.4], there exist singular metrics $\{h_\varepsilon\}$ on $L+\Lambda$ such that

(a) $\sqrt{-1}\Theta_{h_\varepsilon} + \varepsilon p^*\omega_X \geqslant \delta_\varepsilon\,\omega_P$ holds on P for some $\delta_\varepsilon > 0$;
(b) $\{x \in P \mid \nu(h_\varepsilon, x) > 0\}$ is not dominant over X_0.

By Demailly's approximation theorem [Dem92], we can replace $\{h_\varepsilon\}$ with singular metrics with analytic singularities so that h_ε is smooth over a Zariski open set in X by property (b).

Consider the natural inclusion

$$p_*\big(\mathcal{O}_P(K_{P/X} + (r+1)(L+\Lambda)) \otimes \mathcal{I}(h_\varepsilon^{r+1})\big) \subset p_*\big(\mathcal{O}_P(K_{P/X} + (r+1)(L+\Lambda))\big).$$

By (a), the positivity of direct image sheaves [PT18, HPS18] shows that the left-hand side is weakly positively curved. More precisely, the induced L^2-metrics $\{G_\varepsilon\}$ on the left-hand side satisfy $\sqrt{-1}\Theta_{G_\varepsilon} \geqslant -\varepsilon\omega_X \otimes \mathrm{id}$. Moreover, by construction, G_ε is smooth over a non-empty Zariski open subset of X. The above inclusion is generically an isomorphism, since the multiplier ideal sheaf $\mathcal{I}(h_\varepsilon^{r+1})$ is trivial over a Zariski open subset of P. Hence, the right-hand side also admits singular Hermitian metrics $\{H_\varepsilon\}$ with the same properties (see Proposition 2.4).

On the other hand, since $p\colon P_0 \to X_0$ is the projective space bundle associated with $\mathcal{E}|_{X_0}$, we have

$$K_{P/X} + (r+1)(L+\Lambda) \cong L + p^*\det\mathcal{E} \quad \text{ over } X_0,$$

where r is the rank of $\mathcal{E}$. Here we used $\Lambda = 0$ over X_0. This shows that

$$p_*\big(\mathcal{O}_P(K_{P/X} + (r+1)(L+\Lambda))\big) \cong p_*\big(\mathcal{O}_{P_0}(L + p^*\det\mathcal{E})\big) = (\mathcal{E}\otimes\det\mathcal{E})$$

over X_0. Note that this identification holds only over X_0 since $\det\mathcal{E}$ is not necessarily invertible on X.

By the assumption $c_1(\mathcal{E}) = 0$, Proposition 4.7 shows that there exists a smooth metric g on $\det\mathcal{E}|_{X_0}$ such that $\sqrt{-1}\Theta_g = 0$. Set $H_\varepsilon := G_\varepsilon \cdot g^{-1}$. Since $\operatorname{codim}(X\setminus X_0) \geqslant 2$, the metric H_ε extends across $X \setminus X_0$ as a singular Hermitian metric on $\mathcal{E}$, and it satisfies the desired properties.

Finally, the remaining assertions follow from the construction of h_ε and Theorem 4.6. Indeed, the metric h_ε on $L|_{P_0}$ is induced by H_ε and the surjective morphism

$$p^*\mathcal{E}|_{P_0} \to L|_{P_0}.$$

By Theorem 4.6, it extends to a singular metric h_ε on $L+\Lambda$. This completes the proof. $\square$

Let h be the singular metric on $L+\Lambda$ obtained as a limit of $\{h_\varepsilon\}$. We consider the upper level set of Lelong numbers of h:

$$P_+ := \{x \in P \mid \nu(h,x) > 0\}.$$

We divide the argument into two cases:

Case 1: P_+ is not dominant over X.

Case 2: P_+ is dominant over X.

Case 1. Suppose that P_+ is not dominant over X. We will show that $\mathcal{E}^{**}|_{X_{\mathrm{reg}}}$ is Hermitian flat. By applying the positivity of direct image sheaves in [HPS18] (see also [PT18]) once again, we conclude that the direct image sheaf

$$p_*\big(\mathcal{O}_P(K_{P/X} + (r+1)(L+\Lambda)) \otimes \mathcal{I}(h^{r+1})\big) \subset p_*\big(\mathcal{O}_P(K_{P/X} + (r+1)(L+\Lambda))\big)$$

admits a positively curved singular Hermitian metric. Under the assumption of Case 1, the above inclusion is generically an isomorphism. Hence, by Proposition 2.4, we conclude that

$$p_*\big(\mathcal{O}_P(K_{P/X} + (r+1)(L+\Lambda))\big)$$

also admits a positively curved singular Hermitian metric. Arguing as in Lemma 4.8, and using the assumption $c_1(\mathcal{E}) = 0$, we obtain a positively curved singular Hermitian metric H on $\mathcal{E}$.

The determinant metric $\det H$ induces a positively curved singular metric on $\det \mathcal{E}$. By $c_1(\mathcal{E}) = 0$, the weight of $\det H$ is pluriharmonic (hence smooth) on X_0 (see the proof of Proposition 4.7). By [Rau15, Theorem 1.6], the curvature of H is well-defined. Then H is Hermitian flat on $\mathcal{E}|_{X_0}$ by [CP17, Cororally 2.9] or [HIM22, Lemma 3.6]. This shows that $\mathcal{E}|_{X_0}$ is induced by a unitary representation of $\pi_1(X_0)$.

Since X_{reg} is smooth and $\operatorname{codim}(X \setminus X_{\mathcal{E}}) \geqslant 2$, the inclusion $X_0 \hookrightarrow X_{\mathrm{reg}}$ induces an isomorphism

$$\pi_1(X_0) \xrightarrow{\sim} \pi_1(X_{\mathrm{reg}})$$

by the van Kampen theorem. Thus, the unitary representation extends to $\pi_1(X_{\mathrm{reg}})$, and hence $\mathcal{E}^{**}$ is Hermitian flat on X_{reg}.

Case 2. Suppose that P_+ is dominant over X. Assume that $\mathcal{E}$ is stable with respect to a Kähler form ω_X. We will show that this case cannot occur.

The smooth case. To illustrate the idea of the proof, we first consider the case where X is smooth and $\mathcal{E}$ is locally free.

Since $\mathcal{E}$ is stable and X is smooth, by [UY86], there exists a smooth Hermitian–Einstein metric G on $\mathcal{E}$, i.e.

$$\sqrt{-1}\Theta_G(\mathcal{E}) \wedge \omega_X^{n-1} \equiv 0 \quad \text{on } X.$$

Let g be the induced metric on $L := \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. Set $P := \mathbb{P}(\mathcal{E})$, and let $p\colon P \to X$ be the natural projection.

We now construct a destabilizing subsheaf. By Proposition 3.2 and Lemma 4.8 (2), and arguing as in the proof of Lemma 4.8, there exists $0 < \delta < 1$ such that

$$\mathcal{G} := p_*\Big(\mathcal{O}_P\big(K_{P/X} + (r+1)L\big) \otimes \mathcal{I}\big((h^\delta \cdot g^{(1-\delta)})^{r+1}\big)\Big)$$

is a nonzero subsheaf of $\mathcal{E} \otimes \det \mathcal{E}$, and the inclusion is proper. By the construction of g, we have

$$\sqrt{-1}\Theta_g(L) \wedge p^*\omega_X^{n-1} \geqslant 0 \quad \text{over } X.$$

Moreover, the same holds for h since $\sqrt{-1}\Theta_h(L) \geqslant 0$ in the sense of currents. Hence we have

$$\sqrt{-1}\Theta_{h^\delta \cdot g^{(1-\delta)}}(L) \wedge p^*\omega_X^{n-1} \geqslant 0 \quad \text{over } X.$$

Therefore, applying [CCP, Theorem 1.6], we obtain

$$\sqrt{-1}\Theta_{\det H}(\det \mathcal{G}) \wedge \omega_X^{n-1} \geqslant 0 \quad \text{on } X,$$

where H is the L^2-metric on $\mathcal{G}$ induced by $h^\delta \cdot g^{(1-\delta)}$.

By the above curvature inequality, we have

$$\int_X c_1(\mathcal{G}) \wedge \omega_X^{n-1} \geqslant 0.$$

By $c_1(\mathcal{E})=0$, Proposition 4.7 implies

$$\int_X c_1\big(\mathcal{G}\otimes(\det\mathcal{E})^*\big)\wedge\omega_X^{n-1}=\int_X c_1(\mathcal{G})\wedge\omega_X^{n-1}\geqslant 0.$$

By construction, the sheaf $\mathcal{G}\otimes(\det\mathcal{E})^*$ is a nonzero subsheaf of $\mathcal{E}$, and the inclusion is proper. Therefore, the sheaf $\mathcal{G}\otimes(\det\mathcal{E})^*$ destabilizes $\mathcal{E}$, a contradiction.

The singular case. We now treat the general case. Let $\pi\colon\tilde{X}\to X$ be a resolution of singularities of X such that

$$E:=\pi^*\mathcal{E}/\operatorname{tor}$$

is a locally free sheaf on $\tilde{X}$. Let $\omega_{\tilde{X}}$ be a Kähler metric on $\tilde{X}$. By [CGNPPW, Section 3] or [Tom21], the locally free sheaf E is stable with respect to $\pi^*\omega_X+\varepsilon\omega_{\tilde{X}}$ for $0<\varepsilon\ll 1$. Hence there exists a smooth Hermitian–Einstein metric G_ε on E with respect to $\pi^*\omega_X+\varepsilon\omega_{\tilde{X}}$, i.e. it satisfies

$$\sqrt{-1}\Theta_{G_\varepsilon}(E)\wedge(\pi^*\omega_X+\varepsilon\omega_{\tilde{X}})^{n-1}=c_\varepsilon\operatorname{id}_E\cdot(\pi^*\omega_X+\varepsilon\omega_{\tilde{X}})^n.$$

Here c_ε is the constant satisfying $\lim_{\varepsilon\to 0}c_\varepsilon=0$. Let g_ε be the quotient metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$ induced by G_ε. The metric g_ε satisfies

$$\sqrt{-1}\Theta_{g_\varepsilon}(\mathcal{O}_{\mathbb{P}(E)}(1))\wedge\big(\tilde{p}^*(\pi^*\omega_X+\varepsilon\omega_{\tilde{X}})\big)^{n-1}\geqslant c_\varepsilon\big(\tilde{p}^*(\pi^*\omega_X+\varepsilon\omega_{\tilde{X}})\big)^n\quad\text{over }\tilde{X},\tag{4.8.1}$$

where $\tilde{p}\colon\mathbb{P}(E)\to\tilde{X}$ is the natural projection.

Let $\tau\colon\tilde{P}\to P$ be a bimeromorphic morphism such that $\tilde{P}$ is smooth and it satisfies the following commutative diagram:

$$\begin{tikzcd}
\tilde{P} \arrow[d, "\tau"'] \arrow[dr, "\tilde{\pi}"] & \\
\mathbb{P}(E) \arrow[d, "\tilde{p}"'] & P \arrow[d, "p"] \\
\tilde{X} \arrow[r, "\pi"'] & X.
\end{tikzcd}$$

Then, we can write

$$\tilde{L}:=\tilde{\pi}^*(L+\Lambda)+\Lambda_1\cong\tau^*\mathcal{O}_{\mathbb{P}(E)}(1)+\Lambda_2\tag{4.8.2}$$

for some effective divisors Λ_1 and Λ_2 supported in the inverse image of $(X\setminus X_0)$ (see the proof of Proposition 4.7).

Let $\tilde{h}$ be the singular metric on $\tilde{L}$ induced by the metric h on L and the metric defined by the natural section of Λ_1. By construction, the curvature $\sqrt{-1}\Theta_{\tilde{h}}$ is semipositive. On the other hand, thanks to (4.8.2), we obtain the singular metric $\tilde{g}_\varepsilon$ on $\tilde{L}$ induced by the metric g_ε on $\mathcal{O}_{\mathbb{P}(E)}(1)$ and the metric defined by the natural section of Λ_2.

Consider the direct image sheaf

$$\tilde{\mathcal{G}}:=(\tilde{p}\circ\tau)_*\Big(\mathcal{O}_{\tilde{P}}\big(K_{\tilde{P}/\tilde{X}}+(r+1)\tilde{L}\big)\otimes\mathcal{I}\big((\tilde{h}^\delta\cdot\tilde{g}_\varepsilon^{(1-\delta)})^{r+1}\big)\Big)$$

and the induced L^2-metric $\tilde{H}_\varepsilon$ on $\tilde{\mathcal{G}}$. Note that g_ε is smooth, hence the multiplier ideal sheaf $\mathcal{I}\big((\tilde{h}^\delta \cdot \tilde{g}_\varepsilon^{(1-\delta)})^{r+1}\big)$ depends only on δ. In particular, the sheaf $\tilde{\mathcal{G}}$ is independent of ε.

Applying [CCP, Theorem 1.6] and using (4.8.1), we obtain

$$\sqrt{-1}\Theta_{\det \tilde{H}_\varepsilon}(\det \tilde{\mathcal{G}}) \wedge (\pi^*\omega_X + \varepsilon\omega_{\tilde{X}})^{n-1} \geqslant c_\varepsilon(\pi^*\omega_X + \varepsilon\omega_{\tilde{X}})^n \quad \text{on } \tilde{X}.$$

Taking $\varepsilon \to 0$ and using $\lim_{\varepsilon\to 0} c_\varepsilon = 0$, we obtain

$$\int_{\tilde{X}} c_1(\tilde{\mathcal{G}}) \wedge \pi^*\omega_X^{n-1} \geqslant 0. \tag{4.8.3}$$

Since $\pi_*\tilde{\mathcal{G}} \subset \mathcal{E} \otimes \det \mathcal{E}$ on X_0, we have

$$\big(\pi_{[*]}\tilde{\mathcal{G}}\big) \subset \mathcal{E}[\otimes] \det \mathcal{E} \quad \text{on } X.$$

By Proposition 3.2 and Lemma 4.8 (2), there is $\delta > 0$ such that the left-hand side is a nonzero proper subsheaf of $\mathcal{E}[\otimes] \det \mathcal{E}$.

Moreover, the slope $\mu_{\omega_X}\big(\pi_{[*]}\tilde{\mathcal{G}}\big)$ can be computed as

$$\int_{\tilde{X}} c_1\Big(\pi^*\big(\pi_{[*]}\tilde{\mathcal{G}}\big)/\operatorname{tor}\Big) \wedge \pi^*\omega_X^{n-1} = \int_{\tilde{X}} c_1(\tilde{\mathcal{G}}) \wedge \pi^*\omega_X^{n-1}.$$

Here the equality follows since $\int_D \pi^*\omega_X^{n-1} = 0$ for any π-exceptional divisor D, and

$$c_1(\tilde{\mathcal{G}}) - c_1\Big(\pi^*\big(\pi_{[*]}\tilde{\mathcal{G}}\big)/\operatorname{tor}\Big)$$

is supported on the π-exceptional locus (cf. the argument in the proof of Proposition 4.7). Therefore, by (4.8.3), we obtain

$$\mu_{\omega_X}\big(\pi_{[*]}\tilde{\mathcal{G}}\big) \geqslant 0. \tag{4.8.4}$$

This contradicts the stability of $\mathcal{E}$ with respect to ω_X.

□

4.3. **Proof of Theorem 1.1.** In this subsection, we prove Theorem 1.1.

Proof of Theorem 1.1 (i). By Theorem 2.9, there exists a maximally quasi-étale cover

$$\nu\colon X' \to X.$$

We will prove assertion (i) for this maximally quasi-étale cover $\nu\colon X' \to X$. The proof is divided into two steps.

Step 1. The reflexive pullback $\nu^{[*]}\mathcal{E}$ is also pseudo-effective and has vanishing first Chern class. Fix a Kähler form $\omega_{X'}$ on X'. By Proposition 2.7, the sheaf $\nu^{[*]}\mathcal{E}$ is semistable with respect to $\omega_{X'}$. Hence, we can consider the Jordan–Hölder filtration of $\nu^{[*]}\mathcal{E}$:

$$0 =: \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_k := \nu^{[*]}\mathcal{E}. \tag{4.8.5}$$

In this step, we prove that each $(\mathcal{E}_i/\mathcal{E}_{i-1})^{**}$ is a Hermitian flat locally free sheaf on X'.

By construction, the quotient $\mathcal{E}_i/\mathcal{E}_{i-1}$ is stable, and its slope vanishes. The quotient $\mathcal{E}_i/\mathcal{E}_{i-1}$ is torsion-free, hence locally free in codimension 1, so we obtain

$$c_1\big((\mathcal{E}_i/\mathcal{E}_{i-1})^{**}\big) = c_1(\mathcal{E}_i/\mathcal{E}_{i-1}) = 0$$

by the vanishing of the slope and pseudo-effectivity.

We first consider the inclusion $\mathcal{E}_1 \subset \mathcal{E}_k = \nu^{[*]}\mathcal{E}$, which induces the generically surjective morphism $\mathcal{E}_k^* \to \mathcal{E}_1^*$. By Propositions 2.4 and 2.5, the sheaf $\mathcal{E}_1^*$ is pseudo-effective. Applying Theorem 1.2 to $\mathcal{E}_1^*$, we conclude that $\mathcal{E}_1^*$ is a Hermitian flat locally free sheaf on X'_{reg} by $c_1(\mathcal{E}_1^*) = 0$. By Theorem 2.9 (2), it follows that $\mathcal{E}_1^*$ is a Hermitian flat locally free sheaf on X'.

The quotient $\mathcal{E}_k/\mathcal{E}_1$ is also pseudo-effective and has vanishing first Chern class. Considering the inclusion $\mathcal{E}_2/\mathcal{E}_1 \subset \mathcal{E}_k/\mathcal{E}_1$ and repeating the same argument as above, we deduce that $(\mathcal{E}_2/\mathcal{E}_1)^{**}$ is a Hermitian flat locally free sheaf on X'. Iterating this procedure, we see that $(\mathcal{E}_i/\mathcal{E}_{i-1})^{**}$ is a Hermitian flat locally free sheaf on X' for every i.

Step 2. In this step, we prove that $\mathcal{E}_k^{**}$ is a numerically flat locally free sheaf on X' by induction on k. Note that when $k = 1$, there is nothing to prove. Indeed, in this case, the sheaf $\mathcal{E}_1 = \nu^{[*]}\mathcal{E}$ is stable, and hence $\mathcal{E}_1^{**}$ is a Hermitian flat locally free sheaf by Theorem 1.2.

First, we claim that $\mathcal{E}_i$ is pseudo-effective and satisfies $c_1(\mathcal{E}_i) = 0$ for every i. The inclusion $\mathcal{E}_i \subset \mathcal{E}_k = \nu^{[*]}\mathcal{E}$ induces the generically surjective morphism $\mathcal{E}_k^* \to \mathcal{E}_i^*$. By Propositions 2.4 and 2.5, we see that $\mathcal{E}_i^*$ is pseudo-effective. Hence, once we know that $c_1(\mathcal{E}_i) = 0$, the sheaf $\mathcal{E}_i$ is pseudo-effective by Proposition 2.5. To check $c_1(\mathcal{E}_i) = 0$, we consider the exact sequence

$$0 \to \mathcal{E}_{i-1} \to \mathcal{E}_i \to \mathcal{E}_i/\mathcal{E}_{i-1} \to 0. \tag{4.8.6}$$

Since $c_1(\mathcal{E}_i/\mathcal{E}_{i-1}) = 0$, using (4.8.6), we obtain inductively

$$c_1(\mathcal{E}_i) = c_1(\mathcal{E}_{i-1}) + c_1(\mathcal{E}_i/\mathcal{E}_{i-1}) = 0,$$

starting from $c_1(\mathcal{E}_1) = 0$.

By considering (4.8.5) up to $(k-1)$, the induction hypothesis shows that $\mathcal{E}_{k-1}^{**}$ is a numerically flat locally free sheaf on X'. Let $\mathcal{S}$ be the saturation of $\mathcal{E}_{k-1}^{**} \subset \mathcal{E}_k^{**}$, and set $\mathcal{F} := \mathcal{E}_k^{**}$. Then, we have the exact sequence

$$0 \to \mathcal{S} \to \mathcal{F} \to \mathcal{Q} \to 0.$$

This operation does not change (4.8.6) in codimension 1. Indeed, by [IMZ26, Lemma 2.9] (or by the proof of Proposition 3.1), the quotient $\mathcal{E}_k/\mathcal{E}_{k-1}$ is locally free on $X'_0 := X'_{\mathrm{reg}} \cap X'_{\nu^{[*]}\mathcal{E}}$, and (4.8.6) is exact as a sequence of vector bundles on X'_0. Thus, the sheaf $\mathcal{S}$ coincides with $\mathcal{E}_{k-1}^{**}$; in particular, it is locally free. Moreover, we have $c_1(\mathcal{Q}) = c_1(\mathcal{E}_k/\mathcal{E}_{k-1}) = 0$.

By Proposition 3.1, there exists a Zariski closed subset $Z \subset X'$ of codimension $\geqslant 3$ such that $\mathcal{Q}$ is reflexive on $X' \setminus Z$. Since $\mathcal{Q}^{**} = (\mathcal{E}_k/\mathcal{E}_{k-1})^{**}$ is locally free, the sheaf $\mathcal{Q}$

is locally free on $X' \setminus Z$. Consequently, the sheaf $\mathcal{F}$ is locally free on $X' \setminus Z$, and the sequence is exact as a sequence of vector bundles on $X' \setminus Z$. Therefore, the locally free sheaf $\mathcal{F}|_{X' \setminus Z}$ is determined by an extension class

$$\beta \in H^1(X' \setminus Z, \operatorname{Hom}(\mathcal{Q}, \mathcal{S})) \cong H^1(X' \setminus Z, \operatorname{Hom}(\mathcal{Q}^{**}, \mathcal{S}^{**})).$$

By $\operatorname{codim} Z \geqslant 3$ and [ST71, Theorem 1.14], the restriction map

$$H^1\big(X', \operatorname{Hom}(\mathcal{Q}^{**}, \mathcal{S}^{**})\big) \longrightarrow H^1\big(X' \setminus Z, \operatorname{Hom}(\mathcal{Q}^{**}, \mathcal{S}^{**})\big)$$

is an isomorphism. Let $\pi\colon \widetilde{X}' \to X'$ be a desingularisation. Since X' has rational singularities, for any locally free sheaf V on X' we have

$$R^j\pi_*(\pi^*V) = V \otimes R^j\pi_*\mathcal{O}_{\widetilde{X}'} = 0 \quad (j \geqslant 1).$$

By the Leray spectral sequence, we obtain

$$H^1\big(\widetilde{X}', \pi^* \operatorname{Hom}(\mathcal{Q}^{**}, \mathcal{S}^{**})\big) \cong H^1\big(X', \operatorname{Hom}(\mathcal{Q}^{**}, \mathcal{S}^{**})\big) \cong H^1\big(X' \setminus Z, \operatorname{Hom}(\mathcal{Q}^{**}, \mathcal{S}^{**})\big).$$

Hence, the class β lifts to a class

$$\widetilde{\beta} \in H^1\big(\widetilde{X}', \pi^* \operatorname{Hom}(\mathcal{Q}^{**}, \mathcal{S}^{**})\big) \cong H^1\big(\widetilde{X}', \operatorname{Hom}(\pi^*\mathcal{Q}^{**}, \pi^*\mathcal{S}^{**})\big).$$

Let $V_{\widetilde{\beta}}$ be the locally free sheaf on $\widetilde{X}'$ defined by $\widetilde{\beta}$. Since $\widetilde{X}'$ is a compact Kähler manifold and both $\pi^*\mathcal{Q}^{**}$ and $\pi^*\mathcal{S}^{**}$ are flat, Simpson's result [Sim92] (see also [Den21, Theorem 1.1]) implies that $V_{\widetilde{\beta}}$ is flat. By construction, the sheaf $\pi^*\mathcal{F}$ coincides with $V_{\widetilde{\beta}}$ over $X' \setminus Z$. Therefore, the sheaf $\mathcal{F}$ is flat on X'_{reg}, and hence flat on X' by Theorem 2.9 (2). By the local freeness, [Wu22, Corollary] shows that $\mathcal{E}_k^{**} = \mathcal{F}$ is a numerically flat locally free sheaf on X'.

□

We finally prove assertion (ii) of Theorem 1.1.

Proof of Theorem 1.1 (ii).

Step 1. The basic strategy is to descend the flat connection on $\nu^{[*]}\mathcal{E}$ to a flat connection on $\mathcal{E}|_{X_{\mathrm{reg}}}$. However, this is not straightforward. To perform this descent to X_{reg}, we use the explicit construction of flat connections given in [Den21].

We consider

$$f\colon \tilde{X}' \xrightarrow{\ \pi\ } X' \xrightarrow{\ \nu\ } X,$$

where $\nu\colon X' \to X$ is a maximally quasi-étale cover and $\pi\colon \tilde{X}' \to X'$ is a resolution of singularities which is an isomorphism over X'_{reg}. In what follows, we often identify $f^{-1}(X_{\mathrm{reg}}) \subset \tilde{X}'$ with $\nu^{-1}(X_{\mathrm{reg}}) \subset X'$ via $\pi\colon \tilde{X}' \to X'$. Further, we consider the Jordan–Hölder filtration of $\mathcal{E}$:

$$0 =: \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_k := \mathcal{E}.$$

Arguing as in the proof of (i), we see that

$$F_i := (\mathcal{E}_{i+1}/\mathcal{E}_i)^{**}$$

is Hermitian flat on X_{reg}. Let h_i be a Hermitian flat metric on $F_i|_{X_{\mathrm{reg}}}$. Set $F_i' := \nu^{[*]}F_i$ (which is locally free on X' by Theorem 2.9 (2)), and let h_i' be the extension of ν^*h_i to a Hermitian flat metric on F_i'. Moreover, by (i), the reflexive pullback $f^{[*]}\mathcal{E}$ is locally free and can be written as a successive extension of the Hermitian flat vector bundles $(\tilde{F}_i, \tilde{h}_i)$, where

$$(\tilde{F}_i, \tilde{h}_i) := \pi^*(F_i', h_i').$$

By [Sim92], we already know that $f^{[*]}\mathcal{E}$ admits a flat connection. However, in order to descend it to a flat connection on $\mathcal{E}|_{X_{\mathrm{reg}}}$, we need the explicit construction of the flat connection in [Den21].

First, consider the vector bundle $\oplus_{i=1}^k \tilde{F}_i$ equipped with the connection $\tilde{\nabla}$ and the complex structure $\bar{\partial}_\eta$ defined by

$$\tilde{\nabla} := \oplus_{i=1}^k D_{\tilde{h}_i}^{(1,0)} = \begin{pmatrix} D_{\tilde{h}_1}^{(1,0)} & 0 & \cdots & 0 \\ 0 & D_{\tilde{h}_2}^{(1,0)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_{\tilde{h}_k}^{(1,0)} \end{pmatrix}$$

and

$$\bar{\partial}_\eta = \begin{pmatrix} \bar{\partial} & \eta_{1,2} & \cdots & \eta_{1,k} \\ 0 & \bar{\partial} & \ddots & \vdots \\ \vdots & \ddots & \bar{\partial} & \eta_{k-1,k} \\ 0 & \cdots & 0 & \bar{\partial} \end{pmatrix}. \tag{4.8.7}$$

Here $D_{\tilde{h}_i}^{(1,0)}$ denotes the $(1,0)$-part of the Chern connection defined by $\tilde{h}_i$. Our purpose is to construct smooth $\mathrm{Hom}(\tilde{F}_j, \tilde{F}_i)$-valued $(0,1)$-forms

$$\eta_{i,j} \in C^\infty_{(0,1)}(\tilde{X}', \mathrm{Hom}(\tilde{F}_j, \tilde{F}_i))$$

such that:

(1) The holomorphic vector bundle $(\oplus_{i=1}^k \tilde{F}_i, \bar{\partial}_\eta)$ is isomorphic to $f^{[*]}\mathcal{E}$ on $\tilde{X}'$.

(2) The following equalities hold:

$$D^{(1,0)}_{\tilde{h}_j^* \otimes \tilde{h}_i} \eta_{i,j} = 0 \quad \text{and} \quad \bar{\partial}\eta_{i,j} = 0,$$

where $D^{(1,0)}_{\tilde{h}_j^* \otimes \tilde{h}_i}$ is the $(1,0)$-part of the Chern connection on $\mathrm{Hom}(\tilde{F}_j, \tilde{F}_i)$ defined by $\tilde{h}_j$ and $\tilde{h}_i$.

(3) $\eta_{i,j}|_{f^{-1}(X_{\mathrm{reg}})}$ can be written as the pullback of a form on X_{reg}.

Once (1)–(3) are established, we obtain a flat connection on $\mathcal{E}|_{X_{\mathrm{reg}}}$. Indeed, writing $\eta_{i,j} = f^*\xi_{i,j}$ over X_{reg} by (3), the holomorphic vector bundle $\oplus_{i=1}^k F_i$ with $\bar{\partial}_\xi$ defined as in (4.8.7) is isomorphic to $\mathcal{E}|_{X_{\mathrm{reg}}}$ by (1). Moreover, the connection

$$\oplus_{i=1}^k D_{h_i}^{(1,0)} + \bar{\partial}_\xi$$

is flat by (2).

Step 2. As a warm-up, we treat the case $k = 2$. Consider the exact sequence

$$0 \to \tilde{F}_1 \to f^{[*]}\mathcal{E}_2 = f^{[*]}\mathcal{E} \to \tilde{F}_2 \to 0. \tag{4.8.8}$$

We will construct $\eta_{1,2} \in C^\infty_{(0,1)}(\tilde{X}', \mathrm{Hom}(\tilde{F}_2, \tilde{F}_1))$ satisfying properties (1), (2) and (3).

The extension class of (4.8.8) can be represented by a form

$$\alpha \in C^\infty_{(0,1)}(\tilde{X}', \mathrm{Hom}(\tilde{F}_2, \tilde{F}_1))$$

such that

$$\bar\partial_\alpha = \begin{pmatrix} \bar\partial & \alpha \\ 0 & \bar\partial \end{pmatrix}$$

satisfies the property $(1)_\alpha$, obtained from (1) by replacing $\eta_{i,j}$ with α. Since $\bar\partial\alpha = 0$ by $\bar\partial_\alpha^2 = 0$, Hodge theory provides $\psi \in C^\infty(\tilde{X}', \mathrm{Hom}(\tilde{F}_2, \tilde{F}_1))$ such that

$$\eta_{1,2} := \alpha + \bar\partial\psi \in \mathrm{Ker}\, D^{(1,0)}_{\tilde{h}_2^* \otimes \tilde{h}_1}. \tag{4.8.9}$$

Hence, the form $\eta_{1,2}$ satisfies the desired property (1) and

$$D^{(1,0)}_{\tilde{h}_2^* \otimes \tilde{h}_1} \eta_{1,2} = 0. \tag{4.8.10}$$

It remains to verify the descent property (3).

Let $\gamma \in C^\infty_{(0,1)}(X_{\mathrm{reg}}, \mathrm{Hom}(F_2, F_1))$ be a smooth representative of the extension class of $\mathcal{E}_2|_{X_{\mathrm{reg}}}$ defined by

$$0 \to F_1 \to \mathcal{E}_2|_{X_{\mathrm{reg}}} \to F_2 \to 0. \tag{4.8.11}$$

Restricting (4.8.8) to $f^{-1}(X_{\mathrm{reg}}) \cong \nu^{-1}(X_{\mathrm{reg}})$ and comparing with the pullback of (4.8.11), we see that $\nu^*\gamma$ and $\eta_{1,2}|_{\nu^{-1}(X_{\mathrm{reg}})}$ represent the same extension class over X_{reg}. Hence, there exists $\phi \in C^\infty(\nu^{-1}(X_{\mathrm{reg}}), \mathrm{Hom}(F_2', F_1'))$ such that

$$\eta_{1,2}|_{\nu^{-1}(X_{\mathrm{reg}})} = \nu^*\gamma + \bar\partial\phi \quad \text{over } X_{\mathrm{reg}}. \tag{4.8.12}$$

We may assume that $\nu \colon X' \to X$ is a Galois cover with Galois group G. The section defined by the average

$$\frac{1}{\deg \nu} \sum_{g \in G} g^*\phi \in C^\infty(\nu^{-1}(X_{\mathrm{reg}}), \mathrm{Hom}(F_2', F_1'))$$

is G-invariant, hence it descends to a smooth section $\phi_{\mathrm{av}} \in C^\infty(X_{\mathrm{reg}}, \mathrm{Hom}(F_2, F_1))$ satisfying

$$\nu^*\phi_{\mathrm{av}} = \frac{1}{\deg \nu} \sum_{g \in G} g^*\phi \quad \text{over } X_{\mathrm{reg}}.$$

Set

$$s := \nu^*\phi_{\mathrm{av}} - \phi \in C^\infty(\nu^{-1}(X_{\mathrm{reg}}), \mathrm{Hom}(F_2', F_1')).$$

Then, by Lemma 4.9 (which will be proved in the next step), the section s is holomorphic over X_{reg} and bounded with respect to $h_2'^* \otimes h_1'$. The Riemann extension type theorem shows that π^*s extends to a holomorphic section of $\mathrm{Hom}(\tilde{F}_2, \tilde{F}_1)$ on $\tilde{X}'$.

Since $\bar\partial s = 0$ on $\nu^{-1}(X_{\mathrm{reg}})$, we obtain

$$\eta_{1,2} = \nu^*\gamma + \bar\partial\phi = \nu^*\gamma + \bar\partial(\nu^*\phi_{\mathrm{av}} - s) = \nu^*(\gamma + \bar\partial\phi_{\mathrm{av}}) \quad \text{on } \nu^{-1}(X_{\mathrm{reg}}) \tag{4.8.13}$$

from (4.8.12). Thus $\eta_{1,2}$ is the pullback of a form over X_{reg}, proving (3) in the case $k = 2$.

Step 3. In this step, we prove the following lemma.

Lemma 4.9. *Let us consider the same situation as in Step 2. Then the section $s = \nu^*\phi_{\mathrm{av}} - \phi$ is holomorphic over X_{reg} and bounded with respect to $h_2'^* \otimes h_1'$.*

Proof. We first show that

$$D^{(1,0)}_{h_2^*\otimes h_1}(\gamma + \bar\partial\phi_{\mathrm{av}}) = 0, \tag{4.9.1}$$

where $D^{(1,0)}_{h_2^*\otimes h_1}$ is the $(1,0)$-part of the Chern connection on $\mathrm{Hom}(F_2, F_1)$ defined by h_2 and h_1. Let U be a small open subset of X_{reg} such that $\nu^{-1}(U) = \sqcup_{\ell=1}^{\deg\nu} U_\ell$ is a disjoint union of open subsets, and $\nu|_{U_\ell} : U_\ell \to U$ is biholomorphic. Fix U_1 and, for each $1 \leqslant \ell \leqslant \deg\nu$, choose $g_\ell \in G$ such that $g_\ell\colon U_1 \to U_\ell$ is biholomorphic. Since $h_i' = \nu^* h_i$, we have $g_\ell^* h_i' = h_i'$ on U_1. By construction, we have $D^{(1,0)}_{h_2'^*\otimes h_1'}(\eta_{1,2}) = 0$ on $\nu^{-1}(X_{\mathrm{reg}})$ by (4.8.9), and hence, we obtain

$$0 = D^{(1,0)}_{h_2'^*\otimes h_1'}(\nu^*\gamma + \bar\partial\phi) \quad \text{on } U_1 \tag{4.9.2}$$

by (4.8.12). Applying g_ℓ^* to (4.9.2) and using $g_\ell^*(\nu^*\gamma) = \nu^*\gamma$, we obtain

$$0 = D^{(1,0)}_{h_2'^*\otimes h_1'}\big(\nu^*\gamma + \bar\partial(g_\ell^*\phi)\big) \quad \text{on } U_1.$$

Averaging over ℓ yields

$$0 = D^{(1,0)}_{h_2'^*\otimes h_1'}\Big(\nu^*\gamma + \bar\partial\frac{1}{\deg\nu}\sum_{\ell=1}^{\deg\nu} g_\ell^*\phi\Big) = D^{(1,0)}_{h_2'^*\otimes h_1'}\big(\nu^*(\gamma + \bar\partial\phi_{\mathrm{av}})\big) \quad \text{on } U_1.$$

Since ν is étale and h_i' is defined as the pullback of h_i, we obtain (4.9.1) on U.

Now put $s := \nu^*\phi_{\mathrm{av}} - \phi$. From (4.8.10), (4.8.12), and (4.9.1), we have

$$\begin{aligned} D^{(1,0)}_{h_2'^*\otimes h_1'}\bar\partial s &\overset{(4.8.12)}{=} D^{(1,0)}_{h_2'^*\otimes h_1'}\bar\partial\nu^*\phi_{\mathrm{av}} + D^{(1,0)}_{h_2'^*\otimes h_1'}(\nu^*\gamma - \eta_{1,2}) \\ &\overset{(4.8.10)}{=} D^{(1,0)}_{h_2'^*\otimes h_1'}(\bar\partial\nu^*\phi_{\mathrm{av}} + \nu^*\gamma) \\ &= \nu^*\big(D^{(1,0)}_{h_2^*\otimes h_1}\bar\partial\phi_{\mathrm{av}} + \gamma\big) \overset{(4.9.1)}{=} 0 \quad \text{over } X_{\mathrm{reg}}. \end{aligned}$$

Together with the Hermitian flatness of h_2' and h_1', we obtain

$$\begin{aligned} &\sqrt{-1}\,\partial\bar\partial|\,s\,|^2_{h_2'^*\otimes h_1'} \\ =&\sqrt{-1}\left\langle D^{(1,0)}_{h_2'^*\otimes h_1'}s,\, D^{(1,0)}_{h_2'^*\otimes h_1'}s\right\rangle_{h_2'^*\otimes h_1'} + \sqrt{-1}\left\langle\bar\partial s,\,\bar\partial s\right\rangle_{h_2'^*\otimes h_1'} \geqslant 0 \quad \text{over } X_{\mathrm{reg}}. \end{aligned}$$

In particular, the function $|s|^2_{h_2'^*\otimes h_1'}$ is plurisubharmonic over X_{reg}, and hence on X'. Since X' is compact, it is constant, and therefore $\bar\partial s = 0$ and $D^{(1,0)}_{h_2'^*\otimes h_1'} s = 0$ over X_{reg}. This proves the lemma. $\square$

Step 4. In this step, we construct $\eta_{i,j} \in C^\infty_{(0,1)}(\tilde X', \mathrm{Hom}(\tilde F_j, \tilde F_i))$ satisfying (1)–(3) by induction on k. By the induction hypothesis, for every $i < j \leqslant k-1$, there exists $\eta_{i,j}$ satisfying (2) and (3) such that $(\oplus_{i=1}^{k-1}\tilde F_i, \bar\partial_{k-1})$ is isomorphic to $f^{[*]}\mathcal{E}_{k-1}$ on $\tilde X'$, where $\bar\partial_{k-1}$ is the complex structure defined by

$$\bar\partial_{k-1} = \begin{pmatrix} \bar\partial & \eta_{1,2} & \cdots & \eta_{1,k-1} \\ 0 & \bar\partial & \cdots & \eta_{2,k-1} \\ 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & \bar\partial \end{pmatrix}.$$

Our purpose is to construct $\{\eta_{i,k}\}_{i=1}^{k-1}$ satisfying (1), (2), and (3).

Representing the extension class of the corresponding exact sequence, we choose smooth forms $\{\alpha_i\}_{i=1}^{k-1} \subset C^\infty_{(0,1)}(\tilde X', \mathrm{Hom}(\tilde F_k, \tilde F_i))$ such that $(\oplus_{i=1}^{k}\tilde F_i, \bar\partial_\alpha)$ is isomorphic to $f^{[*]}\mathcal{E}$ on $\tilde X'$, where

$$\bar\partial_{\alpha} = \begin{pmatrix} \bar\partial & \eta_{1,2} & \cdots & \eta_{1,k-1} & \alpha_1 \\ 0 & \bar\partial & \cdots & \eta_{2,k-1} & \alpha_2 \\ 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \bar\partial & \alpha_{k-1} \\ 0 & 0 & 0 & 0 & \bar\partial \end{pmatrix}.$$

Then, we have the following two properties:

(i) Since $\bar\partial_\alpha^2 = 0$, we obtain

$$\bar\partial\alpha_{k-1} = 0 \text{ and } \bar\partial\alpha_i = -\sum_{j=1}^{k-1-i} \eta_{i,i+j}\wedge\alpha_{i+j} \quad \text{for every } i < k-1. \tag{4.9.3}$$

(ii) For smooth forms $\{\beta_i\}_{i=1}^{k-1} \subset C^\infty_{(0,1)}(\tilde X', \mathrm{Hom}(\tilde F_k, \tilde F_i))$ on $\tilde X'$, we set the new complex structure

$$\bar\partial_{\beta} = \begin{pmatrix} \bar\partial & \eta_{1,2} & \cdots & \eta_{1,k-1} & \beta_1 \\ 0 & \bar\partial & \cdots & \eta_{2,k-1} & \beta_2 \\ 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \bar\partial & \beta_{k-1} \\ 0 & 0 & 0 & 0 & \bar\partial \end{pmatrix}.$$

The operator $\bar\partial_\beta$ defines the same vector bundle as $\bar\partial_\alpha$ if and only if there are smooth sections $f_i \in C^\infty\big(\tilde X',\ \mathrm{Hom}(\tilde F_k, \tilde F_i)\big)$ such that

$$\begin{cases} \beta_{k-1} = \alpha_{k-1} + \bar\partial f_{k-1}, \\ \beta_{k-2} = \alpha_{k-2} - \eta_{k-2,k-1}\cdot f_{k-1} + \bar\partial f_{k-2}, \\ \qquad \vdots \\ \quad \beta_1 = \alpha_1 - \sum\limits_{2\leqslant j\leqslant k-1} \eta_{1,j}\cdot f_j + \bar\partial f_1. \end{cases} \tag{4.9.4}$$

We now construct $\eta_{k-1,k} \in C^\infty_{(0,1)}(\tilde X', \mathrm{Hom}(\tilde F_k, \tilde F_{k-1}))$ satisfying (2) and (3). By (4.9.3), we have $\bar\partial\alpha_{k-1} = 0$. Thus, by Hodge theory on $\tilde X'$, we can find $\psi_{k-1} \in C^\infty(\tilde X', \mathrm{Hom}(\tilde F_k, \tilde F_{k-1}))$ such that

$$\alpha_{k-1} + \bar\partial\psi_{k-1} \in \mathrm{Ker}\, D^{(1,0)}_{\tilde h_k^*\otimes\tilde h_{k-1}}.$$

By the same argument as in Step 2 (using Lemma 4.9 on $\nu^{-1}(X_{\mathrm{reg}}) \cong f^{-1}(X_{\mathrm{reg}})$), there exists $s_{k-1} \in H^0(\tilde X', \mathrm{Hom}(\tilde F_k, \tilde F_{k-1}))$ such that

$$\eta_{k-1,k} := \alpha_{k-1} + \bar\partial(\psi_{k-1} + s_{k-1}) \in \mathrm{Ker}\, D^{(1,0)}_{\tilde h_k^*\otimes\tilde h_{k-1}},$$

and $\eta_{k-1,k}$ can be written as the pullback of a form on X_{reg}. Then $\eta_{k-1,k}$ satisfies (2) and (3).

Set $f_{k-1} := \psi_{k-1} + s_{k-1}$ and define $(\beta_1, \cdots, \beta_{k-2})$ by

$$(\beta_1, \cdots, \beta_{k-2}) := \big(\alpha_1 - \eta_{1,k-1}\cdot f_{k-1}, \cdots, \alpha_{k-2} - \eta_{k-2,k-1}\cdot f_{k-1}\big).$$

Consider the complex structure

$$\bar\partial_{\beta,\eta} = \begin{pmatrix} \bar\partial & \eta_{1,2} & \cdots & \eta_{1,k-1} & \beta_1 \\ 0 & \bar\partial & \cdots & \eta_{2,k-1} & \beta_2 \\ 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \beta_{k-2} \\ 0 & 0 & 0 & \bar\partial & \eta_{k-1,k} \\ 0 & 0 & 0 & 0 & \bar\partial \end{pmatrix}.$$

By (4.9.4), $(\oplus_{i=1}^k \tilde F_i, \bar\partial_{\beta,\eta})$ is isomorphic to $(\oplus_{i=1}^k \tilde F_i, \bar\partial_\alpha)$, hence to $f^{[*]}\mathcal{E}$ on $\tilde X'$.

We now construct $\eta_{k-2,k} \in C^\infty_{(0,1)}(\tilde X', \mathrm{Hom}(\tilde F_k, \tilde F_{k-2}))$ satisfying (2) and (3). Applying (4.9.3) to $\bar\partial_{\beta,\eta}$, we have

$$\bar\partial(\beta_{k-2}) = -\eta_{k-2,k-1}\wedge\eta_{k-1,k} \quad \text{on } \tilde X'.$$

The right-hand side is $D^{(1,0)}_{\tilde h_k^*\otimes\tilde h_{k-2}}$-closed. On the other hand, by Hodge theory, we can find $\psi_{k-2} \in C^\infty(\tilde X', \mathrm{Hom}(\tilde F_k, \tilde F_{k-2}))$ such that

$$\beta_{k-2} + \bar\partial\psi_{k-2} \in \mathrm{Ker}\, D^{(1,0)}_{\tilde h_k^*\otimes\tilde h_{k-2}}.$$

Moreover, since $\eta_{k-2,k-1} \wedge \eta_{k-1,k}$ is the pullback of a form over X_{reg}, the same argument as in Step 2 yields a section $s_{k-2} \in H^0(\tilde{X}', \mathrm{Hom}(\tilde{F}_k, \tilde{F}_{k-2}))$ such that

$$\eta_{k-2,k} := \beta_{k-2} + \bar{\partial}(\psi_{k-2} + s_{k-2})$$

satisfies (2) and (3).

Set $f_{k-2} := \psi_{k-2} + s_{k-2}$ and define $(\gamma_1, \cdots, \gamma_{k-3})$ by

$$(\gamma_1, \cdots, \gamma_{k-3}) := \big(\beta_1 - \eta_{1,k-2} \cdot f_{k-2}, \cdots, \beta_{k-3} - \eta_{k-3,k-2} \cdot f_{k-2}\big).$$

Consider the complex structure

$$\bar{\partial}_{\gamma,\eta} = \begin{pmatrix} \bar{\partial} & \eta_{1,2} & \cdots & \eta_{1,k-1} & \gamma_1 \\ 0 & \bar{\partial} & \cdots & \eta_{2,k-1} & \gamma_2 \\ 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \gamma_{k-3} \\ 0 & 0 & \cdots & \cdots & \eta_{k-2,k} \\ 0 & 0 & 0 & \bar{\partial} & \eta_{k-1,k} \\ 0 & 0 & 0 & 0 & \bar{\partial} \end{pmatrix}.$$

By (4.9.4), $(\oplus_{i=1}^k \tilde{F}_i, \bar{\partial}_{\gamma,\eta})$ is isomorphic to $(\oplus_{i=1}^k \tilde{F}_i, \bar{\partial}_{\beta,\eta})$, hence to $f^{[*]}\mathcal{E}$ on $\tilde{X}'$.

Repeating this procedure, we construct $\eta_{k-3,k}, \ldots, \eta_{1,k}$ satisfying (2) and (3) such that $\bar{\partial}_\eta$ satisfies (1).

□

5. Numerically flat vector bundles on compact complex manifolds

The aim of this section is to extend Theorem 1.2 in a different direction, namely to the case where X is not necessarily Kähler. This gives a positive answer to the question raised in [DPS94, Remark 1.21].

We first recall some basic notions concerning stability of sheaves on compact complex manifolds. We refer, for example, to [CP25, Section 6] for more details. Let X be a compact complex manifold of dimension n. A Hermitian form ω_g on X is said to be a *Gauduchon metric* if

$$\partial\bar{\partial}\omega_g^{n-1} = 0.$$

The Bott–Chern cohomology group $H^{p,q}_{BC}(X)$ and the Aeppli cohomology group $H^{p,q}_A(X)$ are defined as follows:

$$H^{p,q}_{BC}(X) := \frac{(\mathrm{Ker}\,\partial\colon \mathcal{C}^{p,q}(X) \to \mathcal{C}^{p+1,q}(X)) \cap (\mathrm{Ker}\,\bar{\partial}\colon \mathcal{C}^{p,q}(X) \to \mathcal{C}^{p,q+1}(X))}{\mathrm{Im}\partial\bar{\partial}\colon \mathcal{C}^{p-1,q-1}(X) \to \mathcal{C}^{p,q}(X)},$$

$$H^{p,q}_A(X) := \frac{\mathrm{Ker}\,\partial\bar{\partial}\colon \mathcal{C}^{p,q}(X) \to \mathcal{C}^{p+1,q+1}(X)}{(\mathrm{Im}\partial\colon \mathcal{C}^{p-1,q}(X) \to \mathcal{C}^{p,q}(X)) + (\mathrm{Im}\bar{\partial}\colon \mathcal{C}^{p,q-1}(X) \to \mathcal{C}^{p,q}(X))}.$$

The movable cone

$$\mathrm{Mov}(X) \subset H^{n-1,n-1}_A(X, \mathbb{R})$$

is the closed cone generated by the classes $[\omega_g^{n-1}]$, where ω_g ranges over all Gauduchon metrics on X. The pseudo-effective cone

$$\mathrm{Psef}(X) \subset H^{1,1}_{BC}(X, \mathbb{R})$$

is the closed cone consisting of classes represented by d-closed positive $(1,1)$-currents on X. We next recall some basic properties of stability for torsion-free coherent sheaves on X.

Definition 5.1. Let X be a compact complex manifold of dimension n, and let $\mathcal{E}$ be a coherent torsion-free sheaf on X. Let ω_g be a Gauduchon metric and set $\alpha := \omega_g^{n-1}$. We define the slope of $\mathcal{E}$ with respect to α by

$$\mu_\alpha(\mathcal{E}) := \frac{1}{\operatorname{rank} \mathcal{E}} \int_X c_1(\mathcal{E}) \wedge \omega_g^{n-1}.$$

We say that $\mathcal{E}$ is stable with respect to α if

$$\mu_\alpha(\mathcal{F}) < \mu_\alpha(\mathcal{E})$$

for every subsheaf $\mathcal{F} \subset \mathcal{E}$ satisfying $1 \leqslant \mathrm{rk}\mathcal{F} < \mathrm{rk}\mathcal{E}$.

Then, we prove the following lemma.

Lemma 5.2. *Let X be a compact complex manifold equipped with a Gauduchon metric ω_g. Let E be a numerically flat vector bundle on X, namely, assume that $c_1(E) = 0$ in $H^{1,1}_{BC}(X, \mathbb{R})$ and that E is nef. If E is stable with respect to $\alpha := \omega_g^{n-1}$, then E is Hermitian flat.*

Proof. The proof follows the same idea as that of Theorem 1.2. Recall that E is nef if and only if for every $\varepsilon > 0$, there exists a smooth metric h_ε on $L := \mathcal{O}_{\mathbb{P}(E)}(1)$ such that

$$\sqrt{-1}\Theta_{h_\varepsilon}\big(\mathcal{O}_{\mathbb{P}(E)}(1)\big) \geqslant -\varepsilon\, \omega_{g'}.$$

Here $\omega_{g'}$ is a fixed Hermitian metric on $\mathbb{P}(E)$. Let r be the rank of E. Consider the natural projection $\pi\colon \mathbb{P}(E) \to X$. and the hyperplane bundle $L = \mathcal{O}_{\mathbb{P}(E)}(1)$. Since E is nef and L is relatively ample, there exists a sequence of smooth Hermitian metrics h_ε on L such that

$$\sqrt{-1}\Theta_{h_\varepsilon}(L) \geqslant -\varepsilon \pi^* \omega_g.$$

Fix a Gauduchon metric $\omega_{g'}$ on $\mathbb{P}(E)$. Since $\mathbb{P}(E)$ is compact, the quantities

$$\int_{\mathbb{P}(E)} \big(\sqrt{-1}\Theta_{h_\varepsilon}(L) + \varepsilon \pi^* \omega_g\big) \wedge \omega_{g'}^{n+r-2}$$

are uniformly bounded. After a suitable normalization and after passing to a subsequence, we may assume that $\sqrt{-1}\Theta_{h_\varepsilon}(L)$ weakly converges to a positive current $T \geqslant 0$ such that

$$[T] = c_1(L) \in H^{1,1}_{BC}(\mathbb{P}(E), \mathbb{R}).$$

Let h be a singular metric on L such that $T = \sqrt{-1}\Theta_h(L)$. Set

$$P_+ := \{x \in \mathbb{P}(E) \mid \nu(T, x) > 0\}.$$

We divide the argument into two cases.

Case 1. Suppose that P_+ is not dominant over X. By applying the positivity theorem for direct image sheaves [PT18], the metric h^{r+1} induces an L^2 metric H on

$$\det E \otimes E = p_*\big(\mathcal{O}_P(K_{\mathbb{P}(E)/X} + (r+1)L)\big)$$

such that $(\det E \otimes E, H)$ is positively curved on X. Since $c_1(E) = 0$, it follows that $(\det E \otimes E, H)$ is Hermitian flat. Hence E is also Hermitian flat.

Case 2. Suppose that P_+ is dominant over X. We show that this case cannot occur. Since E is stable and X is smooth, by [LY87], there exists a smooth Hermitian–Einstein metric G on E, namely,

$$\sqrt{-1}\Theta_G(E) \wedge \omega_g^{n-1} \equiv 0 \qquad \text{on } X.$$

The metric G induces a quotient metric h_1 on L.

We now construct a destabilizing subsheaf. By Proposition 3.2, there exists $0 < \delta < 1$ such that

$$\mathcal{G} = \pi_*\Big(\mathcal{O}_P(K_{\mathbb{P}(E)/X} + (r+1)L) \otimes \mathcal{I}\big((h^\delta h_1^{1-\delta})^{r+1}\big)\Big)$$

is a nonzero proper subsheaf of $E \otimes \det E$. By construction, we have

$$\sqrt{-1}\Theta_{h_1}(L) \wedge \pi^*\omega_g^{n-1} \geqslant 0 \qquad \text{on } \mathbb{P}(E).$$

Together with $\sqrt{-1}\Theta_h(L) \geqslant 0$, this yields

$$\sqrt{-1}\Theta_{h^\delta h_1^{1-\delta}}(L) \wedge \pi^*\omega_g^{n-1} \geqslant 0.$$

Applying [CCP, Theorem 1.6], we obtain

$$\sqrt{-1}\Theta_{\det H}(\det \mathcal{G}) \wedge \omega_g^{n-1} \geqslant 0 \qquad \text{on } X,$$

where H denotes the induced L^2 metric on $\mathcal{G}$. Therefore, we obtain

$$\int_X c_1(\mathcal{G}) \wedge \omega_g^{n-1} \geqslant 0.$$

Since $\mathcal{G} \otimes (\det E)^*$ is a nonzero proper subsheaf of E, it destabilizes E, contradicting the stability of E.

$\square$

Combining Lemma 5.2 with the argument of [DPS94, Section 2], we obtain the main theorem of this section.

Theorem 5.3. *Let X be a compact complex manifold, and let E be a locally free sheaf on X. If E is numerically flat, then E admits a filtration by holomorphic subbundles*

$$\{0\} = E_0 \subset E_1 \subset \cdots \subset E_k = E \tag{5.3.1}$$

such that each graded quotient E_i/E_{i-1} is Hermitian flat. In particular, all Chern classes $c_k(E)$ vanish.

Proof. The proof is the same as that of Theorem 1.1, but is considerably simpler in the present setting, where X is smooth and E is locally free. Fix a Gauduchon metric ω_g. Since E is numerically flat, we can show that E is semistable with respect to ω_g^{n-1}. Let $\mathcal{F} \subset \mathcal{O}_X(E)$ be a reflexive ω_g^{n-1}-stable subsheaf. By the same argument as in [DPS94, Theorem 1.18, Step 1 and Lemma 1.20], the sheaf $\mathcal{F}$ is in fact a numerically flat subbundle of E. Note that the proof of [DPS94, Theorem 1.18, Step 1 and Lemma 1.20] does not require the Kähler assumption. Applying Lemma 5.2 to $\mathcal{F}$, we conclude that $\mathcal{F}$ is Hermitian flat. The theorem then follows by induction on the rank, since $E/\mathcal{F}$ is numerically flat again. $\square$

It should be possible to generalize Theorem 5.3 to certain singular settings. However, the following question seems particularly interesting.

Question 5.4. *Let X be a compact complex manifold, and let E be a locally free sheaf on X. If E is numerically flat, does E admit a holomorphic flat connection compatible with the filtration* (5.3.1)*?*

Recall that, when X is Kähler, Question 5.4 has an affirmative answer by [Sim92, Den21]; the key tool is the $\partial\bar{\partial}$-lemma. It is unclear whether Question 5.4 remains true for manifolds that do not satisfy the $\partial\bar{\partial}$-lemma.

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