Local systems on special varieties have two-step nilpotent monodromy

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Joint works with **Cadorel** and **Yamanoi**;

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Simons Collaboration on Moduli of Varieties

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Special varieties

- Campana special varieties: no fibration with orbifold base of general type.
- Opposite to varieties of general type.
- Any variety admits a fibration with special fibers and general type orbifold base (core map).
- Examples: varieties with $\bar{\kappa} = 0$, rationally connected varieties.
- For any $d > 0$ and $k \in \{-\infty, 0, \ldots, d-1\}$, there exists special projective varieties of dimension d and Kodaira dimension k.

Definition: orbifold base

Let (X, D) be a log smooth projective pair, with D a reduced SNC divisor. Let $f : X \to Y$ be an algebraic fiber space. Let $\Delta \subset Y$ be a prime divisor. We write

$$f^*(\Delta) = \sum_j m_j E_j + R,$$

where R is f-exceptional and $f(E_j) = \Delta$. We define

$$m(f, D; \Delta) = \begin{cases} \infty, & \text{if } f^{-1}(\Delta) \subset D, \\ \min\{m_j \mid E_j \not\subset D\}, & \text{otherwise.} \end{cases}$$

The orbifold base of f is defined as

$$(Y, \Delta(f, D)) := \left(Y, \sum_\Delta \left(1 - \frac{1}{m(f, D; \Delta)}\right)\Delta\right),$$

where Δ ranges over all prime divisors of Y.

- **Convention**: Each variety is assumed to be smooth defined $/\mathbb{C}$.

Definition

- A quasi-projective variety X_0 is special if, for any log pair (X, D) such that $X \setminus D \to X_0$ is a proper birational morphism, and for any algebraic fiber space $f : X \to Y$ over a projective variety Y, one has

$$\kappa\Big(K_Y + \Delta(f, D)\Big) < \dim Y.$$

- X_0 is Brody special if there exists a Zariski dense $f : \mathbb{C} \to X_0$.

- Being (Brody) special: a birational invariant & stable under étale covers.

- **Conjecture:** Special $\Leftrightarrow$ Brody special.

Fundamental groups of special varieties: conjecture

- Actively studied from the viewpoints of:
  - arithmetic geometry (density of rational points),
  - hyperbolicity (dense entire curves, vanishing Kobayashi pseudo-distance),
  - topology (fundamental groups).

Abelianity Conjecture (Campana 04')

A special variety has almost abelian fundamental group.

- A group G is said to be almost P if it contains a subgroup of finite index that has property P.

Campana & Yamanoi's theorems

Theorem (Campana 04', Yamanoi 10')
On a special or Brody special projective variety X, any complex local system has almost abelian monodromy. In particular, if $\pi_1(X)$ is linear, the abelianity conjecture holds.

- (Cadorel–D-Yamanoi '22) Abelianity conjecture fails in the quasi-projective case.
- **Example** (Aguilar–Campana): L ample on an elliptic curve B,

$$X := L \setminus \{\text{zero section}\}$$

is a log Calabi–Yau surface, $\mathbb{C}^*$-fibration on B.

- (Gysin sequence)

  $$0 \to H^1(B) \to H^1(X) \to H^0(B) \xrightarrow{c_1(L)} H^2(B) \to H^2(X) \to H^1(B).$$

- $c_1(L) \neq 0 \implies H^1(X, \mathbb{Z}) = H^1(B, \mathbb{Z})$.
- Hence

  $$1 \to \pi_1(\mathbb{C}^*) \simeq \mathbb{Z} \to \pi_1(X) \to \pi_1(B) \simeq \mathbb{Z}^2 \to 1$$

  is central $\Rightarrow$ nilpotent of class 2.

Definition

G is nilpotent if it admits a central series

$$\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G, \quad G_{i+1}/G_i \subset Z(G/G_i).$$

The minimal n is nilpotency class. G is (at most) n-step nilpotent if its class is $\leq n$.

Conjectures-revised

Nilpotency Conjecture

A special quasi-projective variety has almost nilpotent fundamental group.

Question

If X is quasi-projective and $\pi_1(X)$ is nilpotent, must it be 2-step nilpotent?

2-Nilpotency Conjecture

A special quasi-projective variety has almost 2-step nilpotent fundamental group.

Main Theorems

Theorem (Cadorel–D–Yamanoi, 22')
Let X be a special or Brody special quasi-projective variety. Then any complex local system on X has almost nilpotent monodromy.

- In particular, the Nilpotency Conjecture holds whenever there exists a faithful representation $\pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$.
- The proof relies on hyperbolicity results, themselves based on earlier work of Brotbek, D-, Daskalopoulos, and Mese on harmonic maps to Euclidean buildings (extending the theorem of Gromov–Schoen).

Main theorems

Theorem (D–Yamanoi, 24')

Let X be a special or Brody special quasi-projective variety. Then any local system in positive characteristic on X has almost abelian monodromy.

Theorem (Cao–D–Hacon–Păun, 26')

Let X be a special quasi-projective variety. Then any complex local system on X has almost 2-step nilpotent monodromy.

- The proofs require developing deformation theory of local systems on quasi-compact Kähler manifolds, together with Hodge-theoretic tools for such local systems.

Hyperbolicity via fundamental groups

Theorem (Cadorel–D–Yamanoi, 22')

Let Y be quasi-projective. If there exists big, Zariski-dense representation

$$\sigma : \pi_1(Y) \to G(\mathbb{C}),$$

with G semisimple, then Y is of log general type, and pseudo-Brody hyperbolic (i.e. $\exists\, Z \subsetneq Y$ containing all entire curves).

- ϱ is big if for any closed $Z \subset X$ through VG point,

$$\varrho\Big(\mathrm{Im}\Big[\pi_1(Z) \to \pi_1(X)\Big]\Big) \text{ is infinite.}$$

- **Strategy:** 1. rigid & integral $\implies$ $\mathbb{Z}$-VHS;

- 2. rigid & non-integral
- 3. non-rigid
- 2 & 3 $\implies$ $\exists$ big, Zariski dense, unbounded $\tau : \pi_1(X) \to G(K)$, G almost simple, K NA local field.
- Using harmonic map to Euclidean building, find a ramified Galois cover $X^{\mathrm{sp}} \to X$, that is of log general type.
- X^{sp} has maximal quasi-Albanese dimension. Use Nevanlinna theory, prove it is pseudo Brody hyperbolic .
- Spread hyperbolicity and positivity from X^{sp} to X. $\square$

Proof of nilpotency conjecture

- **Goal:** X special / Brody special $\implies$ $\forall\ \varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$, $\mathrm{Im}(\varrho)$ almost nilpotent.
- First show $G := \overline{\varrho(\pi_1(X))}^{\mathrm{Zar}}$ is solvable.
- Assume G not solvable. Let $R(G)$ be the radical. $G/R(G)$ is semisimple, $\sigma : \pi_1(X) \to (G/R(G))(\mathbb{C})$ Zariski-dense.
- By Campana, Kollár: $\exists\, f : X \to Y$ dominant with $f^*\tau = \sigma$, τ big, Zariski dense.

- Hyperbolicity thm $\Rightarrow Y$ log general type, pseudo–Brody hyperbolic.
- Contradiction $\Rightarrow G = R(G)$, hence G solvable.
- Exact sequence:

$$0 \to U \to G \to T \to 0$$

  ($U = R_u(G)$ unipotent, T maximal torus)
- Conjugation action $T \curvearrowright U/U'$. ($U' = [U, U]$)

- Trivial action $\Rightarrow G \simeq T \times U$ (nilpotent)
- Relate this action to monodromy of VMHS from quasi-Albanese $a : X \to A$ (after restriction to the smooth locus)
- **Lemma.** The map a is dominant with connected general fibers and π_1-exact:

$$\pi_1(F) \longrightarrow \pi_1(X) \longrightarrow \pi_1(A) \longrightarrow 0$$

$\implies a$ behaves like a smooth fibration.

- **Deligne:** algebraic monodromy group of VMHS has unipotent radical.

- $T \to \mathrm{Aut}(U/U')$ factors through a unipotent group.
- **Lemma.** Torus $\to$ unipotent group, is trivial.
- Hence $G \simeq U \times T$, G nilpotent. $\square$
- **Next goal:** new proof of Campana & Yamanoi abelianity (projective varieties).

1-step phenomenon

- Using Deligne–Goldman–Millson and Simpson's NHAT: a 1-step phenomenon

Theorem (Cao–D–Hacon–Păun)

Let $h : Y \to X$ be holomorphic between compact Kähler manifolds, and (V, ∇) a semisimple flat bundle on X with monodromy $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$. If

1. $h^*\varrho$ is trivial,
2. $h^* : H^1(X, \mathrm{End}(V)) \to H^1(Y, h^*\mathrm{End}(V))$ is zero,

then for any deformation τ of ϱ, $h^*\tau$ is trivial.

Campana & Yamanoi's theorems: new proof

- $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$, X projective special / Brody special.
- **Goal:** $\mathrm{Im}(\varrho)$ abelian.
- CDY thm:

$$\varrho = (\varrho_1, \varrho_2), \quad \mathrm{Im}(\varrho_1) \text{ abelian, } \mathrm{Im}(\varrho_2) \text{ unipotent.}$$

- Reduce to ϱ unipotent:

$$\mathrm{Im}(\varrho) = \{1\}.$$

- Albanese $a : X \to A$ is π_1-exact; F general fiber, $\iota : F \hookrightarrow X$.
-

$$\iota^* H^1(X, \mathbb{C}) = 0.$$

- (V, ∇) trivial flat bundle. Then

$$\iota^* : H^1(X, \mathrm{End} V) \to H^1(F, \mathrm{End} V|_F) = 0.$$

- ϱ unipotent $\implies$ deformation of the trivial ϱ_0.
- 1-step phenomenon:

$$\varrho\Big(\mathrm{Im}[\pi_1(F) \to \pi_1(X)]\Big) = \varrho_0\Big(\mathrm{Im}[\pi_1(F) \to \pi_1(X)]\Big) = 0.$$

- π_1-exact Albanese

$$\pi_1(F) \to \pi_1(X) \to \pi_1(A) \to 0$$

$\implies$ ϱ factors through $\pi_1(A)$ $\implies$ abelian image. □

2-step phenomenon

- Quasi-Kähler case: 1-step fails $\Rightarrow$ 2-step phenomenon.

Theorem (CDHP)

Let X be quasi-Kähler. There exists a local system $\mathcal{V}_N$ with 2-step unipotent monodromy such that $\forall$ extendable holomorphic map $h : Y \to X$ from a quasi-Kähler Y, $h^*\mathcal{V}_N$ is constant $\implies$ any deformation τ of rank-N trivial local system, $h^*\tau$ is trivial.

- **Proof strategy**: explicit construction of a canonical universal connection, which represents the universal deformation of the trivial local system, by developing analytic Hodge-theoretic methods for local systems on quasi-Kähler manifolds.

Proof of 2-step nilpotency

- $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$, X quasi-projective special.
- **Goal:** $\varrho(\pi_1(X))$ 2-step nilpotent.

Theorem (CDHP)

A general fiber Y of quasi-Albanese map $a : X \to A$ is special.

- Reduce to ϱ unipotent.
- Y special $\Rightarrow$ quasi-Albanese $a_Y : Y \to A_Y$ is π_1-exact.
- Let F be a general fiber of a_Y. Using explicit $\mathcal{V}_N$,

$$\mathcal{V}_N|_F \text{ has trivial monodromy.}$$

- 2-step phenomenon $\Rightarrow \iota_F^* \varrho$ trivial.
- π_1-exactness of a_Y:

$$\pi_1(F) \to \pi_1(Y) \to \pi_1(A_Y) \to 0$$

$\Rightarrow \iota_Y^* \varrho$ factors through $\pi_1(A_Y) \Rightarrow$ abelian image.

- Exact sequence

$$\pi_1(Y) \to \pi_1(X) \to \pi_1(A) \to 0$$

$\Rightarrow \varrho(\pi_1(X))$ is 2-step nilpotent. □

Question

Does 2-step nilpotency hold for Brody special varieties?

Thank you!