L^2-VANISHING THEOREM AND A CONJECTURE OF KOLLÁR

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Abstract. In 1995, Kollár conjectured that a complex projective n-fold X with generically large fundamental group has Euler characteristic $\chi(X, K_X) \geq 0$. In this paper, we confirm the conjecture assuming X has linear fundamental group, i.e., there exists an almost faithful representation $\pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$. We deduce the conjecture by proving a stronger L^2 vanishing theorem: for the universal cover $\widetilde{X}$ of such X, its L^2-Dolbeaut cohomology $H^{n,q}_{(2)}(\widetilde{X}) = 0$ for $q \neq 0$. The main ingredients of the proof are techniques from the linear Shafarevich conjecture along with some analytic methods.

Contents

1. Introduction

1.1. **Main theorem.** In the study of Shafarevich maps, Kollár made the following conjecture ([Kol95, Conjecture 18.12.1]).

Conjecture 1.1 (Kollár). *Let X be a smooth complex projective variety. If X has generically large fundamental group, then $\chi(X, K_X) \geq 0$.*

Following the notations of [Kol95], we say that X *has generically large fundamental group* (resp. $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ is a *generically large representation*) if for any irreducible positive-dimensional subvariety Z of X passing through a very general point, the image $\mathrm{Im}[\pi_1(Z) \to \pi_1(X)]$ (resp. $\rho(\mathrm{Im}[\pi_1(Z) \to \pi_1(X)])$) is infinite (see [Kol95, Definition 2.4]

2020 *Mathematics Subject Classification.* 32J25, 14D07, 53C23, 32L20.

Key words and phrases. Kollár conjecture, L^2-cohomology, Vanishing theorem, Harmonic map, Bruhat-Tits building, Linear Shafarevich conjecture, Variation of mixed Hodge structures, Generically large fundamental group.

for the precise meaning of "very general"). Note that in [CDY22, DYK23, DY24], generically large is called *big*.

In [GL87], Green and Lazarsfeld proved Kollár's conjecture when X has maximal Albanese dimension. In this paper, we will use methods of L^2-cohomology to study Conjecture 1.1. For a compact Kähler manifold (X, ω), we denote by $\pi_X : \widetilde{X} \to X$ its universal cover, and for any non-negative integer p and q, let $H^{p,q}_{(2)}(\widetilde{X})$ be the L^2-Dolbeault cohomology group with respect to the metric $\pi_X^*\omega$. Note that its definition does not depend on the choice of ω. Our main theorem is the following:

Theorem A. *Let X be a smooth projective variety of dimension n. If there exists a generically large representation $\pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$, then the following statements hold.*

(i) $H^{p,0}_{(2)}(\widetilde{X}) = 0$ *for* $0 \le p \le n-1$ *and* $H^{n,q}_{(2)}(\widetilde{X}) = 0$ *for* $1 \le q \le n$.
(ii) *The Euler characteristic* $\chi(X, K_X) \ge 0$.
(iii) *If the strict inequality* $\chi(X, K_X) > 0$ *holds, then*
  - *(a) there exists a nontrivial L^2-holomorphic n-form on $\widetilde{X}$;*
  - *(b) X is of general type.*

In particular, we prove Conjecture 1.1 assuming $\pi_1(X)$ is linear, i.e. there exists an almost faithful representation $\pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$.

The most difficult aspect of proving Theorem A is showing that $H^{p,0}_{(2)}(\widetilde{X}) = 0$ for $0 \le p \le n-1$. We outline the proof strategy at the beginning of Section 4. The remaining conclusions of Theorem A can be derived from this L^2-vanishing theorem, using Atiyah's L^2-index theorem and Kollár's theorem.

1.2. **Some histories and comparison with previous works.** In this subsection, (X, ω) is a compact Kähler manifold of dimension n and we denote by $\pi_X : \widetilde{X} \to X$ the universal cover. In [Gro91], Gromov introduced the notion of *Kähler hyperbolicity* in his study of the Hopf conjecture. Recall that X is Kähler hyperbolic if there is a smooth 1-form β such that $\pi_X^*\omega = d\beta$ and the norm $|\beta|_{\pi_X^*\omega}$ is bounded from above by a constant. He then proved the vanishing of k-th L^2-Betti numbers of $\widetilde{X}$ for $k \ne n$.

Gromov's idea was later extended by Eyssidieux [Eys97], in which more general notions of *weakly Kähler hyperbolicity* are introduced. Eyssidieux's work was recently generalized by Bei, Claudon, Diverio, Eyssidieux, and Trapani [BDET24, BCDT24], who studied a birational analog of Kähler hyperbolicity. Also as generalizations of Gromov's work, Cao-Xavier [CX01] and Jost-Zuo [JZ00] introduced the notion of *Kähler parabolicity*. They independently observed that the arguments of Gromov concerned with the vanishing of L^2-Betti numbers work also under the weaker assumption that β is a smooth 1-form such that $|\beta(x)|_{\pi_X^*\omega}$ has *sub-linear growth*. In other words, there exists a constant $c > 0$ such that $|\beta(x)|_{\pi_X^*\omega} \le c(1 + d_{\widetilde{X}}(x, x_0))$.

The formulation and proof of our partial L^2-vanishing theorem in Theorem 2.4 are inspired by the aforementioned works, and is introduced specifically for the proof of Theorem A. As a result, it may appear technically involved.

In another aspect of the proof of Theorem A, we extensively use techniques from the study of the reductive and linear Shafarevich conjectures in [DYK23, EKPR12]. Recall that the Shafarevich conjecture predicts that a complex projective variety with a large fundamental group has its universal cover holomorphically convex. This conjecture was proved by Eyssidieux, Katzarkov, Pantev and Ramachandran [Eys04, EKPR12] for smooth projective varieties with linear fundamental groups, and was recently extended by Yamanoi and the first author in [DYK23] to projective normal varieties with reductive fundamental groups.

1.3. **Notation and Convention.**

- All varieties in this paper are defined over $\mathbb{C}$.
- Let $(X.\omega)$ be a Kähler manifold. We denote by $\pi_X : \widetilde{X} \to X$ the universal cover of X.
- Let (X, ω) be a compact Kähler manifold. Unless otherwise specified, $d_{\widetilde{X}}(x, x_0)$ stands for the Riemannian distance between x and the base point x_0 in $\widetilde{X}$ with respect to the metric $\pi_X^*\omega$.
- For a complex space Z, Z^{norm} denotes its normalization, and Z^{reg} denotes its regular locus.
- We use the standard abbreviations VHS and VMHS for *variation of Hodge structures* and *variation of mixed Hodge structures* respectively.
- By convention, a closed positive $(1,1)$-current T on a complex manifold is *semi-positive*.
- Plurisubharmonic functions are abbreviated as psh functions.
- A positive closed $(1,1)$-current T has *continuous potential* if locally we have $T = \sqrt{-1}\partial\bar{\partial}\psi$ with ψ a continuous psh function.

1.4. **Acknowledgment.** We would like to express our gratitude to Patrick Brosnan, Junyan Cao, Simone Diverio, Philippe Eyssidieux, Mihai Păun, Pierre Py, Christian Schnell, Xu Wang, and Mingchen Xia for their inspiring discussions. We extend special thanks to Chikako Mese for her invaluable discussion and expertise on harmonic maps into Euclidean buildings, and to Yuan Liu and Xueyuan Wan for reading the first version of the paper and their very helpful remarks. YD acknowledges the partial support of the French Agence Nationale de la Recherche (ANR) under reference ANR-21-CE40-0010. BW thanks Peking University and BICMR for the generous hospitality and support during the writing of this paper. He is partially supported by a Simons fellowship.

2. A PARTIAL L^2-VANISHING THEOREM

A closed positive $(1,1)$-current T on a complex manifold X can be seen as a $(1,1)$-form with positive measure coefficients. Note that positive measures admit a Lebesgue decomposition into an absolutely continuous part (with respect to the Lebesgue measure on X) and a singular part. We therefore get a decomposition of T itself into an absolutely continuous part T_{ac} and a singular part T_{sing}. We begin with the following definition in [Bou02, §2.3].

Definition 2.1 (Lebesgue decomposition)**.** The decomposition $T = T_{\mathrm{ac}} + T_{\mathrm{sing}}$ is called the *Lebesgue decomposition* of T .

Definition 2.2 (Semi-Kähler form)**.** Let (X, ω_X) be a compact Kähler manifold. A smooth closed real $(1,1)$-form ω_{sk} on X is *semi-Kähler* if ω_{sk} is semipositive everywhere and is strictly positive on a Zariski dense open subset of X.

Before proving a key L^2-vanishing theorem, we first recall the definition of L^2-cohomology (cf. [BDIP02, Definition 12.3]).

Definition 2.3 (L^2-cohomology)**.** Let (Y,ω) be a complete Kähler manifold. Let $L^{p,q}_{(2)}(Y)$ be the space of L^2-integrable (p,q)-forms with respect to the metric ω. A section u is said to be in $\mathrm{Dom}\,\bar\partial$ if $\bar\partial u$ calculated in the sense of distributions is still in L^2. Then the L^2-Dolbeault cohomology is defined as

$$H^{p,q}_{(2)}(Y)=\ker\bar\partial\,/\,\overline{\mathrm{Im}\,\bar\partial\cap\mathrm{Dom}\,\bar\partial}.$$

If (X,ω) is a compact Kähler manifold, then $H^{p,q}_{(2)}(\widetilde{X})$ denotes the L^2-cohomology with respect to the metric $\pi_X^*\omega$.

Let us state and prove our partial L^2-vanishing theorem.

Theorem 2.4. *Let (X,ω) be a compact Kähler n-fold. Let $f:X\to Y$ be a proper surjective holomorphic map with connected fibers over a compact Kähler normal space Y of dimension m. Denote by $\tilde f:\widetilde{X}\to\widetilde{Y}$ the lift of f to universal covers. Assume that there exists a 1-form β on $\widetilde{X}$ with L^1_{loc}-coefficients, and a continuous quasi-psh function ψ on X satisfying the following properties.*

(1) The 1-form β is smooth on an open subset of $\widetilde{X}$ whose complement has zero Lebesgue measure, and ψ is smooth on a Zariski dense open subset of X. Moreover, there is a constant $C>0$ such that

$$|\beta(x)|_{\pi_X^*\omega}\leq_{\text{a.e.}} C(d_{\widetilde{X}}(x,x_0)+1)\tag{1}$$

where x_0 is a base point of $\widetilde{X}$.

(2) The sum $d\beta+\pi_X^*dd^{\mathrm{c}}\psi$ is a closed positive $(1,1)$-current, satisfying

$$\pi_X^*f^*\omega_Y\leq_{\text{a.e.}}(d\beta+\pi_X^*dd^{\mathrm{c}}\psi)_{\mathrm{ac}}\leq_{\text{a.e.}}\pi_X^*\omega\tag{2}$$

where ω_Y is a semi-Kähler form on Y.

Then

(i) *For any $p\in\{0,\ldots,m-1\}$, we have $H^{p,0}_{(2)}(\widetilde{X})=0$.*

Assume that there exists a non-zero $\alpha\in H^{p,0}_{(2)}(\widetilde{X})$ for some $p<n$. Let Y° be the Zariski open subset of Y^{reg}, over which f is a proper submersion and ω_Y is Kähler.

(ii) *we have*

$$\alpha|_{\widetilde{X}^\circ}\in H^0(\widetilde{X}^\circ,\tilde f^*\Omega^m_{\widetilde{Y}^\circ}\otimes\Omega^{p-m}_{\widetilde{X}^\circ}),\tag{3}$$

where $\widetilde{Y}^\circ:=\pi_Y^{-1}(Y^\circ)$ and $\widetilde{X}^\circ:=\tilde f^{-1}(\widetilde{Y}^\circ)$.

(iii) *For any $y\in\widetilde{Y}^\circ$, let α_y be the holomorphic $(p-m)$-form on $\tilde f^{-1}(y)$ induced by α under the isomorphism*

$$\tilde f^*\Omega^m_{\widetilde{Y}^\circ}\otimes\Omega^{p-m}_{\widetilde{X}^\circ}|_{\tilde f^{-1}(y)}\simeq\Omega^{p-m}_{\tilde f^{-1}(y)}.$$

Then $\alpha|_{\tilde f^{-1}(y)}$ is d-closed.

Proof. To lighten the notation, we write ω instead of $\pi_X^*\omega$ abusively.

Step 1. Since the sectional curvature of the complete Kähler manifold $(\widetilde{X}, \pi_X^*\omega)$ is uniformly bounded, by a result of W. Shi (see e.g. [Hua19, Theorem 1.2]), there exists a constant $C > 0$ and a smooth exhausting function r on $\widetilde{X}$ such that

$$d_{\widetilde{X}}(x, x_0) + 1 \leq r(x) \leq d_{\widetilde{X}}(x, x_0) + C,$$

$|dr|_\omega(x) \leq C$ and $|dd^c r|_\omega(x) \leq C$ for all $x \in \widetilde{X}$. Hence by (1), we have

$$|\beta|_\omega(x) \leq_{\text{a.e.}} Cr(x).$$

Let $\varrho : \mathbb{R} \to \mathbb{R}$ be a smooth function with $0 \leq \varrho \leq 1$ such that

$$\varrho(t) = \begin{cases} 1, & \text{if } t \leq 0; \\ 0, & \text{if } t \geq 1. \end{cases}$$

We consider the compactly supported function

$$f_j(x) = \varrho(r(x) - j + 1), \tag{4}$$

where j is a positive integer. Then

$$\operatorname{Supp}(f_j) \subset \{x \in \widetilde{X} \mid r(x) \leq j\},$$
$$df_j(x) = \varrho'(r(x) - j + 1)dr,$$

and

$$\sqrt{-1}\partial\bar{\partial} f_j = \varrho'(r - j + 1)\sqrt{-1}\partial\bar{\partial} r + \sqrt{-1}\varrho''(r - j + 1)\partial r \wedge \bar{\partial} r.$$

Then there exists some constant $c_1 > 0$ such that

$$|dd^c f_j(x)|_\omega \leq c_1 \text{ and } |df_j(x)|_\omega \leq c_1. \tag{5}$$

for any $x \in \widetilde{X}$.

Let α be a holomorphic $(p, 0)$-form which is L^2 with respect to ω with $0 \leq p \leq n-1$. Then

$$i^{p^2} \int_{\widetilde{X}} \alpha \wedge \bar{\alpha} \wedge \omega^{n-p} = \frac{(n-p)!}{n!} \int_{\widetilde{X}} |\alpha|_\omega \omega^n < \infty. \tag{6}$$

Claim 2.5. *The smooth $(p+k, p+k)$-form $i^{p^2}\alpha \wedge \bar{\alpha} \wedge \omega^k$ is closed and* positive *in the sense of* [Dem12, Chapter 3, Definition 1.1].

Proof of Claim 2.5. Since ω and α are both closed, it is obvious that $i^{p^2}\alpha \wedge \bar{\alpha} \wedge \omega^k$ is closed. By [Dem12, Chapter 3, Example 1.2], $i^{p^2}\alpha \wedge \bar{\alpha}$ is a positive (p, p)-form. By [Dem12, Chapter 3, Corollary 1.9], ω is a *strongly positive* $(1, 1)$-form in the sense of [Dem12, Chapter 3, Definition 1.1] (it is different from the notion "strictly positive" $(1, 1)$-form!). We apply [Dem12, Chapter 3, Proposition 1.11] to conclude that $i^{p^2}\alpha \wedge \bar{\alpha} \wedge \omega^k$ is *positive* $(p+k, p+k)$-forms for any $k = 1, \ldots, n-p$. The claim is proved. □

We have

$$\begin{aligned} i^{p^2} \int_{\widetilde{X}} (\pi_X^*\psi dd^c f_j - df_j \wedge \beta) \wedge \alpha \wedge \bar{\alpha} \wedge \omega^{n-p-1} &= i^{p^2} \int_{\widetilde{X}} f_j (d\beta + dd^c \pi_X^*\psi) \wedge \alpha \wedge \bar{\alpha} \wedge \omega^{n-p-1} \\ &= i^{p^2} \int_{\widetilde{X}} f_j \alpha \wedge \bar{\alpha} \wedge \omega^{n-p-1} \wedge T_{\text{ac}} + i^{p^2} \int_{\widetilde{X}} f_j T_{\text{sing}} \wedge \alpha \wedge \bar{\alpha} \wedge \omega^{n-p-1} \end{aligned} \tag{7}$$

where $T = d\beta + dd^{\mathrm{c}}\pi_X^*\psi$, and $T = T_{\mathrm{ac}} + T_{\mathrm{sing}}$ is the Lebesgue decomposition as in Definition 2.1.

Claim 2.5 implies that

$$i^{p^2}\int_{\widetilde{X}} f_j T_{\mathrm{sing}} \wedge \alpha \wedge \bar{\alpha} \wedge \omega^{n-p-1} \geq 0 \tag{8}$$

since T_{sing} is a positive $(1,1)$-current and $f_j \geq 0$. By Item 2, $T_{\mathrm{ac}} \leq_{\mathrm{a.e.}} \omega$. Since $0 \leq f_j \leq 1$ and $\lim_{j\to\infty} f_j(x) = 1$ for any $x \in \widetilde{X}$, by Claim 2.5 and Lebesgue's dominated convergence theorem, we have

$$\begin{aligned} 0 \leq \lim_{j\to\infty} i^{p^2}\int_{\widetilde{X}} f_j\alpha \wedge \bar{\alpha} \wedge \omega^{n-p-1} \wedge T_{\mathrm{ac}} = \\ i^{p^2}\int_{\widetilde{X}} \alpha \wedge \bar{\alpha} \wedge \omega^{n-p-1} \wedge T_{\mathrm{ac}} \leq i^{p^2}\int_{\widetilde{X}} \alpha \wedge \bar{\alpha} \wedge \omega^{n-p} \overset{(6)}{<} +\infty. \end{aligned} \tag{9}$$

Denote by

$$B_j := \{x \in \widetilde{X} \mid j-1 \leq r(x) \leq j\}.$$

In what follows, for any smooth form γ on $\widetilde{X}$, we denote by $|\gamma|$ its norm with respect to the metric ω. Denote $\mathrm{dvol} := \frac{\omega^n}{n!}$. Since ψ is continuous, we have $\sup_X |\psi| \leq c_2$ for some constant $c_2 > 0$. Then by $\mathrm{supp}(dd^{\mathrm{c}} f_j) \subset B_j$, we have

$$\begin{aligned} \left|\int_{\widetilde{X}} \pi_X^*\psi dd^{\mathrm{c}} f_j \wedge \alpha \wedge \bar{\alpha} \wedge \omega^{n-p-1}\right| &\leq \int_{B_j} \left|\pi_X^*\psi dd^{\mathrm{c}} f_j \wedge \alpha \wedge \bar{\alpha} \wedge \omega^{n-p-1}\right| \mathrm{dvol} \\ &\leq c_2 \int_{B_j} |dd^{\mathrm{c}} f_j|\, |\alpha|^2\, |\omega^{n-p-1}|\, \mathrm{dvol} \\ &\overset{(5)}{\leq} c_1 c_2 \int_{B_j} |\alpha|^2\, \mathrm{dvol}\,. \end{aligned} \tag{10}$$

Note that $|\beta(x)| \leq_{\mathrm{a.e.}} Cr(x)$ for some constant $C > 0$. Hence by $\mathrm{supp}(df_j) \subset B_j$, one has

$$\begin{aligned} \left|\int_{\widetilde{X}} df_j \wedge \beta \wedge \alpha \wedge \bar{\alpha} \wedge \omega^{n-p-1}\right| &\leq \int_{B_j} \left|df_j \wedge \beta \wedge \alpha \wedge \bar{\alpha} \wedge \omega^{n-p-1}\right| \mathrm{dvol} \\ &\leq \int_{B_j} |df_j|\, |\beta|\, |\alpha|^2\, |\omega^{n-p-1}|\, \mathrm{dvol} \\ &\overset{(5)}{\leq} c_1 C j \int_{B_j} |\alpha|^2\, \mathrm{dvol} \end{aligned} \tag{11}$$

Claim 2.6. *There exists a subsequence $\{j_i\}_{i\geq 1}$ such that*

$$\lim_{i\to\infty} j_i \int_{B_{j_i}} |\alpha(x)|^2\, \mathrm{dvol} = 0. \tag{12}$$

Proof. If not, then there exists a positive constant c' and a positive integer n_0 such that

$$j\int_{B_j} |\alpha(x)|^2\, \mathrm{dvol} \geq c' > 0$$

for any $j \geq n_0$. This yields

$$\begin{aligned}\int_{\widetilde{X}} |\alpha(x)|^2 \,\mathrm{dvol} &\geq \sum_{j=n_0}^{\infty} \int_{B_j} |\alpha(x)|^2 \,\mathrm{dvol} \\ &\geq c' \sum_{j=n_0}^{+\infty} \frac{1}{j} = +\infty,\end{aligned}$$

which leads to a contradiction that α is an L^2-form with respect to ω. □

Claim 2.6 implies that there exists a subsequence $\{j_i\}_{i\geq 1}$ for which (12) holds. (7), (10), (11) and (12) imply that

$$\lim_{k\to\infty} i^{p^2} \int_{\widetilde{X}} f_{j_k} \alpha \wedge \bar{\alpha} \wedge \omega^{n-p-1} \wedge T_{\mathrm{ac}} + i^{p^2} \int_{\widetilde{X}} f_{j_k} T_{\mathrm{sing}} \wedge \alpha \wedge \bar{\alpha} \wedge \omega^{n-p-1} = 0.$$

Together with (8), (9), we obtain that

$$i^{p^2} \int_{\widetilde{X}} \alpha \wedge \bar{\alpha} \wedge \omega^{n-p-1} \wedge T_{\mathrm{ac}} = 0.$$

Since $T_{\mathrm{ac}} \geq \pi_X^* f^* \omega_Y$, this equality along with Claim 2.5 imply that

$$i^{p^2} \int_{\widetilde{X}} \alpha \wedge \bar{\alpha} \wedge \omega^{n-p-1} \wedge \pi_X^* f^* \omega_Y = 0.$$

By Claim 2.5 again, $i^{p^2} \alpha \wedge \bar{\alpha} \wedge \pi_X^* f^* \omega_Y$ is a positive $(p+1, p+1)$-form. The above equality implies that

$$i^{p^2} \alpha \wedge \bar{\alpha} \wedge \pi_X^* f^* \omega_Y \equiv 0. \tag{13}$$

Step 2. We abusively denote ω_Y for $\pi_Y^* \omega_Y$. Let y be any point in $\widetilde{Y}^\circ$ and let $x \in \widetilde{X}$ be any point in $\tilde{f}^{-1}(y)$. One can then find coordinate open subsets $(U; z_1, \ldots, z_{n+m})$ centered at x and $(V; w_1, \ldots, w_m)$ centered at y, such that $\tilde{f}(z_1, \ldots, z_{n+m}) = (z_1, \ldots, z_m)$. Furthermore, we assume that $(V; w_1, \ldots, w_m)$ is orthonormal at y with respect to ω_Y. Then $\omega_Y(y) = \sqrt{-1} \sum_{j=1}^m dw_j \wedge d\bar{w}_j$.

For any subset $I \subset \{1, \ldots, n+m\}$, we denote $\omega_I := \wedge_{i\in I} \sqrt{-1} dz_i \wedge d\bar{z}_i$. Let $\alpha \in H^{p,0}_{(2)}(\widetilde{X})$. If we express $\alpha|_U = \sum_{|I|=p} \alpha_I dz_I$, where $dz_I := dz_{i_1} \wedge \cdots \wedge dz_{i_p}$ with I consisting of $i_1 < \ldots < i_p$, then we have

$$i^{p^2} \alpha \wedge \bar{\alpha} \wedge \tilde{f}^* \omega_Y(x) = \sum_{j \in \{1,\ldots,m\}\backslash I} |\{1, \ldots, m\}\backslash I| \cdot |\alpha_I|^2(x) \omega_{I\cup j}.$$

Here $|\{1, \ldots, m\}\backslash I|$ represents the cardinality of the set $\{1, \ldots, m\}\backslash I$. By (13), we have $\{1, \ldots, m\} \subset I$ for each I satisfying $\alpha_I(x) \neq 0$. Since x is an arbitrary point in the Zariski dense open subset $\widetilde{X}^\circ$ of $\widetilde{X}$, this establishes Theorem 2.4.(i). If $n > m$ and $p \geq m$, this also establishes (3). Hence Theorem 2.4.(ii) is proved.

Step 3. Let us prove Theorem 2.4.(iii). For any $y \in \widetilde{Y}^\circ$, we denote $F_y := \tilde{f}^{-1}(y)$. Each non-trivial m-form η in $\Omega^m_{\widetilde{Y}^\circ, y}$, induces a *unique* holomorphic $(p-m)$-form $\alpha_y \in H^0(F_y, \Omega^{p-m}_{F_y})$ on F_y via the isomorphism of locally free sheaves on F_y as follows

$$\tilde{f}^*\Omega^m_{\widetilde{Y}^\circ} \otimes \Omega^{p-m}_{\widetilde{X}^\circ}|_{\tilde{f}^{-1}(y)} \stackrel{\otimes \tilde{f}^*\eta^\vee}{\to} \Omega^{p-m}_{\widetilde{X}^\circ}|_{\tilde{f}^{-1}(y)} \to \Omega^{p-m}_{\tilde{f}^{-1}(y)}.$$

Such α_y can be written explicitly. We use the coordinate systems $(U; z_1, \ldots, z_{n+m})$ and $(V; w_1, \ldots, w_m)$ introduced in Step 2. The canonical form $dw_1 \wedge \cdots \wedge dw_m$ in V is a nowhere-vanishing. For any set $I \supset \{1, \ldots, m\}$, denote by $\tilde{I} := I \backslash \{1, \ldots, m\}$. By Theorem 2.4.(ii), we have

$$\alpha|_U = \sum_{|I|=p, I \supset \{1,\ldots,m\}} \alpha_I dz_{\tilde{I}} \wedge dz_1 \wedge \cdots \wedge dz_m. \tag{14}$$

Then the holomorphic $(p-m)$-form α_y on $F_y \cap U$ induced by $dw_1 \wedge \cdots \wedge dw_m$ is defined by

$$\alpha_y := \sum_{|I|=p, I \supset \{1,\ldots,m\}} \alpha_I|_{F_y} dz_{\tilde{I}}. \tag{15}$$

It can be verified that the above definition depends only on the m-form $dw_1 \wedge \cdots \wedge dw_m(y)$. Therefore, if we choose different coordinate open subsets covering F_y, they glue together into a well-defined holomorphic $(p-m)$-form α_y on F_y.

Equation (14) implies that

$$0 = d\alpha = \sum_{|I|=p} \sum_{j=m+1}^{n} \frac{\partial \alpha_I}{\partial z_j} dz_j \wedge dz_{\tilde{I}} \wedge dz_1 \wedge \cdots \wedge dz_m. \tag{16}$$

This yields

$$d\alpha_y = \sum_{|I|=p, I \supset \{1,\ldots,m\}} \sum_{j=m+1}^{n} \frac{\partial \alpha_I}{\partial z_j}|_{F_y} dz_j \wedge dz_{\tilde{I}} = 0.$$

Theorem 2.4.(iii) is proved. We complete the proof of the theorem. □

We will need the following consequence of Theorem 2.4 in the proof of Theorem A.

Corollary 2.7. *Let (X, ω) be a compact Kähler n-fold, and let $f : X \to A$ be a holomorphic map to an abelian variety A such that* $\dim f(X) = n$. *Let $\widetilde{X}_1$ be a connected component of $X \times_A \widetilde{A}$, where $\widetilde{A} \to A$ is the universal cover of A. Then for any infinite Galois cover $\widetilde{X'} \to X$ dominating $\widetilde{X}_1$, we have $H^{p,0}_{(2)}(\widetilde{X'}) = 0$ for any $p \in \{0, \ldots, n-1\}$.*

Proof. Denote by $\pi' : \widetilde{X'} \to X$ the covering map. We take global linear coordinates $(z_1, \ldots, z_m)$ for $\widetilde{A}$ such that $\sqrt{-1}\sum_{i=1}^m dz_i \wedge d\bar{z}_i$ descends to a Kähler form ω_A on A. Since $\widetilde{X'}$ dominates $\widetilde{X}_1$, there is a holomorphic map $g : \widetilde{X'} \to \widetilde{A}$ that lifts $f : X \to A$. Let $g_i := z_i \circ g$. Consider smooth 1-form

$$\beta := g^*\sqrt{-1}\bar{\partial}(\sum_{i=1}^m |z_i|^2) = \sqrt{-1}\sum_{i=1}^n g_i(x)\overline{\partial g_i(x)}.$$

Note that $\partial g_i(x)$ descends to a holomorphic 1-form on X. Thus, there is a constant $C > 0$ such that $|\partial g_i(x)|_{\pi'^*\omega} \le C$ for any $x \in \widetilde{X'}$ and any $i = 1, \ldots, n$. Therefore, g is Lipschitz. Fix a base point $x_0 \in \widetilde{X'}$. Then there is a constant $C' > 0$ such that

$$|g_i(x)| \le C'\big(d_{\widetilde{X'}}(x, x_0) + 1\big)$$

for any $x \in \widetilde{X}$. This implies that

$$|\beta(x)|_{\pi'^*\omega} \le nCC'\big(d_{\widetilde{X'}}(x, x_0) + 1\big).$$

Note that $d\beta = \pi'^* f^* \omega_A$, which is a semi-Kähler form on X. By Theorem 2.4.(i), we conclude the desired L^2-vanishing theorem. □

3. Constructing 1-forms via harmonic maps to Euclidean buildings

Let K be a non-archimedean local field of characteristic zero and let G be a reductive algebraic group defined over K. There exists a Euclidean building associated with G, which is called the Bruhat-Tits building and denoted by $\Delta(G)$. We refer the readers to [KP23, Rou23] for the definition and properties of Bruhat-Tits buildings.

Let (V, W, Φ) be the root system associated with $\Delta(G)$. It means that V is a real Euclidean space endowed with a Euclidean metric and W is an affine Weyl group acting on V. Namely, W is a semidirect product $T \rtimes W^v$, where W^v is the *vectorial Weyl group*, which is a finite group generated by reflections on V, and T is a translation group of V. Here Φ is the root system of V. It is a finite set of $V^*\backslash\{0\}$ such that W^v acts on Φ as a permutation. Moreover, Φ generates V^*. From the reflection hyperplanes of W we obtain a decomposition of V into facets. Let $\mathcal{H}$ be set of hyperplanes of V defined by $w \in W$. The maximal facets, called *chambers*, are the open connected components of $V \backslash \cup_{H\in\mathcal{H}}$.

For any apartment A in $\Delta(G)$, there exists an isomorphism $i_A : A \to V$, which is called a chart. For two charts $i_{A_1} : A_1 \to V$ and $i_{A_2} : A_2 \to V$, if $A_1 \cap A_2 \neq \varnothing$, it satisfies the following properties:

(a) $Y := i_{A_2}(i_{A_1}^{-1}(V))$ is convex.
(b) There is an element $w \in W$ such that $w \circ i_{A_1}|_{A_1\cap A_2} = i_{A_2}|_{A_1\cap A_2}$.

The charts allow us to map facets into $\Delta(G)$ and their images are also called facets. The axioms guarantee that these notions are chart independent.

Let X be a compact Kähler manifold and let $\varrho : \pi_1(X) \to G(K)$ be a Zariski dense representation. By the work of Gromov-Schoen [GS92] (see also [BDDM22] for the quasi-projective case), there exists a ϱ-equivariant harmonic mapping $u : \widetilde{X} \to \Delta(G)$ where $\Delta(G)$ is the (enlarged) Bruhat-Tits building of G (see [KP23, Definition 4.3.2] for the definition). Such u is moreover pluriharmonic. We denote by $R(u)$ the regular set of harmonic map u. That is, for any $x \in R(u)$, there exists an open subset Ω_x containing x such that $u(\Omega_x) \subset A$ for some apartment A. Since $G(K)$ acts transitively on the apartments of $\Delta(G)$, we know that $R(u)$ is the pullback of an open subset X' of X. By [GS92], $X\backslash X'$ has *Hausdorff codimension* at least two.

We fix an orthogonal coordinates $(x_1, \ldots, x_N)$ for V. Define smooth real functions on Ω_x by setting

$$u_{A,i} := x_i \circ i_A \circ u. \tag{17}$$

The pluriharmonicity of u implies that $\sqrt{-1}\partial\bar{\partial}u_{A,i} = 0$. We consider a smooth real semi-positive $(1,1)$-form on Ω_x defined by

$$\sqrt{-1}\sum_{i=1}^{N}\partial u_{A,i}\circ\bar{\partial}u_{A,i}.$$

By [Eys04, §3.3.2], such real $(1,1)$-form does not depend on the choice of A, and is invariant under $\pi_1(X)$-action. Therefore, it descends to a smooth real closed semi-positive $(1,1)$-form on X'. It is shown in [Eys04] that it extends to a positive closed $(1,1)$-current T_ϱ on X with continuous potential.

Definition 3.1 (Canonical current)**.** The above closed positive $(1,1)$-current T_ϱ on X is called the *canonical current* of ϱ.

By [Eys04, CDY22, DYK23], one has a proper fibration $s : X \to S_\varrho$ associated with ϱ, which is called *Katzarkov-Eyssidieux reduction map*. It has the following properties.

Proposition 3.2 ([Eys04, CDY22])**.** *Let X be a smooth projective variety and let $\rho : \pi_1(X) \to G(K)$ be a Zariski dense representation where G is a reductive group over a non-archimedean local field K. Then there exists a proper morphism $s_\rho : X \to S_\rho$ (so-called Katzarkov-Eyssidieux reduction map) onto a normal projective variety with connected fibers such that for any irreducible closed subvariety $Z \subset X$, the following properties are equivalent:*

(1) $\rho(\mathrm{Im}[\pi_1(Z_{\mathrm{norm}}) \to \pi_1(X)])$ *is bounded;*
(2) $\rho(\mathrm{Im}[\pi_1(Z) \to \pi_1(X)])$ *is bounded;*
(3) $s_\rho(Z)$ *is a point.*

Moreover, there exists a $(1,1)$-current T'_ϱ with continuous potential on S_ϱ such that $s_\varrho^* T'_\varrho = T_\varrho$. □

Proposition 3.3. *Let x_0 be a fixed base point in $\widetilde{X}$. For the canonical current T_ϱ defined in Definition 3.1, we have*

$$\sqrt{-1}\partial\bar{\partial}d^2_{\Delta(G)}(u(x), u(x_0)) \geq \pi_X^* T_\varrho, \tag{18}$$

where $d_{\Delta(G)}(\bullet,\bullet)$ is the distance function on $\Delta(G)$. Moreover, the above equality holds over a dense open subset $\widetilde{X}^\circ$ such that $\widetilde{X}\backslash\widetilde{X}^\circ$ has zero Lebesgue measure.

Proof. We assume that $x_0 \in R(u)$. Let ω be a Kähler metric on X. For any $x \in R(u)$ where $R(u)$ is the regular set of the harmonic map u, there exists an open subset Ω_x containing x such that $u(\Omega_x) \subset A$ for some apartment A.

Note that $d_{\Delta(G)}$ is $G(K)$-invariant. Let $\iota : C \to \Omega_x$ be any holomorphic curve. Then by [GS92, Proposition 2.2 in p. 191], we have

$$\Delta d^2(u\circ\iota(x), u(x_0)) \geq |\nabla u\circ\iota|^2,$$

where the $|\nabla u\circ\iota|^2$ is the norm defined with respect to $d_{\Delta(G)}$ and ω. It follows that

$$|\nabla u\circ\iota|^2\iota^*\omega = \iota^*(\sqrt{-1}\sum_{i=1}^{N}\partial u_{A,i}\wedge\bar{\partial}u_{A,i})$$

where $u_{A,i}$ is defined in (17). Therefore, over Ω_x we have

$$\sqrt{-1}\partial\bar{\partial}d^2_{\Delta(G)}(u(x),u(x_0)) \geq \sqrt{-1}\sum_{i=1}^{m}\partial u_{A,i}\wedge\bar{\partial}u_{A,i}.$$

By Definition 3.1, we have (18) over the whole $R(u)$.

Claim 3.4. (18) *holds over the whole* $\widetilde{X}$.

Proof. Since u is Lipschitz, it follows that $d^2_{\Delta(G)}(u(x),u(x_0))$ is a continuous function on $\widetilde{X}$. Recall that T_ϱ is a positive closed $(1,1)$-current with continuous potential. This implies that for any point $x\in\widetilde{X}$, there exist a neighborhood Ω_x and a continuous function ϕ on Ω_x such that

$$\sqrt{-1}\partial\bar{\partial}d^2_{\Delta(G)}(u(x),u(x_0)) - \pi_X^*T_\varrho = \sqrt{-1}\partial\bar{\partial}\phi$$

on Ω_x. Note that $\sqrt{-1}\partial\bar{\partial}\phi \geq 0$ over $\Omega_x\cap R(u)$. Since the complement of $\Omega_x\cap R(u)$ has Hausdorff codimension at least two in Ω_x, we apply the extension theorem in [Shi72, Theorem 3.1(i)] to conclude that there is a psh function ϕ' on Ω_x such that $\phi'|_{\Omega_x\cap R(u)} = \phi|_{\Omega_x\cap R(u)}$. Since ϕ is continuous, and ϕ' is upper semi-continuous, it follows that $\phi=\phi'$ on Ω_x. This shows that (18) holds over Ω_x, hence over the whole $\widetilde{X}$. The claim is proved. □

Let $\widetilde{X}^\circ$ be the set of points x in $\widetilde{X}$ such that there exists an open neighborhood Ω_x containing x such that $u(\Omega_x)$ is contained in the closure of a chamber C of the building $\Delta(G)$. By [DM24], $\widetilde{X}^\circ$ is a dense open subset such that $\widetilde{X}\backslash\widetilde{X}^\circ$ has zero Lebesgue measure. Note that there exists an apartment A containing both $\overline{C}$ and x_0. It follows that for any $y\in\Omega_x$, we have

$$d^2_{\Delta(G)}(u(y),u(x_0)) = \sum_{i=1}^{N}|u_{A,i}(y)-u_{A,i}(x_0)|^2. \tag{19}$$

Therefore,

$$\sqrt{-1}\partial\bar{\partial}d^2_{\Delta(G)}(u(y),u(x_0)) = \sqrt{-1}\sum_{i=1}^{m}\partial u_{A,i}(y)\wedge\bar{\partial}u_{A,i}(y)$$

by the pluriharmonicity of $u_{A,i}$. This proves that over $\widetilde{X}^\circ$, we have

$$\sqrt{-1}\partial\bar{\partial}d^2_{\Delta(G)}(u(x),u(x_0)) = \pi_X^*T_\varrho(x).$$

The proposition is proved. □

Remark 3.5. Note that we cannot expect the equality (18) holds over $R(u)$. Here is an example. Consider a tree $T\subset\mathbb{R}^2$ defined by $T=\mathbb{R}\times\{0\}\cup 0\times\mathbb{R}_{\geq 0}$ and thus $(0,0)$ is the vertice of T. Consider a pluriharmonic map $u:\mathbb{D}\to T$ defined by $z\mapsto(\mathrm{Re}(z),0)$. Let $P:=(0,1)$ be a base point in T. Then $d^2(u(z),P)=(|\mathrm{Re}(z)|+1)^2$, where $d(\bullet,\bullet)$ denotes the distance function on the tree T. In this case, we note that $\sqrt{-1}\partial\bar{\partial}d^2(u(z),P)$ does not has absolutely continuous coefficients on $\mathbb{D}$: on the line $\mathbb{D}\cap(\mathrm{Re}(z)=0)$, the coefficients of $\sqrt{-1}\partial\bar{\partial}d^2(u(z),P)$ have non-trivial singular part for its Lebesgue decomposition. However, the regular locus $R(u)$ is the whole disk $\mathbb{D}$.

Let us denote by

$$\beta_\varrho(x) := i\bar{\partial} d^2_{\Delta(G)}(u(x), u(x_0)). \tag{20}$$

Since $d^2_{\Delta(G)}(u(x), u(x_0))$ is a psh function on $\widetilde{X}$ by Proposition 3.3, by [GZ17, Theorem 1.46], β_ϱ has L^1_{loc}-coefficients.

Lemma 3.6. *There exists a dense open subset $\widetilde{X}^\circ$ of $\widetilde{X}$ whose complement has zero Lebesgue measure such that β_ϱ is smooth. Moreover, there exists a number $c > 0$ such that for any $x \in \widetilde{X}^\circ$, we have*

$$|\beta_\varrho(x)|_{\pi_X^*\omega} \leq c(1 + d_{\widetilde{X}}(x, x_0)).$$

Here ω is a Kähler metric on X.

Proof. Let $\widetilde{X}^\circ$ be the open subset defined in the proof of Proposition 3.3. Then $\widetilde{X}\backslash\widetilde{X}^\circ$ has zero Lebesgue measure. For any $x \in \widetilde{X}^\circ$, it has an open neighborhood Ω_x and an apartment A such that $u(\Omega_x) \subset A$ and $u(x_0) \in A$. Then by (19) and (20), one has

$$\beta_\varrho(y) = \sqrt{-1}\sum_{i=1}^N (u_{A,i}(y) - u_{A,i}(x_0))\bar{\partial} u_{A,i}(y).$$

Since u is Lipschitz and ϱ-equivariant, it follows that there exists a uniform constant $c_1 > 0$ such that for any $y \in \widetilde{X}$, we have

$$d_{\Delta(G)}(u(y), u(x_0)) \leq c_1(1 + d_{\widetilde{X}}(y, x_0)).$$

Note that for any $y \in \Omega_x$, we have

$$|u_{A,i}(y) - u_{A,i}(x_0)| \leq d_{\Delta(G)}(u(y), u(x_0)).$$

On the other hand, by the Lipschitz condition of u, there exists another uniform constant $c_2 > 0$ such that for any $y \in \Omega_x$, we have

$$\sum_{i=1}^N |\bar{\partial} u_{A,i}(y)|_{\pi^*\omega} \leq c_2.$$

In conclusion, we have

$$|\beta_\varrho(y)|_{\pi_X^*\omega} \leq c_1 c_2(1 + d_{\widetilde{X}}(y, x_0)) \quad \text{for any } y \in \Omega_x.$$

Since x is any point in $\widetilde{X}^\circ$, the above inequality holds for any $x \in \widetilde{X}^\circ$. The lemma is proved. □

4. L^2-VANISHING THEOREM AND GENERICALLY LARGE LOCAL SYSTEMS

In this section we will prove Theorem A. In Section 4.1, we address the case where ϱ is semisimple, utilizing techniques from the proof of the reductive Shafarevich conjecture in [DYK23]. The desired 1-form β, as required in Theorem 2.4, arises from 1-forms β_τ associated with $\tau : \pi_1(X) \to \mathrm{GL}_N(K)$ defined in (20), where K is a non-archimedean local field. We then reduce the proof to Theorem 2.4.(i).

For the proof of the general cases of Theorem A, we will apply techniques from the proof of the linear Shafarevich conjecture in [EKPR12]. Using similar techniques as in the semi-simple case, we first construct a suitable fibration $f : X \to Y$ (the reductive Shafarevich morphism

$\mathrm{sh}_M^0 : X \to \mathrm{Sh}_M^0(X)$) such that the conditions in Theorem 2.4 are fulfilled. Moreover, by the structure of the linear Shafarevich morphism, for *almost every* smooth fiber of f, the conditions in Corollary 2.7 are satisfied.

Therefore, if there exists a non-trivial $\alpha \in H^{p,0}_{(2)}(\widetilde{X})$ for some $p \in \{0, \dots, \dim X - 1\}$, we can apply Theorem 2.4.(i) to show that $p \geq m := \dim Y$, and use Theorem 2.4.(iii) to obtain a non-trivial L^2 holomorphic $(p-m)$-form on the universal cover of almost every smooth fiber of f. Finally, we apply Corollary 2.7 to obtain a contradiction.

Section 4.1 is covered by Section 4.2, and readers can skip it if they prefer to proceed directly to the proof of the general case of Theorem A.

4.1. **Case of semi-simple local systems.**

Theorem 4.1. *Let X be a smooth projective variety. Let $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ be a semisimple representation. If ϱ is generically large, then $H^{p,0}_{(2)}(\widetilde{X}) = 0$ for $0 \leq p \leq n-1$.*

We will use techniques in proving the reductive Shafarevich conjecture in [DYK23, Eys04]. We summarize the main results needed in proving Theorem 4.1 as follows.

Theorem 4.2 ([DYK23, Proof of Theorem 4.31])**.** *Let X be a smooth projective variety. Let $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ be a semisimple representation. If ϱ is generically large, then after replacing X by a finite étale cover, there exist*

(1) a family of Zariski dense representations $\{\tau_i : \pi_1(X) \to G_i(K_i)\}_{i=1,\dots,\ell}$ where each G_i is a reductive group over a non-archimedean local field K_i of characteristic zero,

(2) a $\mathbb{C}$-VHS $\mathcal{L}$ on X,

such that there exists a birational morphism $\mu : X \to Y$ onto a normal projective variety Y such that

$$\{T_{\tau_1} + \cdots + T_{\tau_\ell} + \sqrt{-1}\mathrm{tr}(\theta \wedge \theta^*)\} = \{\mu^*\omega_Y\} \in H^{1,1}(X, \mathbb{R}),$$

where ω_Y is a Kähler form on Y. Here

- *T_{τ_i} is the canonical current on X associated with τ_i defined in Definition 3.1.*
- *θ is the Higgs field of the Hodge bundle relative to $\mathcal{L}$ and θ^* is the adjoint of θ with respect to the Hodge metric.* □

Note that in (20), we construct certain 1-forms β_{τ_i} associated with the non-archimedean representations τ_i in Theorem 4.2. For the $\mathbb{C}$-VHS $\mathcal{L}$, there is also another way to construct similar 1-forms. This was established by Eyssidieux in [Eys97] and we recollect some facts therein.

Proposition 4.3 ([Eys97, Proposition 4.5.1])**.** *Let X be a smooth projective variety and let $\mathcal{L}$ be a $\mathbb{C}$-VHS on X. Let $\mathscr{D}$ be the period domain of $\mathcal{L}$ and let $f : \widetilde{X} \to \mathscr{D}$ be the period map. Let $q : \mathscr{D} \to \mathscr{R}$ be the natural quotient where $\mathscr{R}$ is the corresponding Riemannian symmetric space of $\mathscr{D}$. Define $\omega_{\mathcal{L}} := \sqrt{-1}\mathrm{tr}(\theta \wedge \theta^*)$ as in Theorem 4.2, which is a smooth closed positive $(1,1)$-form. Then there exists a smooth function $\psi_{\mathscr{D}} : \mathscr{R} \to \mathbb{R}_{>0}$ and constants $C, c > 0$ depending only on $\mathscr{D}$ such that the function $\phi := \psi_{\mathscr{D}} \circ q \circ f$ is smooth and plurisubharmonic and we have*

$$\sqrt{-1}\partial\phi \wedge \bar{\partial}\phi \leq \pi_X^*\omega_{\mathcal{L}}, \tag{21}$$

$$c\pi_X^*\omega_{\mathcal{L}} \leq \sqrt{-1}\partial\bar{\partial}\phi \leq C\pi_X^*\omega_{\mathcal{L}}. \tag{22}$$

□

Proof of Theorem 4.1. After replacing X by a finite étale cover, by Theorem 4.2 there exist Zariski dense representations $\{\tau_i : \pi_1(X) \to G_i(K_i)\}_{i=1,\ldots,\ell}$ where each G_i is a reductive algebraic group over a non-archimedean field K_i, along with a $\mathbb{C}$-VHS $\mathcal{L}$ satisfying the stated properties. Let $\mu : X \to Y$ be the birational morphism in Theorem 4.2. Then

$$\{T_{\tau_1} + \cdots + T_{\tau_\ell} + \omega_{\mathcal{L}}\} = \{\mu^*\omega_Y\}$$

where ω_Y is a Kähler form on Y. Here $\omega_{\mathcal{L}} := \sqrt{-1}\mathrm{tr}(\theta \wedge \theta^*)$ is a smooth $(1,1)$-form defined in Theorem 4.2. We choose a Kähler form ω_X on X such that $\mu^*\omega_Y \leq \omega_X$. We also denote by ω_X its pullback on the universal cover $\widetilde{X}$ abusively.

Since each T_{τ_i} has continuous local potential, there exist a continuous quasi-psh function ψ on X, such that

$$T_{\tau_1} + \cdots + T_{\tau_\ell} + \omega_{\mathcal{L}} = \mu^*\omega_Y - \sqrt{-1}\partial\bar{\partial}\psi. \tag{23}$$

Note that ψ is continuous and smooth outside a proper Zariski closed subset.

In what follows, for any form η we shall denote by $|\eta|$ its norm with respect to ω_X. Let $u_i : \widetilde{X} \to \Delta(G_i)$ be the τ_i-equivariant pluriharmonic map whose existence is ensured by [GS92], where $\Delta(G_i)$ is the Bruhat-Tits building of G_i. By (20), if we define

$$\beta_i(x) = \sqrt{-1}\bar{\partial}d^2_{\Delta(G_i)}(u_i(x), u_i(x_0)), \tag{24}$$

then by Lemma 3.6, β_i has L^1_{loc}-coefficients and is smooth outside a set of zero Lebesgue measure. Moreover, there exists a constant $c_1 > 0$ with $|\beta_i(x)| \leq_{\text{a.e.}} c_1(d_{\widetilde{X}}(x, x_0) + 1)$ for any i.

Let $f : \widetilde{X} \to \mathscr{D}$ be the period map of the $\mathbb{C}$-VHS $\mathcal{L}$ and let $\phi = \psi_{\mathscr{D}} \circ q \circ f$ be the smooth plurisubharmonic function defined in Proposition 4.3. By (21) and (22), we have

$$\sqrt{-1}\partial\phi \wedge \bar{\partial}\phi \leq \pi_X^*\omega_{\mathcal{L}} \quad \text{and} \quad c_2\pi_X^*\omega_{\mathcal{L}} \leq \sqrt{-1}\partial\bar{\partial}\phi \leq c_3\pi_X^*\omega_{\mathcal{L}} \tag{25}$$

for some constant $0 < c_2 < c_3$. The first inequality implies that there exists a constant $c_4 > 0$ such that

$$|(\bar{\partial}\phi)(x)| \leq c_4 \tag{26}$$

for any $x \in \widetilde{X}$. Define $\beta := \frac{i\bar{\partial}\phi}{c_2} + \sum_{i=1}^{\ell}\beta_i$. Then β has L^1_{loc}-coefficients. By Lemma 3.6 and (26), it satisfies

$$|\beta(x)| \leq_{\text{a.e.}} c_5(1 + d_{\widetilde{X}}(x, x_0)) \tag{27}$$

for some constant $c_5 > 0$. By Proposition 3.3 and (25), we have

$$\pi_X^*\mu^*\omega_Y = \pi_X^*(\sum_{i=1}^{\ell} T_{\tau_i} + \omega_{\mathcal{L}} + \sqrt{-1}\partial\bar{\partial}\psi) \leq d\beta + \sqrt{-1}\partial\bar{\partial}\pi_X^*\psi,$$

and over a dense open subset $\widetilde{X}^\circ$ of $\widetilde{X}$ whose complement has zero Lebesgue measure, we have

$$d\beta + \sqrt{-1}\partial\bar{\partial}\pi_X^*\psi = \pi_X^*(\sum_{i=1}^{\ell} T_{\tau_i}) + \frac{1}{c_2}\sqrt{-1}\partial\bar{\partial}\phi + \pi_X^*\sqrt{-1}\partial\bar{\partial}\psi \leq \pi_X^*\mu^*\omega_Y + \frac{c_3 - c_2}{c_2}\pi_X^*\omega_{\mathcal{L}}.$$

Since μ is birational, $\omega_{\rm sk} := \mu^*\omega_Y$ is a semi-Kähler form on X. It then follows that

$$\pi_X^*\omega_{\rm sk} \leq (d\beta + \sqrt{-1}\partial\bar{\partial}\pi_X^*\psi)_{\rm ac} \leq \pi_X^*\left(\omega_{\rm sk} + \frac{c_3 - c_2}{c_2}\omega_{\mathcal{L}}\right) \leq c_6\pi_X^*\omega_X \tag{28}$$

for some constant $c_6 > 0$. Therefore, for the 1-form β on $\widetilde{X}$ and the function ψ on X, they satisfy the conditions in Theorem 2.4. We conclude the desired L^2-vanishing theorem. □

Remark 4.4. If we compare the proof of Theorem 4.1 with that of the reductive Shafarevich conjecture in [DYK23, Eys04], we observe striking similarities in their approaches. Indeed, in [Eys04, Proposition 4.1.1], Eyssidieux proved the following result: let X be a compact Kähler normal variety. If there exist a continuous plurisubharmonic function $\phi : \widetilde{X} \to \mathbb{R}_{>0}$ and a positive closed $(1,1)$-current T on X with continuous potential such that $\{T\}$ is a Kähler class and $\sqrt{-1}\partial\bar{\partial}\phi \geq \pi_X^*T$, then $\widetilde{X}$ is Stein. Therefore, if X is a smooth projective variety endowed with a semisimple and large representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$, then we can prove that $\widetilde{X}$ is Stein using Theorem 4.2 as follows.

Consider the continuous function

$$\phi_0 := \sum_{i=1}^{\ell} d^2_{\Delta(G_i)}(u_i(x), u_i(x_0)) + \frac{\phi}{c_2}$$

on $\widetilde{X}$, where u_i and ϕ are defined in the proof of Theorem 4.2. We then have

$$\sqrt{-1}\partial\bar{\partial}\phi_0 \geq \pi_X^*(\sum_{i=1}^{\ell} T_\ell + \omega_{\mathcal{L}}).$$

Note that $\sum_{i=1}^{\ell}\{T_\ell + \omega_{\mathcal{L}}\}$ is a Kähler class in X if ϱ is large by Theorem 4.2. Therefore, by the above Eyssidieux's criterion, we conclude that $\widetilde{X}$ is Stein.

4.2. **Proof of Theorem A.** In this subsection we will prove Theorem A.

Theorem 4.5. *Let X be a smooth projective variety of dimension n. Let $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ be a linear representation. If ϱ is generically large, then $H^{p,0}_{(2)}(\widetilde{X}) = 0$ for $0 \leq p \leq n-1$.*

Proof. **Step 1.** Let $M := M_{\rm B}(\pi_1(X), \mathrm{GL}_N)(\mathbb{C})$ be the character variety of $\pi_1(X)$. By [DYK23, Proof of Theorem 3.29], there exists

- a family of Zariski dense representations $\{\tau_i : \pi_1(X) \to G(K_i)\}_{i=1,\ldots,\ell}$ where each G_i is a reductive group over a non-archimedean local field K_i of characteristic zero;
- a $\mathbb{C}$-VHS $\mathcal{L}$ with the period domain $\mathscr{D}_1$;

such that the following properties hold. We define $\widetilde{H^0_M}$ to be the intersection of the kernels of all semisimple representations $\pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$. Denote by $\widetilde{X^0_M} := \widetilde{X}/\widetilde{H^0_M}$ and $\pi_0 : \widetilde{X^0_M} \to X$ the projection map. Then the period map of $\mathcal{L}$ descends to $\phi : \widetilde{X^0_M} \to \mathscr{D}_1$. For the holomorphic map

$$\begin{aligned}\Phi_0 : \widetilde{X^0_M} &\to \prod_{i=1}^{\ell} S_{\tau_i} \times \mathscr{D}_1\\ x &\mapsto (s_{\tau_1}\circ\pi_0, \ldots, s_{\tau_\ell}\circ\pi_0, \phi(x)),\end{aligned}$$

each connected component of the fiber of Φ_0 is compact. Here $s_{\tau_i}: X \to S_{\tau_i}$ is the Katzarkov-Eyssidieux reduction for τ_i defined in Proposition 3.2. By [DYK23, Proof of Theorem 3.29], Φ_0 factors through

$$\widetilde{X_M^0} \xrightarrow{r_M^0} \widetilde{S_M^0}(X) \xrightarrow{r_0} \prod_{i=1}^{\ell} S_{\tau_i} \times \mathscr{D}_1$$

where r_M^0 is a proper surjective holomorphic fibration and r_0 is holomorphic map with each fiber being a discrete set. Moreover, $\widetilde{S_M^0}(X)$ does not contain compact subvarieties. Therefore, the Galois group $\mathrm{Aut}(\widetilde{X_M^0}/X)$ induces an action on $\widetilde{S_M^0}(X)$ which is properly discontinuous, and such that r_M^0 is equivariant with respect to the action by $\mathrm{Aut}(\widetilde{X_M^0}/X)$. By [DYK23, Lemma 3.28], replacing X by a finite étale cover, we can assume that such an action on $\widetilde{S_M^0}(X)$ is free. Taking the quotient of r_M^0 by $\mathrm{Aut}(\widetilde{X_M^0}/X)$, we obtain

$$\begin{tikzcd}
\widetilde{X_M^0} \arrow[r, "\pi_0"] \arrow[d, "r_M^0"] & X \arrow[d, "\mathrm{sh}_M^0"] \\
\widetilde{S_M^0}(X) \arrow[r, "q_0"] \arrow[d, "r_0"] & \mathrm{Sh}_M^0(X) \\
\prod_{i=1}^{\ell} S_{\tau_i} \times \mathscr{D}_1 &
\end{tikzcd}$$

Here $\mathrm{Sh}_M^0(X)$ is called the *reductive Shafarevich morphism* associated with M.

We shall use [EKPR12] to deal with the linear representation case. According to [EKPR12, §5.2], there is a $\mathbb{R}$-VMHS $\mathcal{M}$ of weight length 1 with the mixed period domain $\mathscr{M}$ (cf. [EKPR12, Lemma 5.4]) and an infinite Galois étale cover $\pi_1 : \widetilde{X_M^1} \to X$ (cf. [EKPR12, p. 1575] for the definition) factorizing through $\pi_0 : \widetilde{X_M^0} \to X$ such that

(a) the mixed period domain descends to $\varpi : \widetilde{X_M^1} \to \mathscr{M}$;
(b) for the holomorphic map

$$\begin{aligned}\Phi_1 : \widetilde{X_M^1} &\to \prod_{i=1}^{\ell} S_{\tau_i} \times \mathscr{D}_1 \times \mathscr{M} \\ x &\mapsto (s_{\tau_1} \circ \pi_1, \ldots, s_{\tau_\ell} \circ \pi_1, \phi(x), \varpi(x)),\end{aligned}$$

each connected component of the fiber of Φ is compact.

Here we abusively use $\phi : \widetilde{X_M^1} \to \mathscr{D}_1$ to denote by the composite of $\phi : \widetilde{X_M^0} \to \mathscr{D}_1$ with $\widetilde{X_M^1} \to \widetilde{X_M^0}$. By [EKPR12, p. 1576], Φ_1 factors through

$$\widetilde{X_M^1} \xrightarrow{r_M^1} \widetilde{S_M^1}(X) \xrightarrow{r_1} \prod_{i=1}^{\ell} S_{\tau_i} \times \mathscr{D}_1 \times \mathscr{M}$$

where r^1_M is a proper surjective holomorphic fibration and r_1 is holomorphic map with each fiber being a discrete set.

By [EKPR12, Lemma 5.7], $\widetilde{S^1_M}(X)$ does not contain compact subvarieties. Therefore, the Galois group $\mathrm{Aut}(\widetilde{X^1_M}/X)$ induces an action on $\widetilde{S^1_M}(X)$ which is properly discontinuous, and such that r^1_M is equivariant with respect to the action by $\mathrm{Aut}(\widetilde{X^1_M}/X)$. Replacing X by a finite étale cover, we assume that such an action is free. Taking the quotient of r^1_M by $\mathrm{Aut}(\widetilde{X^1_M}/X)$, we obtain:

$$\begin{tikzcd}
\widetilde{X^1_M} \arrow[r, "\pi_1"] \arrow[d, "r^1_M"] \arrow[dd, bend right=60, "\Phi_1"'] & X \arrow[r, equal] \arrow[d, "\mathrm{sh}^1_M"] & X \arrow[d, "\mathrm{sh}^0_M"] & \widetilde{X^0_M} \arrow[l, "\pi_0"'] \arrow[d, "r^0_M"] \arrow[dd, bend left=60, "\Phi_0"] \\
\widetilde{S^1_M}(X) \arrow[r, "q_1", "\text{étale}"'] \arrow[d, "r_1"] & \mathrm{Sh}^1_M(X) \arrow[r, "g"] & \mathrm{Sh}^0_M(X) & \widetilde{S^0_M}(X) \arrow[l, "q_0"', "\text{étale}"] \arrow[d, "r_0"] \\
\prod_{i=1}^{\ell} S_{\tau_i} \times \mathscr{D}_1 \times \mathscr{M} & & & \prod_{i=1}^{\ell} S_{\tau_i} \times \mathscr{D}_1
\end{tikzcd} \tag{29}$$

By [EKPR12, p. 1549], if ϱ is generically large, then sh^1_M is a bimeromorphic map.

Step 2. By [DYK23, Proof of Theorem 4.31] (which is exactly Theorem 4.2), there exists a Kähler form ω_T on the normal projective variety $\mathrm{Sh}^0_M(X)$ such that

$$\{T_{\tau_1} + \cdots + T_{\tau_\ell} + \sqrt{-1}\mathrm{tr}(\theta \wedge \theta^*)\} \in H^{1,1}(X, \mathbb{R}) = \{(\mathrm{sh}^0_M)^* \omega_T\}. \tag{30}$$

Here T_{τ_i} is the canonical current for τ_i defined in Definition 3.1 and $\sqrt{-1}\mathrm{tr}(\theta \wedge \theta^*)$ is the semi-positive $(1,1)$-form over X defined in Theorem 4.2. Since each T_{τ_i} has continuous local potential, there exist a continuous function ψ on X, such that

$$T_{\tau_1} + \cdots + T_{\tau_\ell} + \omega_{\mathcal{L}} = (\mathrm{sh}^0_M)^* \omega_T - \sqrt{-1}\partial\bar{\partial}\psi.$$

Let $\mathscr{D}_2$ be the graded period domain of $\mathscr{M}$. Note that $\mathscr{M} \to \mathscr{D}_2$ is a holomorphic vector bundle (cf. [Car87]). Let Z be any fiber of $X \to \mathrm{Sh}^0_M(X)$. Let $\widetilde{Z^1_M}$ be any connected component of the inverse image $\pi_1^{-1}(Z)$. Then there exists some $P \in \mathscr{D}_2$ such that for the fiber V of $\mathscr{M} \to \mathscr{D}_2$ at $P \in \mathscr{D}_2$, we have $\varpi|_{\widetilde{Z^1_M}} : \widetilde{Z^1_M} \to V$. Moreover, by [EKPR12, p. 1575-1576], after we replace Z by a finite étale cover, there exists a map $a : Z^{\mathrm{sn}} \to A$, where A is an abelian variety and Z^{sn} is the semi-normalization of Z such that, $q := \varpi|_{\widetilde{Z^1_M}}$ factors as

$$\begin{tikzcd}
\widetilde{Z^1_M}^{\mathrm{sn}} \arrow[r, "p"] \arrow[d] \arrow[dr] & \widetilde{A} \arrow[d] \\
\widetilde{Z^1_M} \arrow[r, "q"] & V
\end{tikzcd}$$

where $\widetilde{Z^1_M}^{\mathrm{sn}}$ is the semi-normalization of $\widetilde{Z^1_M}$, $\widetilde{A}$ is the universal cover of A, p is the lift of a which is proper, and $\widetilde{A} \to V$ is a linear injective map. Since r_1 has discrete fibers, sh^1_M is proper bimeromorphic, and the image of $\widetilde{Z_M^1}$ under the composite map

$$\widetilde{S_M^1}(X) \xrightarrow{r_1} \prod_{i=1}^{\ell} S_{\tau_i} \times \mathscr{D}_1 \times \mathscr{M} \to \prod_{i=1}^{\ell} S_{\tau_i} \times \mathscr{D}_1$$

is constant, it follows that $\dim p(\widetilde{Z_M^1}^{\mathrm{sn}}) = \dim \widetilde{Z_M^1}^{\mathrm{sn}}$ if Z is a general fiber of $X \to \mathrm{Sh}_M^0(X)$. Here $\prod_{i=1}^{\ell} S_{\tau_i} \times \mathscr{D}_1 \times \mathscr{M} \to \prod_{i=1}^{\ell} S_{\tau_i} \times \mathscr{D}_1$ is the natural projection map. This implies the following result.

Claim 4.6. *For a general fiber Z of $X \to \mathrm{Sh}_M^0(X)$, there is a map $a : Z^{\mathrm{sn}} \to A$ to an abelian variety A such that*

$$\dim Z^{\mathrm{sn}} = \dim a(Z^{\mathrm{sn}}). \tag{31}$$

Step 3. For each i, let β_i be 1-form on $\widetilde{X}$ defined in (24). By Proposition 3.3, it is generically smooth and satisfies that $d\beta_i \geq \pi_X^* T_{\tau_i}$, with equality holding outside a closed subset of zero Lebesgue measure. We fix a Kähler metric ω_X on X and, by slight abuse of notation, also denote its lift on the universal cover $\widetilde{X}$ by ω_X. By Lemma 3.6, there exists a number $c_0 > 0$ and a dense open set $\widetilde{X}^\circ \subset \widetilde{X}$ whose complement has zero Lebesgue measure such that for any $i \in \{1, \ldots, \ell\}$ and $x \in \widetilde{X}^\circ$, we have

$$|\beta_i(x)|_{\omega_X} \leq c_0(1 + d_{\widetilde{X}}(x, x_0)). \tag{32}$$

Consider the period map $p : \widetilde{X} \to \mathscr{D}_1$ of $\mathcal{L}$ and let $\phi = \psi_{\mathscr{D}_1} \circ q \circ p$ be the positive smooth plurisubharmonic function defined in Proposition 4.3. By eq. (21) and (22), we have

$$i\partial\phi \wedge \bar{\partial}\phi \leq \pi_X^* \omega_{\mathcal{L}},$$
$$c_1 \pi_X^* \omega_{\mathcal{L}} \leq \sqrt{-1}\partial\bar{\partial}\phi \leq c_2 \pi_X^* \omega_{\mathcal{L}}$$

for some constant $0 < c_1 < c_2$. The first inequality implies that there exists a constant $c_3 > 0$ such that

$$|(\bar{\partial}\phi)(x)|_{\omega_X} \leq c_3 \tag{33}$$

for any $x \in \widetilde{X}$. Write

$$\beta := \beta_1 + \cdots + \beta_\ell + \frac{i}{c_1}\bar{\partial}\phi.$$

Then β has L^1_{loc}-coefficients and is smooth outside a set of zero Lebesgue measure. By (32), we have

$$|\beta(x)|_{\omega_X} \leq_{\mathrm{a.e.}} c_4(1 + d_{\widetilde{X}}(x, x_0)) \tag{34}$$

for some constant $c_4 > 0$.

By (30), one has

$$\pi_X^*(\mathrm{sh}_M^0)^*\omega_T \leq (d\beta + \psi))_{\mathrm{ac}} \leq \pi_X^*\left((\mathrm{sh}_M^0)^*\omega_T + \frac{c_2 - c_1}{c_1}\omega_{\mathcal{L}}\right) \leq c_6 \pi_X^* \omega_X \tag{35}$$

for some constant $c_6 > 0$. We write $f : X \to Y$ for $\mathrm{sh}_M^0 : X \to Y$. Therefore, for the 1-form β and the function ψ, they satisfy the conditions in Theorem 2.4.

Step 4. We will apply Theorem 2.4 and use its notations as defined therein, without re-explaining their meanings. Assume by contradiction that, for some $p \in \{0, \ldots, n-1\}$, there is a non-trivial $\alpha \in H^{p,0}_{(2)}(\widetilde{X})$. By Theorem 2.4, we have $n > m$ and $p > m$. Furthermore, over the Zariski open subset Y° of Y^{reg}, we have

$$\alpha|_{\widetilde{X}^\circ} \in H^0(\widetilde{X}^\circ, \tilde{f}^*\Omega^m_{\widetilde{Y}^\circ} \otimes \Omega^{p-m}_{\widetilde{X}^\circ}),$$

where $\widetilde{Y}^\circ := \pi_Y^{-1}(Y^\circ)$ and $\widetilde{X}^\circ := \tilde{f}^{-1}(\widetilde{Y}^\circ)$. We pick any $y_0 \in Y^\circ$, and choose a simply connected coordinate open subset $(V; w_1 \ldots, w_m)$ centered at y_0. We abusively denote by V a connected component of $\pi_Y^{-1}(V)$. Then $dw_1 \wedge \cdots \wedge dw_m$ is a nonwhere-vanishing holomorphic m-form on V. Let Ω be a connected component of $\tilde{f}^{-1}(V)$. We denote by $g : \Omega \to V$ the restriction of $\tilde{f}$ on Ω. Then g is a submersion with connected fibers. For any $y \in V$, by Step 3 in the proof of Theorem 2.4, $dw_1 \wedge \cdots \wedge dw_m$ induces a unique holomorphic $(p-m)$-form α_y on $g^{-1}(y)$ defined in (15) within suitable coordinate open subset $(U; z_1, \ldots, z_n)$ of Ω. Then by (14) and (15) together with the Fubini theorem, we have

$$\begin{aligned}0 &\leq \int_V \left(\int_{g^{-1}(y)} i^{(p-m)^2}\alpha_y \wedge \overline{\alpha_y} \wedge (\omega_X|_{g^{-1}(y)})^{n-p}\right) idw_1 \wedge d\bar{w}_1 \wedge \cdots \wedge idw_m \wedge d\bar{w}_m \\ &= \int_\Omega i^{p^2}\alpha \wedge \bar{\alpha} \wedge \omega_X^{n-p} \leq \int_{\widetilde{X}} |\alpha|^2 \,\mathrm{dvol} < +\infty.\end{aligned}$$

Therefore, there is a subset Z of V with zero Lebesgue measure, such that for any $y \in V \backslash Z$, we have

$$\int_{g^{-1}(y)} i^{(p-m)^2}\alpha_y \wedge \overline{\alpha_y} \wedge (\omega_X|_{g^{-1}(y)})^{n-p} < \infty. \tag{36}$$

Thus, we construct an L^2 holomorphic $(p-m)$-form α_y on $g^{-1}(y)$ for any $y \in V \backslash Z$, which is also d-closed by Theorem 2.4.(iii). We note that for any $x \in \tilde{f}^{-1}(y)$, $\alpha_y(x) = 0$ if and only if $\alpha(x) = 0$.

Denote by $X_y := f^{-1}(y)$. By Claim 4.6, if we choose a general point $y \in V \backslash Z$, then there exists a morphism $a : X_y \to A$ to an abelian variety A such that $\dim X_y = \dim a(X_y)$. Moreover, for a connected component $\widetilde{X_y'}$ of $\pi_1^{-1}(X_y)$, there is a lift $\widetilde{X_y'} \to \widetilde{A}$ of a. Note that $g^{-1}(y)$ is a connected component of $\pi_X^{-1}(X_y)$. Then $g^{-1}(y)$ dominates $\widetilde{X_y}'$. The conditions in Corollary 2.7 are fulfilled. It follows that $\alpha_y = 0$. Hence, $\alpha(x) = 0$ almost everywhere. By continuity, we conclude that $\alpha(x) = 0$ everywhere. This yields a contradiction. Therefore, $H^{p,0}_{(2)}(\widetilde{X}) = 0$ for any $p \in \{0, \ldots, n-1\}$. The theorem is proved. □

We will apply Theorem 4.5 to prove Theorem A.

Proof of Theorem A. Let $\mathcal{H}^{n,q}_{(2)}(\widetilde{X})$ be the L^2-harmonic (n,q)-forms with respect to the metric ω_X. By the Lefschetz theorem in [Gro91, Theorem 1.2.A]), for any $q \in \{1, \ldots, n\}$,

$$\begin{aligned}\mathcal{H}^{n-q,0}_{(2)}(\widetilde{X}) &\to \mathcal{H}^{n,q}_{(2)}(\widetilde{X}) \\ \alpha &\mapsto \omega^q \wedge \alpha\end{aligned}$$

is an isomorphism. Since we have an isomorphism $\mathcal{H}^{n,q}_{(2)}(\widetilde{X}) \simeq H^{n,q}_{(2)}(\widetilde{X})$, this establishes that $H^{n,q}_{(2)}(\widetilde{X})$ for $q \in \{1, \ldots, n\}$.

We denote by $\Gamma = \pi_1(X)$ and $\dim_\Gamma H^{n,q}_{(2)}(\widetilde{X})$ the Von Neumann dimension of $H^{n,q}_{(2)}(\widetilde{X})$ (cf. [Ati76] for the definition). By Atiyah's L^2-index theorem along with Theorem A.(i), we have

$$\chi(X, K_X) = \sum_{q=0}^{n} (-1)^q \dim_\Gamma H^{n,q}_{(2)}(\widetilde{X}) = \dim_\Gamma H^{n,0}_{(2)}(\widetilde{X}) \geq 0. \tag{37}$$

Theorem A.(ii) is proved.

If the strict inequality (37) holds, then $H^{n,0}_{(2)}(\widetilde{X}) \neq 0$. We then apply [Kol95, Corollary 13.10] to conclude that K_X is big. Theorem A.(iii) follows. The theorem is proved. □

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