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Hyperbolicity, the Shafarevich conjecture and fundamental groups of complex quasi-projective varieties

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Dedicated to the memory of Jean-Pierre Demailly (1957-2022)

HYPERBOLICITY, THE SHAFAREVICH CONJECTURE AND FUNDAMENTAL GROUPS OF COMPLEX QUASI-PROJECTIVE VARIETIES

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CONTENTS

INTRODUCTION

Fundamental groups of algebraic varieties have been extensively studied. In this memoir, our main focus lies on the following problem: how do representations of the fundamental group of a complex quasi-projective variety into general linear groups over fields of arbitrary characteristic influence the algebro-geometric properties of the variety? This topic leads to a fascinating interplay between non-abelian Hodge theories, harmonic maps, hyperbolicity, Nevanlinna theory, and the Shafarevich conjecture. We will organize the main results in this memoir from various perspectives, exploring their implications and connections.

In Chapter 1, we develop some tools in non-abelian Hodge theories in the non-archimedean setting using the techniques of harmonic mappings of infinite energy into non-positively-curved spaces and Bruhat-Tits buildings. Non-abelian Hodge theories are robust tools in studying the fundamental groups of algebraic varieties. Consider a complex quasi-projective manifold X and a reductive representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$. When X is compact and $K = \mathbb{C}$, in the 90s Simpson [Sim88, Sim92] systematically established the theory of Higgs bundles and discovered its relation with variations of Hodge structure. This theory was later extended to the cases where X is non-compact by Mochizuki in a series of difficult works [Moc06, Moc07a, Moc07b]. On the other hand, when X is compact and K is a non-archimedean local field, Gromov-Schoen [GS92] constructed harmonic mappings from the universal covering of X to the Euclidean buildings, with significant applications ranging from p-adic super-rigidity [GS92], Simpson's motivic conjecture [CS08], Shafarevich conjecture [Eys04, EKPR12] and beyond.

The first result presented in Chapter 1 extends Gromov-Schoen's theory to non-compact cases, involving the treatment of infinite energy harmonic mappings into Euclidean buildings. As an application, we construct logarithmic symmetric differentials in the presence of unbounded linear representations of $\pi_1(X)$ in non-archimedean local fields. Moreover, we prove that a complex quasi-projective manifold X admits logarithmic symmetric differentials provided there is a linear representation $\pi_1(X) \to \mathrm{GL}_N(\mathbb{K})$ with infinite image, where $\mathbb{K}$ can be any field. This extends a previous work by Klingler, Brunebarbe, and Totaro to the quasi-projective setting. Another significant application is a reduction theorem for unbounded linear representations of $\pi_1(X)$ in non-archimedean local fields. The main results in this chapter lay the groundwork for subsequent works.

Chapter 2 is on the hyperbolicity of complex quasi-projective normal varieties in the presence of a local system with *big* monodromy. The motivation stems from the question: how does fundamental groups of algebraic varieties determine its hyperbolicity property. We are mainly interested in three notions of hyperbolicity from different aspects. In the algebraic setting, a quasi-projective variety X is *strongly of log general type* if there exists a proper Zariski closed subset Ξ such that all subvarieties not being of log general type is contained in Ξ. In the analytic setting, X is *pseudo-Picard hyperbolic* (resp. *pseudo-Brody hyperbolic*) if there exists a a proper Zariski closed subset Ξ of X such that any holomorphic map $f : \mathbb{D}^* \to X$ from the punctured disk with essential singularity at the origin (resp. any non-constant holomorphic map $f : \mathbb{C} \to X$) has image contained in Ξ.

The first result in Chapter 2 addresses the initial question. Given a quasi-projective normal variety, if there exists a big representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ such that the Zariski closure of $\varrho(\pi_1(X))$ is a semisimple algebraic group, where K can be any field, then X is both strongly of log general type and pseudo Picard hyperbolic (hence also pseudo Brody hyperbolic).

The second result in Chapter 2 is a confirmation of the *generalized Green-Griffiths-Lang conjecture* in the presence of a big local system: assuming the existence of a big representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$, which is further required to be reductive in cases where $\mathrm{char}\, K = 0$, we prove that X is of log general type if and only if it is strongly of log general type, or pseudo Picard or Brody hyperbolic. This result can be refined further by identifying the non-hyperbolicity loci (special loci) from different perspectives.

In Chapters 3 and 4, we address the Shafarevich conjecture, which stipulates that the universal covering of a complex projective variety is holomorphically convex. This conjecture was studied extensively in 90s building on the tools from non-abelian Hodge theories described in Chapter 1. The most significant breakthrough on this conjecture came from Eyssidieux [Eys04], where he proved the Shafarevich conjecture for *smooth* projective varieties whose fundamental groups admit a faithful reductive representation into a complex general linear group.

Subsequently, in [EKPR12] it was further asked whether Eyssidieux's theorem can be extended to singular projective varieties.

The first result of Chapter 3 addresses this question: the universal covering of a *projective normal variety* is holomorphically convex if there exists a faithful reductive representation of its fundamental group into a complex general linear group. Building on our work on non-abelian Hodge theories in non-archimedean setting shown in Chapter 1, we also construct the *Shafarevich morphism* $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$ associated with $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$, where X is any quasi-projective normal variety and K is any field (if $\mathrm{char}\, K = 0$ we assume additionally that ϱ is reductive). It has the property that for any closed subvariety Z of X, $\mathrm{sh}_\varrho(Z)$ is a point if and only if $\varrho(\mathrm{Im}[\pi_1(Z) \to \pi_1(X)])$ is finite. Additionally, we prove that a projective normal variety whose Γ-dimension (defined by Campana) is at most two is holomorphically convex if there exists a faithful reductive representation of its fundamental group into a general linear group in positive characteristic.

In Chapter 5, we present several applications in algebraic geometry of the results established in previous chapters. The first application concerns Claudon-Höring-Kollár's conjecture, which asserts that a complex projective manifold with quasi-projective universal covering has a locally trivial Albanese map up to a finite étale cover. Their conjecture was proven under the assumption of the abundance conjecture by Claudon-Höring-Kollár, or varieties with virtually abelian fundamental groups by Claudon-Höring. Our result confirms their conjecture in cases where the fundamental groups admit a faithful linear representation into the general linear group of any characteristic.

The second application addresses Campana's tantalizing abelianity conjecture, which predicts that a special or Brody special complex projective manifold has a virtually abelian fundamental group. Here, a complex projective manifold X is *special* if it does not admit a fibration onto an orbifold of general type, with the orbifold structure on the base being given by the divisor of multiple fibers. X is *Brody special* if it has a Zariski dense entire curve. Special or Brody special varieties are known for their non-hyperbolic nature. Campana [Cam04] and Yamanoi [Yam10] proved that any complex linear quotient of the fundamental group of a special or Brody special projective manifold is virtually abelian. Our first result extends their theorems to linear quotients in positive characteristic of the fundamental groups of quasi-projective varieties. Additionally, we establish that any complex linear quotient of the fundamental group of a special or Brody special quasi-projective manifold is virtually nilpotent. We further provide illustrative examples to demonstrate the sharpness of this result. The last result is a structure theorem for algebraic varieties in the presence of a big local system, addressing a conjecture by Kollár.

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Articles presented in this memoir

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(2) Y. Deng. *Big Picard theorems and algebraic hyperbolicity for varieties admitting a variation of Hodge structures.* Épijournal de Géométrie Algébrique, Volume 7(2023).

(3) Y. Deng & K. Yamanoi. *Linear Shafarevich Conjecture in positive characteristic, hyperbolicity and applications*. arXiv:2403.16199, (2024).

(4) Y. Deng, K. Yamanoi & L. Katzarkov. *Reductive Shafarevich Conjecture*. arXiv:2306.03070, (2023).

(5) G. Daskalopoulos, Y. Deng & C. Mese. *Representations of fundamental groups and logarithmic symmetric differential forms*. arXiv:2206.11835, (2022).

Articles not presented

(6) Y. Deng & B. Wang. *Linear Singer-Hopf Conjecture*. arXiv:2405.12012, (2024).

(7) Y. Deng. *On the nilpotent orbit theorem of complex variations of Hodge structure*. Forum Math. Sigma, 11(2023): Paper No. e106, 20.

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(11) Y. Deng. *A characterization of complex quasi-projective manifolds uniformized by unit balls*. Math. Ann. (2022).

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CHAPTER 1

EXISTENCE OF HARMONIC MAPPINGS INTO BRUHAT-TITS BUILDINGS

1.1. Introduction

Let X be a complex smooth quasi-projective variety, and let G be a semisimple algebraic group defined over a field K. In this chapter, we mainly focus on representations $\varrho : \pi_1(X) \to G(K)$, where K can be either the field of complex numbers or an algebraic number field, or a non-archimedean local field. We refer to ϱ as *Zariski dense* if the Zariski closure of its image is the whole group G.

Non-abelian Hodge theories play a significant role in the study of the geometric properties of the above representations $\varrho : \pi_1(X) \to G(K)$. These theories are based on harmonic mappings to symmetric spaces in the archimedean cases (when K is complex or an algebraic number field) or to Bruhat-Tits buildings in the non-archimedean cases (when K is a non-archimedean local field). In the archimedean setting, these theories have been well-established, with notable works including [Don87, Cor88, Sim92, Moc07b].

However, in the non-archimedean case, while the theory has been established for algebraic varieties that are projective, thanks to the work of Gromov-Schoen [GS92], its full generalization to quasi-projective varieties remained a challenge over the past three decades. The main difficulty is that, we will encounter with harmonic maps with *infinite energy*.

Infinite energy harmonic maps between manifolds previously appeared in the work of Lohkamp and Wolf. Lohcamp [Loh90] proved the existence of a harmonic map in a given homotopy class of maps between two non-compact manifolds, provided that a certain simplicity condition is satisfied. The most important case is when the domain is metrically a product near infinity. Wolf [Wol91] studied harmonic maps of infinite energy when the domain is a nodal Riemann surface and applied this study to describe degenerations of surfaces in the Riemann moduli space.

In this chapter I will present the main result in [BDDM22]. Precisely, we establish the existence of equivariant (infinite energy) harmonic maps to Bruhat-Tits buildings, which are associated with representations of fundamental groups of quasi-projective varieties into semisimple algebraic groups over non-archimedean local fields. Additionally, we prove the pluriharmonicity of these harmonic maps and provide estimates for their energy growth at infinity. This result extends the Gromov-Schoen theory to quasi-projective varieties.

I then present applications of the aforementioned result concerning existence theorem of harmonic maps. The first application involves constructing logarithmic symmetric differentials on quasi-projective varieties X in the presence of a representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$, where K is a non-archimedean local field, with the condition that $\varrho(\pi_1(X))$ is unbounded. By leveraging this result, we establish the existence of logarithmic symmetric differentials for quasi-projective varieties, provided their fundamental groups possess an infinite linear quotient. Notably, this theorem extends the work of Brunebarbe, Klinger, and Totaro in [BKT13].

The second application concerns a reduction map theorem for the linear representation of fundamental groups of quasi-projective varieties into algebraic groups over non-archimedean local fields, as developed in [CDY22]. This result generalize previous work by Katzarkov [Kat97] and Eyssidieux [Eys04].

These main results in this chapter are foundational cornerstones for the subsequent chapters.

1.2. Definition of harmonic maps

1.2.1. NPC spaces and Euclidean buildings. — For the definitions in this subsection, we refer the readers to [BH99, Rou09, KP23, BDDM22].

Definition 1.2.1 **(Geodesic space). —** Let (X, d_X) be a metric space. A curve $\gamma : [0, \ell] \to X$ into X is called a geodesic if the length $d_X(\gamma(a), \gamma(b)) = b - a$ for any subinterval $[a, b] \subset [0, \ell]$. A metric space (X, d_X) is a *geodesic space* if there exists a geodesic connecting every pair of points in X.

Definition 1.2.2 **(NPC space). —** An NPC (non-positively curved) space (X, d_X) is a complete geodesic space that satisfies the following condition: for any three points $P, Q, R \in X$ and a geodesic $\gamma : [0, \ell] \to X$ with $\gamma(0) = Q$

and $\gamma(\ell) = R$

$$d_X^2(P, Q_t) \le (1-t)d^2(P,Q) + td^2(P,R) - t(1-t)d^2(Q,R)$$

where $Q_t = \gamma(t\ell)$.

A smooth Riemannian manifold with nonpositive sectional curvature is an NPC space. Among these, Bruhat-Tits buildings $\Delta(G)$ associated with semisimple algebraic groups G defined over non-archimedean local fields K are noteworthy examples of NPC spaces. We will not provide the lengthy definition of Bruhat-Tits buildings here, but interested readers can find precise definitions in references such as [Rou09] and [KP23]. It's noteworthy that $G(K)$ acts isometrically on the building $\Delta(G)$ and transitively on its set of apartments. Here, $G(K)$ denotes the group of K-points of G. The dimension of $\Delta(G)$ equals the K-rank of the algebraic group G, which is the dimension of a maximal split torus in G.

1.2.2. Harmonic maps to NPC spaces. — Consider a map $f : \Omega \to Z$ from an n-dimensional Riemannian manifold (Ω, g) to an NPC space (Z, d_Z). When the target space Z is a smooth Riemannian manifold of nonpositive sectional curvature, the energy of a smooth map $f : \Omega \to Z$ is

$$E^f = \int_\Omega |df|^2 d\,\mathrm{vol}_g$$

where (Ω, g) is a Riemannian domain and $d\,\mathrm{vol}_g$ is the volume form of Ω. We say $f : \Omega \to Z$ is harmonic if it is locally energy minimizing; i.e. for any $p \in \Omega$, there exists $r > 0$ such that the restriction $u|_{B_r(p)}$ minimizes energy amongst all maps $v : B_r(p) \to \widetilde{Z}$ with the same boundary values as $u|_{B_r(p)}$.

In this paper, we mainly consider the target Z to be NPC spaces, not necessarily smooth. Let us recall the definition of harmonic maps in this context (cf. [KS93] for more details).

Let (Ω, g) be a bounded Lipschitz Riemannian domain. Let Ω_ϵ be the set of points in Ω at a distance least ϵ from $\partial\Omega$. Let $B_\epsilon(x)$ be a geodesic ball centered at x and $S_\epsilon(x) = \partial B_\epsilon(x)$. We say $f : \Omega \to Z$ is an L^2-map (or that $f \in L^2(\Omega, Z)$) if

$$\int_\Omega d^2(f, P) d\mathrm{vol}_g < \infty$$

For $f \in L^2(\Omega, Z)$, define

$$e_\epsilon : \Omega \to \mathbb{R}, \quad e_\epsilon(x) = \begin{cases} \int_{y \in S_\epsilon(x)} \frac{d^2(f(x), f(y))}{\epsilon^2} \frac{d\sigma_{x,\epsilon}}{\epsilon^{n-1}} & x \in \Omega_\epsilon \\ 0 & \text{otherwise} \end{cases}$$

where $\sigma_{x,\epsilon}$ is the induced measure on $S_\epsilon(x)$. We define a family of functionals

$$E_\epsilon^f : C_c(\Omega) \to \mathbb{R}, \quad E_\epsilon^f(\varphi) = \int_\Omega \varphi e_\epsilon d\mathrm{Vol}_g.$$

We say f has finite energy (or that $f \in W^{1,2}(\Omega, Z)$) if

$$E^f := \sup_{\varphi \in C_c(\Omega), 0 \le \varphi \le 1} \limsup_{\epsilon \to 0} E_\epsilon^f(\varphi) < \infty.$$

weakly to a measure which is absolutely continuous with respect to the Lebesgue measure. Therefore, there exists a function $e(x)$, which we call the energy density, such that $e_\epsilon(x) d\mathrm{vol}_g \to e(x) d\,\mathrm{vol}_g$. In analogy to the case of smooth targets, we write $|\nabla f|^2(x)$ in place of $e(x)$. Hence $|\nabla f|^2(x) \in L^1_{\mathrm{loc}}(\Omega)$. In particular, the (Korevaar-Schoen) energy of f in Ω is

$$E^f[\Omega] = \int_\Omega |\nabla f|^2 d\,\mathrm{vol}_g\,.$$

***Definition 1.2.3* (Harmonic maps). —** We say a continuous map $f : \Omega \to Z$ from a Lipschitz domain Ω is harmonic if it is locally energy minimizing; more precisely, at each $p \in \Omega$, there exists a neighborhood Ω_p of p such that all comparison maps which agree with u outside of this neighborhood have no less energy.

For $V \in \Gamma\Omega$ where $\Gamma\Omega$ is the set of Lipschitz vector fields on Ω, $|f_*(V)|^2$ is similarly defined. The real valued L^1 function $|f_*(V)|^2$ generalizes the norm squared on the directional derivative of f. The generalization of the pull-back metric is the continuous, symmetric, bilinear, non-negative and tensorial operator

$$\pi_f(V, W) = \Gamma\Omega \times \Gamma\Omega \to L^1(\Omega, \mathbb{R})$$

where

$$\pi_f(V, W) = \frac{1}{2}|f_*(V+W)|^2 - \frac{1}{2}|f_*(V-W)|^2.$$

We refer to [KS93] for more details.

Let $(x^1, \ldots, x^n)$ be local coordinates of (Ω, g) and $g = (g_{ij})$, $g^{-1} = (g^{ij})$ be the local metric expressions. Then energy density function of f can be written (cf. [KS93, (2.3vi)])

$$|\nabla f|^2 = g^{ij} \pi_f(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$$

Next assume (Ω, g) is a Hermitian domain and let $(z^1 = x^1 + ix^2, \ldots, z^n = x^{2n-1} + ix^{2n})$ be local complex coordinates.

***Definition 1.2.4* (Locally Lipschitz). —** A continuous map $f : \Omega \to Z$ is called *locally Lipschitz* if for any $p \in \Omega$, there exists $B_\varepsilon(p) \subset \Omega$ and a constant $C > 0$ such that $d(f(x), f(y)) \le Cd(x, y)$ for $x, y \in B_\varepsilon(p)$.

The following typical example shows the connection between classical Hodge theory and non-abelian Hodge theory.

***Remark 1.2.5*. —** The name of "non-abelian Hodge theory" generalizes aspects of Hodge theory from abelian cohomology to nonabelian cohomology. Let (X, ω) be a compact Kähler manifold and we consider the first cohomology of X with real coefficients $H^1(X, \mathbb{R})$. Note that $H^1(X, \mathbb{R}) = \mathrm{Hom}(\pi_1(X), \mathbb{R})$. The classical Hodge theory shows that each cohomology class $\alpha \in H^1(X, \mathbb{R})$ contains a unique harmonic form η. Taking the integral of η, we obtain a harmonic map $u_\eta : \widetilde{X} \to \mathbb{R}$. If we consider α as a representation $\pi_1(X) \to \mathbb{R}$, such u_η is α-equivariant, where the action of $\mathbb{R}$ on $\mathbb{R}$ is given by translation. Note that u_η is *pluriharmonic*: the $(1, 0)$-part $d_{\mathbb{C}}^{1,0} u_\eta$ of the complexified differential $d_{\mathbb{C}} u_\eta$ of u_η is a holomorphic 1-form. Since u_η is α-equivariant, such $d_{\mathbb{C}}^{1,0} u_\eta$ descends to a holomorphic form on X. Such a holomorphic form is nothing but $\eta^{1,0}$ where $\eta^{1,0}$ is the $(1, 0)$-part of the harmonic form η.

1.3. Main theorem: existence of harmonic maps

***Theorem 1.A* ([BDDM22, Theorem A]). —** *Let X be a complex smooth quasi-projective variety, and let G be a semisimple algebraic group defined over a non-archimedean local field K. If $\varrho : \pi_1(X) \to G(K)$ is a Zariski dense representation, then we have:*

(i) *There exists a ϱ-equivariant, locally Lipschitz harmonic map (with respect to any Kähler metric on X) $u : \widetilde{X} \to \Delta(G)$ from the universal cover $\widetilde{X}$ of X to the Bruhat-Tits building $\Delta(G)$ of G.*
(ii) *This map u is pluriharmonic and has logarithmic energy growth.*
(iii) *The local energy around points at infinity are* finite *provided that the corresponding local monodromies are quasi-unipotent.*
(iv) *If $f : Y \to X$ is a morphism from a smooth projective variety Y, then $u \circ \tilde{f} : \widetilde{Y} \to \Delta(G)$ is $f^*\varrho$-equivariant harmonic map with respect to any Kähler metric on Y. Furthermore, $u \circ \tilde{f}$ has logarithmic energy growth. Here $\tilde{f} : \widetilde{Y} \to \widetilde{X}$ denotes the lift of f between the universal coverings of Y and X.*

Let us explain some notion in the theorem.

First, since G is semisimple, $\Delta(G)$ is a Euclidean building without a Euclidean factor. By [Par00, Theorem 4.1], for any $g \in G(K)$, g is either elliptic or hyperbolic. Thus, there exists $P_0 \in \Delta(G)$ such that $\min_{P \in \Delta(G)} d(P, gP) = d(P_0, gP_0)$.

***Definition 1.3.1* (logarithmic energy growth). —** Let X be a smooth quasi-projective variety and let $\overline{X}$ be a smooth projective compactification of X such that $\Sigma := \overline{X} \backslash X$ is a simple normal crossing divisor. Let $\varrho : \pi_1(X) \to G(K)$ be a Zariski dense representation where G is a semisimple algebraic group. A ϱ-equivariant pluriharmonic map $\tilde{u} : \widetilde{X} \to \Delta(G)$ has *logarithmic energy growth* if for any holomorphic map $f : \mathbb{D}^* \to X$ with no essential singularity at the origin (i.e. f extends to a holomorphic map $\bar{f} : \mathbb{D} \to \overline{X}$), there is a positive constant C such that for any $r \in]0, \frac{1}{2}[$ one has

$$-\frac{L_\gamma^2}{2\pi} \log r \le E^{u \circ f}[\mathbb{D}_{r, \frac{1}{2}}] \le -\frac{L_\gamma^2}{2\pi} \log r + C, \tag{1.3.1}$$

where L_γ is called the *translation length* of $\varrho([\gamma])$ where $\gamma \in \pi_1(X)$ is the element corresponding to the loop around the smooth component Σ_i of the divisor Σ; i.e.

$$L_\gamma := \inf_{P \in \Delta(G)} d(\varrho([\gamma])P, P). \tag{1.3.2}$$

The constant C does not depend on r.

We give a summary of the proof of Theorems 1.A.(i) and 1.A.(ii). We consider a non-empty closed minimal convex $\rho(\pi_1(X))$-invariant subset C of X, meaning that there does not exist a non-empty convex strict subset of C invariant under $\rho(\pi_1(X))$. The existence of such C is guaranteed by [CM09, Theorem 4.3, (Aii)]. As a convex subset of an NPC space, C is itself an NPC space. Thus, when $\dim X = 1$ or $\dim X = 2$, we can apply the existence theorems of an equivariant harmonic map into an NPC space proved in [DM23b, Theorems 1.1 and 1.2] and [DM23a, Theorem 1] respectively.

Next we prove a uniqueness result:

***Claim 1.3.2*. —** *Any ϱ-equivariant harmonic mapping u from $\widetilde{X}$ to C is unique provided that*

(a) *u has logarithmic energy growth at infinity;*
(b) *$\varrho(\pi_1(X))$ does not fix a point at infinity.*

Note that the guarantee of Item (b) can be established when ϱ is Zariski dense. The harmonic map u from a punctured Riemann surface $\mathcal{R} = \bar{\mathcal{R}} \backslash \{p_1, \dots, p_n\}$ satisfies a logarithmic growth estimate towards a puncture in the form of

$$\frac{L^2}{2\pi} \log \frac{1/2}{r} \leq E^u[\Delta_{r,\frac{1}{2}}] \leq \frac{L^2}{2\pi} \log \frac{1/2}{r} + c, \quad 0 < r \leq \frac{1}{2} \tag{1.3.3}$$

where Δ is a holomorphic disk of $\bar{\mathcal{R}}$ at a puncture, $\Delta_{r,1}$ is an annulus with inner and outer radius r and 1 respectively, $E^u[\Delta_{r,1}]$ is the energy contained in $\Delta_{r,1}$ and L is the translation length of the isometry $\rho(\gamma)$ where $\gamma \in \pi_1(X)$ is the element corresponding the loop $\partial\Delta$ around the puncture. One can interpret L as the length of the minimum geodesic homotopic to the image loop $u(\partial\Delta)$ in the quotient metric space $X/\rho(\pi_1(X))$, and the lower bound of (1.3.3) is a consequence of the fact that the energy of the (parameterized) geodesic loop is at least $\frac{L^2}{2\pi}$. Indeed,

$$\frac{L^2}{2\pi} \log \frac{1}{r} \leq \int_r^1 \frac{L^2}{2\pi r} dr = \int_0^{2\pi} \int_r^1 \frac{1}{r^2} \left| \frac{\partial u}{\partial \theta} \right|^2 r dr d\theta \leq E^u[\Delta_{r,1}].$$

(Note that $L = 0$ is the case when the loop $u(\partial\Delta)$ is homotopically trivial.)

The upper bound in (1.3.3) comes from the construction of the harmonic map which we review now. The idea is to first construct a Lipschitz map v with controlled energy growth towards any puncture. This can be accomplished by defining $\theta \mapsto v(r,\theta)$ to be a parameterized geodesic loop corresponding to $\rho(\gamma)$ for each r where (r,θ) is the polar coordinates of the disk Δ. We call such a map a *prototype map*. Next, let $\mathcal{R}_r$ be a Riemann surface with Δ_r (disk of radius r) removed around each puncture. Let u_r be the Dirichlet solution with boundary values $v|_{\partial\mathcal{R}_r}$. We thus construct a family of harmonic maps u_r and prove that the sequence of harmonic maps u_{r_i} converges uniformly to a harmonic map u on every compact subset of $\mathcal{R}$ as $r_i \to 0$. The upper bound on the energy growth towards a puncture of u can be deduced from the energy growth of the prototype map v. The details of the above argument can be found in [DM23b].

In the case of quasi-projective surfaces, the existence also relies on the construction of a prototype map but is technically more complicated because the points at infinity are the normal crossing divisors and hence much more complicated than isolated points. The details of this construction can be found in [DM23a]. By the (more elaborated) Bochner technique, the harmonic map is actually pluriharmonic; in other words, its restriction to any complex curve is harmonic.

The proof of the existence in higher dimension relies on the existence results in dimensions 1 and 2. We use Lefschetz hyperplane theorem to prove Theorem 1.A by induction on $\dim X$. First, assume that $\dim_{\mathbb{C}} X = 3$. The existence result in dimension 2 guarantees that there is a pluriharmonic map defined on a general hyperplane on X. The uniqueness result implies that the pluriharmonic maps defined on the two different hyperplanes agrees along the intersection. Thus, we can define a pluriharmonic map on X. To define a pluriharmonic map in any dimensions, we proceed by induction on the dimension.

1.4. Application (I): logarithmic symmetric differentials and fundamental groups

We will now present significant applications of Theorem 1.A, which are cornerstones for the subsequent works discussed in Chapters 2 to 5.

1.4.1. Constructing logarithmic symmetric differentials. — In previous works [Kat97, Eys04, Kli13], for projective varieties whose fundamental groups admitting unbounded reductive representations into non-archimedean local fields, global symmetric differentials are constructed. In [BDDM22] we apply Theorem 1.A to extend this result to quasi-projective varieties.

***Theorem 1.B* ([BDDM22, Theorem B]). —** *Let X be a complex quasi-projective manifold and K be a non-archimedean local field. If there exists an unbounded representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$, then X has non-zero global logarithmic symmetric forms.*

Proof of Theorem 1.B (sketch). — Note that we do not assume that ϱ is reductive and thus we cannot apply Theorem 1.A. We apply the following lemma in [DYK23, Lemma 3.5].

***Claim 1.4.1.* —** *Let Γ be a finitely generated group and let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a representation where K be a non-archimedean local field. Then ϱ is unbounded if and only if its semi-simplification $\varrho^{ss} : \pi_1(X) \to \mathrm{GL}_N(\bar{K})$ is unbounded.*

This result enables us to replace ϱ by its semi-simplification and K by a finite extension such that we may assume that ϱ is semisimple, and thus reductive.

For simplicity, we assume that the Zariski closure $\varrho(\pi_1(X))$ is a semisimple algebraic group defined over K (after replacing K by a finite extension). By Theorem 1.A, there exists a ϱ-equivariant harmonic map $u : \widetilde{X} \to \Delta(G)$, where $\Delta(G)$ is the Bruhat-Tits building of G. Here $p : \widetilde{X} \to X$ is the universal covering map.

Recall that a point $x \in \widetilde{X}$ is said to be a *regular point* of u if there exists a neighborhood $\mathcal{N}$ of x and an apartment $A \subset \Delta(G)$ such that $u(\mathcal{N}) \subset A$. A *singular point* of u is a point in X that is not a regular point. Note that if $x \in \widetilde{X}$ is a regular point (resp. singular point) of u, then every point of $p^{-1}(p(x))$ is a regular point (resp. singular point) of u. We denote by $\widetilde{X}^\circ \subset \widetilde{X}$ the set of all regular points of u. Then there exists a dense open set $X^\circ \subset X$ such that $p^{-1}(X^\circ) = \widetilde{X}^\circ$. A deep and difficult theorem by Gromov-Schoen [GS92] shows that the complement of X° is a closed subset that has Hausdorff codimensional at least two.

We mention that over the regular locus $\widetilde{X}^\circ$ of u, u is locally a holomorphic map from an open set Ω of $\widetilde{X}$ to some apartment A of $\Delta(G)$ (such A is not unique in general). Note that A is locally isometric to $\mathbb{R}^N$. Therefore, if $\alpha : A \to \mathbb{R}$ is a linear map, then the $(1,0)$-part of the complexified differential $d^{1,0}_{\mathbb{C}}(\alpha \circ u)$ is a holomorphic one form on Ω since u is pluriharmonic. We can choose a generalized linear coordinates $(\alpha_1, \ldots, \alpha_\ell)$ for A such that

- $(d^{1,0}_{\mathbb{C}}(\alpha_1 \circ u), \ldots, d^{1,0}_{\mathbb{C}}(\alpha_\ell \circ u))$ does not depend on the choice of the apartment;
- these holomorphic 1-forms glue together into a mutivalued holomorphic 1-form defined over $\widetilde{X}^\circ$;
- this multivalued holomorphic 1-form is invariant under the $\pi_1(X)$-action, hence descends to a mutivalued 1-form $\{\omega_1, \ldots, \omega_\ell\}$ on a dense open set $X^\circ \subset X$ whose complement is a closed subset that has Hausdorff codimensional at least two;
- $\mathrm{Span}\{\alpha_1, \ldots, \alpha_\ell\} = \mathrm{Hom}(A, \mathbb{R})$ since G is semisimple.

We need to show that $\{\omega_1, \ldots, \omega_\ell\}$ is non-trivial. Otherwise, u is constant, and thus $\varrho(\pi_1(X))$ fixes a point in $\Delta(G)$, which is thus bounded, contradicting to our assumption that ϱ is unbounded.

We show that these multivalued 1-forms $\{\omega_1, \ldots, \omega_\ell\}$ induce a symmetric form on X°. Let T be a formal variable. Then

$$\prod_{k=1}^{m} (T - \omega_i) =: T^m + \sigma_1 T^{m-1} + \cdots + \sigma_m \tag{1.4.1}$$

is well defined. Its coefficients $\sigma_k \in \Gamma(X^\circ, \mathrm{Sym}^k \Omega_X)$.

Let us show that σ_k extends over X. By Theorem 1.A, u is locally Lipschitz. By our construction, there is a uniform constant $C_k > 0$ such that

$$|\sigma_k|_\omega \leq C_k |\nabla u|^k_\omega \quad \text{over } X^\circ, \tag{1.4.2}$$

where ω is a complete metric on X of Poincaré type at infinity. Since $X \subset X^\circ$ has Hausdorff codimension at least two, by (1.4.2) we apply some generalized Riemann extension theorem by Shiffman [Shi68] to conclude that σ_k extends to a symmetric form in $\Gamma(X, \mathrm{Sym}^k \Omega_X)$.

To show that σ_k has only logarithmic poles, we need to apply the energy estimate of u at infinity in Theorem 1.A. The proof is based on some complex analysis. For more details we refer to [BDDM22]. □

1.4.2. Simpson's integrality conjecture. — Let Γ be a finitely generated group. We say $\varrho : \Gamma \to \mathrm{GL}_N(\mathbb{C})$ is *rigid* if its image in the character variety $M_{\mathrm{B}}(\Gamma, \mathrm{GL}_N)$ is isolated. Say ϱ is *integral* if there exists a number field k such that ϱ factors through $\varrho : \Gamma \to \mathrm{GL}_N(O_K)$.

In [Sim92], Simpson conjectured that any rigid representation of the fundamental group of a smooth projective varieties is integral. This conjecture is solved for rank 2 rigid representations of fundamental groups of smooth projective varieties by Corlette-Simpson [CS08]. Based on Theorem 1.B, in [BDDM22] we prove the following theorem.

***Theorem 1.C* ([BDDM22, Theorem C]). —** *Let X be a complex quasi-projective manifold which does not admit any global logarithmic symmetric differentials. Then any reductive representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ is* rigid *and* integral*. Moreover, ϱ is a complex direct factor of a $\mathbb{Z}$-variation of Hodge structures.*

Note that Klingler [Kli13] had previously established Theorem 1.C when considering the case where X is a projective manifold. He utilized non-abelian Hodge theories in the archimedean setting, involving harmonic mappings to symmetric spaces, to establish the rigidity property in Theorem 1.C for compact Kähler manifolds. Precisely, his proof is based on the following result by Arapura.

***Theorem 1.4.2* ([Ara02]). —** *Let X be a smooth projective variety without global symmetric differentials. Then for any $N \in \mathbb{Z}_{>0}$, the character variety $M_{\mathrm{B}}(\pi_1(X), \mathrm{GL}_N)$ is zero dimensional.*

Let us explain the idea of Theorem 1.4.2 as it is elegant and simple. By the work of Simpson [Sim94a, Sim94b], the semistable Higgs bundles of rank N with vanishing Chern classes over X exists, denoted by $M_{\mathrm{Dol}}(X, N)$, is a complex quasi-projective variety. He proved that there exists a homeomorphism between $M_{\mathrm{Dol}}(X, N)$ and $M_{\mathrm{B}}(\pi_1(X), \mathrm{GL}_N)$.

On the other hand, there exists a proper holomorphic fibration $M_{\mathrm{Dol}}(X, N) \to \oplus_{k>0} H^0(X, \mathrm{Sym}^k \Omega_X)$ (so-called *Hitchin fibration*). Since we assume that $\oplus_{k>0} H^0(X, \mathrm{Sym}^k \Omega_X)$ is trivial, it follows that $M_{\mathrm{Dol}}(X, N)$ is compact. Hence $M_{\mathrm{B}}(\pi_1(X), \mathrm{GL}_N)$ is compact. However, it is an affine variety. This implies that $M_{\mathrm{B}}(\pi_1(X), \mathrm{GL}_N)$ is zero dimensional. Theorem 1.4.2 is proved.

Such a nice proof by Arapura cannot be used when X is quasi-projective since the moduli space of Higgs bundles is missing. Nevertheless, we can use Gauge theoretical arguments (i.e. Uhlenbeck's compactness) together with Mochizuki's work [Moc06, Moc07b] to extend Theorem 1.4.2 to the quasi-projective setting. This is shown in the first arXiv version of [BDDM22].

Later we discovered that we can use Theorem 1.A to simultaneously address both aspects of rigidity and integrality. Furthermore, it's noteworthy that our approach exclusively works within non-archimedean local fields of *characteristic zero*. This represents a novel aspect compared to the work by Corlette and Simpson in [CS08]. In their work, they interpret rigidity in rank 2 cases as the representation of $\mathrm{SL}_2(\mathbb{C}(t))$ into a compact subgroup. This introduces an additional layer of complexity since the Bruhat-Tits tree for $\mathrm{SL}_2(\mathbb{C}(t))$ is not locally compact. Consequently, they had to employ the method of reduction modulo p to bring it down to the case of representations in $\mathrm{SL}_2(\mathbb{F}_q(t))$.

Let us explain the idea of the proof of Theorem 1.C.

Proof of Theorem 1.C (sketch). — We recall a nice theorem by Yamanoi. Let Γ be a finitely generated group. Let $R(\Gamma, \mathrm{GL}_N)$ be the representation variety of $\mathrm{Hom}(\pi_1(X), \mathrm{GL}_N)$. It is an affine scheme over $\mathbb{Z}$ such that $R(\Gamma, \mathrm{GL}_N)(k)$ is the set of representation $\Gamma \to \mathrm{GL}_N(k)$ for any field k. The reductive algebraic group GL_N acts on $R(\Gamma, \mathrm{GL}_N)$ by conjugation. Let $\pi : R(\Gamma, \mathrm{GL}_N) \to M_{\mathrm{B}}(\Gamma, \mathrm{GL}_N)$ be the GIT quotient by GL_N.

***Lemma 1.4.3* ([Yam10]).** — *Let K be a finite extension of $\mathbb{Q}_p$ with p any prime. Denote by $R_0 \subset R(\Gamma, \mathrm{GL}_N)(K)$ be the set of bounded representations. Then $\pi(R_0)$ is a compact subset (in the analytic topology) of $M_{\mathrm{B}}(\Gamma, \mathrm{GL}_N)(K)$.*

Write $M_{\mathrm{B}}(X, N) := M_{\mathrm{B}}(\pi_1(X), \mathrm{GL}_N)$ for short. Assume that $M \subset M_{\mathrm{B}}(X, N)$ is a geometric irreducible component that is not zero dimensional. Since $M(\bar{\mathbb{Q}}_p)$ is not compact, it follows from Lemma 1.4.3 that there exists a finite extension K of $\mathbb{Q}_p$ such that there exists an unbounded representation $\sigma : \pi_1(X) \to \mathrm{GL}_N(K)$ with $[\sigma] \in M(K)$. Here we write $[\sigma] := \pi(\sigma)$. We apply Theorem 1.B to conclude that X has some non-trivial logarithmic symmetric differentials and obtain a contradiction. Hence $M_{\mathrm{B}}(X, N)$ is zero dimensional. This implies that ϱ is rigid.

Note that $R(\pi_1(X), \mathrm{GL}_N)(\bar{\mathbb{Q}})$ is dense in $R(\pi_1(X), \mathrm{GL}_N)(\mathbb{C})$. ϱ can thus be deformed into a representation $\pi_1(X) \to \mathrm{GL}_N(\bar{\mathbb{Q}})$. Since ϱ is rigid, for any small continuous deformation ϱ', its semi-simplification is conjugate to ϱ. Hence after replacing ϱ by some conjugate, there exists some number field k such that $\varrho : \pi_1(X) \to \mathrm{GL}_N(k)$.

Let v be any non-archimedean place of k. We denote by k_v the non-archimedean completion of k with respect to v. Consider the representation $\varrho_v : \pi_1(X) \to \mathrm{GL}_N(k_v)$ by composing ϱ with $k \hookrightarrow k_v$. If ϱ_v is unbounded, we apply Theorem 1.B to conclude that X has some logarithmic symmetric differential, contradicting to our assumption. Hence ϱ_v is bounded for each v. It follows that ϱ factors through $\varrho : \pi_1(X) \to \mathrm{GL}_N(O_K)$. This proves that ϱ is integral.

Let $w : k \to \mathbb{C}$ be any archimedean place of k. Consider the representation $\varrho_w : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ by composing $\varrho : \pi_1(X) \to \mathrm{GL}_N(k)$ with $w : k \hookrightarrow \mathbb{C}$. Since $M_{\mathrm{B}}(X, N)$ is zero dimensional, ϱ_w is rigid. By [Sim92], ϱ underlies a complex variation of Hodge structures ($\mathbb{C}$-VHS for short). We apply [CS08] to conclude that ϱ is a direct factor of a $\mathbb{Z}$-VHS. □

1.4.3. Infinite fundamental group and logarithmic symmetric forms. — As an application of Theorem 1.C, we extend a theorem by [BKT13] to the quasi-projective setting.

***Theorem 1.D* ([BDDM22, Theorem D]).** — *Let X be a smooth quasi-projective variety, and let $\overline{X}$ be a smooth projective compactification of X such that $\Sigma := \overline{X} \backslash X$ is a simple normal crossing divisor. Let $\mathbb{K}$ be any field of any characteristic. If there is a linear representation $\tau : \pi_1(X) \to \mathrm{GL}_N(\mathbb{K})$ such that the image is infinite, then there exists a positive integer k such that*

$$H^0(\overline{X}, \mathrm{Sym}^k \Omega_{\overline{X}}(\log \Sigma)) \neq 0$$

Theorem 1.D was proved by Klingler, Brunebarbe and Totaro [BKT13] in the case where X is a compact Kähler manifold.

Proof of Theorem 1.D (sketch). — Assume by contradiction that X does not have logarithmic symmetric differentials. By Theorem 1.C, ϱ is a complex direct factor of a $\mathbb{Z}$-variation of Hodge structures σ. Given that ϱ has an infinite image, it follows that σ does as well. Consider the period map of σ. After replacing σ by a finite étale cover, we may assume that it is torsion free by Selberg's lemma. By a theorem of Griffiths, the period mapping $p : X \to \mathscr{D}/\Gamma$ extends to a proper mapping $X' \to \mathscr{D}/\Gamma$, where $X' \subset \overline{X}$ is a Zariski open set containing X, and $\mathscr{D}$ is the period domain and $\Gamma := \sigma(\pi_1(X))$ is the monodromy group. Let Z be the image $p(X')$ that is positive dimensional since σ has infinite image. For simplicity we may assume that Z is smooth. One can prove that Z is quasi-projective and $p : X' \to Z$ is algebraic. The curvature computation of period domain yields that Z has non-zero logarithmic symmetric differentials. We pullback these differentials to X' via p. The theorem is proved. □

1.5. Application (II): a reduction theorem

For another crucial application of Theorem 1.A is the following reduction theorem. It will have further applications in Chapters 2 and 3.

***Theorem 1.E* ([CDY22, Theorem 0.10]).** — *Let X be a complex smooth quasi-projective variety, and let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a reductive representation where K is non-archimedean local field. Then there exists a* *quasi-projective normal variety S_ϱ and a dominant morphism $s_\varrho : X \to S_\varrho$ with connected general fibers, such that for any connected Zariski closed subset T of X, the following properties are equivalent:*

(a) *the image $\rho(\mathrm{Im}[\pi_1(T) \to \pi_1(X)])$ is a bounded subgroup of $G(K)$.*
(b) *For every irreducible component T_o of T, the image $\rho(\mathrm{Im}[\pi_1(T_o^{\mathrm{norm}}) \to \pi_1(X)])$ is a bounded subgroup of $G(K)$.*
(c) *The image $s_\varrho(T)$ is a point.*

It is worth noting that if X is projective, the equivalence between Item (a) and Item (c) has been established by Katzarkov [Kat97], Eyssidieux [Eys04, Proposition 1.4.7]. Hence when X is compact, the above $s_\varrho : X \to S_\varrho$ is called the *Katzarkov-Eyssidieux reduction*.

The proof of Theorem 1.E relies on Theorem 1.A together with the following result on *partial Albanese morphism*.

Let X be a smooth quasi-projective variety. Let $\{\eta_1, \ldots, \eta_m\}$ be a set of logarithmic holomorphic 1-forms of X. Consider the quasi-Albanese map $f : X \to A_X$. Then there exists logarithmic holomorphic 1-forms $\{\omega_1, \ldots, \omega_m\}$ of A_X such that $f^*\omega_i = \eta_i$. Let B be the largest semi-abelian subvariety of A_X such that $\omega_i|_B \equiv 0$ for each i. Write $A := A_X/B$ which is also a semi-abelian variety. Denote by $a : X \to A$ the composite of f and the quotient map $A_X \to A$. This map a is called the *partial quasi-Albanese map* associated with $\{\eta_1, \ldots, \eta_m\}$. It has the following property:

Claim 1.5.1. — *Let Z be any closed subvariety of X. Then $a(Z)$ is a point if and only if $\eta_i|_Z \equiv 0$.*

Now let us outline the idea of the proof of Theorem 1.E.

Proof of Theorem 1.E (sketch). — For simplicity, we assume that the Zariski closure $\varrho(\pi_1(X))$ is a semisimple algebraic group defined over K (after replacing K by a finite extension). By Theorem 1.A, there exists a ϱ-equivariant harmonic map $u : \widetilde{X} \to \Delta(G)$, where $\Delta(G)$ is the Bruhat-Tits building of G. We use the same notations in the proof of Theorem 1.B.

Recall that we constructed some multivalued 1-forms $\{\omega_1, \ldots, \omega_\ell\}$ over X°. These multivalued 1-forms correspond to the $(1,0)$-part of the complexified differential of the harmonic map u. Based on Theorem 1.B, we can prove that there exists a (in general ramified) finite Galois covering $\pi : X^{\mathrm{sp}} \to X$ with Galois group H such that $\pi^*\{\omega_1, \ldots, \omega_\ell\}$ becomes single valued; i.e. there exists forms $\{\eta_1, \ldots, \eta_\ell\} \subset H^0(\overline{X^{\mathrm{sp}}}, \pi^*\Omega^1_{\overline{X}}(\log D))$ such that $\{\eta_1, \ldots, \eta_\ell\}$ coincides with $\pi^*\{\omega_1, \ldots, \omega_\ell\}$ over X°. Here $\overline{X}$ is a smooth projective compactification of X with $D := \overline{X}\backslash X$ a simple normal crossing divisor and π extends to a finite Galois cover $\overline{X^{\mathrm{sp}}} \to \overline{X}$. Such $X^{\mathrm{sp}} \to X$ is called *spectral covering* associated with the representation ϱ. Its ramification locus is well-behaved.

Claim 1.5.2. — *Let us denote by* $\mathrm{Ram}(\pi)$ *be the ramification locus of π. Then we have*

$$\mathrm{Ram}(\pi) \subset \bigcup_{\eta_i \neq \eta_j} (\eta_i - \eta_j = 0).$$

Moreover, we have

Claim 1.5.3. — *$\{\eta_1, \ldots, \eta_\ell\}$ is invariant under the Galois group H.*

Consider the partial quasi-Albanese map $a : X^{\mathrm{sp}} \to \mathrm{A}$ induced by $\{\eta_1, \ldots, \eta_\ell\}$. Claim 1.5.2 implies that there exists a Galois action H on A such that a is H-equivariant. Taking the quotient by H and we obtain a morphism $b : X \to A/H$. Let $s_\varrho : X \to S_\varrho$ be the quasi-Stein factorization of b. In [CDY22], we prove that s_ϱ the desired factorization in Theorem 1.E. Indeed, by the property of the harmonic map u, $s_\varrho(Z)$ is a point if and only if $u(Z')$ is a point in $\Delta(G)$, where Z' is a connected component $\pi_0^{-1}(Z)$ with $\pi_0 : \widetilde{X} \to X$ being universal covering map. Note that $u(Z')$ is a point in $\Delta(G)$ if and only if $\varrho(\mathrm{Im}[\pi_1(Z) \to \pi_1(X)])$ fixes a point $P \in \Delta(G)$, hence is a bounded subgroup. □

1.6. Further applications

Further applications of Theorems 1.A, 1.B and 1.E include the hyperbolicity of quasi-projective varieties whose fundamental groups admit a large representation (see Chapter 2), as well as constructing the Shafarevich morphism for linear representations of the fundamental groups of quasi-projective varieties (see Chapters 3 and 4). There are some further work in progress and let me mention a few without giving details. In [EKPR12] Eyssidieux et. al. proved the Shafarevich conjecture for *smooth* complex projective varieties with complex linear fundamental group and they proposed to extend their work to the singular setting. In a forthcoming work, we proved that complex projective *normal* varieties have universal coverings holomorphically convex provided that their fundamental groups are complex linear. In another joint work with Botong Wang, we applied techniques from non-abelian Hodge theories to establish the following *Singer-Hopf conjecture* in the linear case:

Theorem 1.F **([WB20]).** — *Let X be a complex projective manifold. If there exists a large linear representation $\rho : \pi_1(X) \to \mathrm{GL}_N(K)$ where K is any field, then for any perverse sheaf $\mathscr{P}$ on X, its Euler characteristic $\chi(X, \mathscr{P}) \geq 0$.*

A direct consequence is the following result.

Corollary 1.G **([WB20]). —** *Let X be a complex projective manifold. Assume that X is aspherical, i.e., its universal covering is contractible. If there exists an almost faithful representation $\rho : \pi_1(X) \to \mathrm{GL}_N(K)$ where K is any field, then $(-1)^{\dim \mathrm{X}}\chi(X) \geq 0$.*

CHAPTER 2

HYPERBOLICITY AND FUNDAMENTAL GROUPS OF QUASI-PROJECTIVE VARIETIES

2.1. Notions of hyperbolicity

The notion of hyperbolicity originates from Picard's great theorem and Picard's little theorem on the range of an analytic function.

***Theorem 2.1.1* (Little Picard theorem). —** *Any holomorphic map $f:\mathbb{C}\to\mathbb{P}^1\backslash\{0,1,\infty\}$ must be constant.*

This theorem is a significant strengthening of Liouville's theorem which states that the image of an entire non-constant function must be unbounded.

***Theorem 2.1.2* (Great Picard theorem). —** *Any holomorphic map $f:\mathbb{D}^*\to\mathbb{P}^1\backslash\{0,1,\infty\}$ from the punctured unit disk does not have essential singularity at the origin.*

This is a substantial strengthening of the Casorati–Weierstrass theorem, which only guarantees that the range of a holomorphic function defined over $\mathbb{D}^*$ with essential singularity at the origin has image dense in $\mathbb{C}$. One can see that Theorem 2.1.2 implies Theorem 2.1.1.

The complex algebraic varieties that have the similar properties as described in Theorems 2.1.1 and 2.1.2 is called *hyperbolic*. Precisely, we have the following definition.

***Definition 2.1.3* (Hyperbolicity). —** Let X be a complex quasi-projective variety.

(i) X is *pseudo Picard hyperbolic* if there is a proper Zariski closed subset $\Xi\subsetneq X$ such that any holomorphic map $f:\mathbb{D}^*\to X$ from the punctured disk $\mathbb{D}^*$ to X with $f(\mathbb{D}^*)\not\subset\Xi$ extends to a holomorphic map from the disk $\mathbb{D}$ to a projective compactification $\overline{X}$ of X.

(ii) X is *pseudo Brody hyperbolic* if there is a proper Zariski closed subset $\Xi\subsetneq X$ such that any non-constant holomorphic map $f:\mathbb{C}\to X$ has image in Ξ.

Note that every pseudo Picard hyperbolic variety is pseudo Brody hyperbolic. While we conjecture the converse to hold true, as of now, we lack both a proof and any counter-example of our conjecture.

In the algebraic setting, we introduce the following definition.

***Definition 2.1.4* (Strongly of log general type). —** Let X be a complex quasi-projective variety. X is *strongly of log general type* if there is a proper Zariski closed subset $\Xi\subsetneq X$ such that any positive-dimensional closed subvariety of X not contained in Ξ is of log general type.

Lang conjectured that a complex quasi-projective variety is strongly of log general type if it is of log general type. To my knowledge, this conjecture remains open even for complex surfaces.

2.2. Generalized Green-Griffiths-Lang conjecture

To characterize algebraic varieties falling into the hyperbolic category, we will start by examining cases where C is a smooth quasi-projective curve, with $\overline{C}$ as its compactification, and D representing the complement of C within $\overline{C}$. Here is a classification: As we can see from the table provided, we can make the following observations: from an algebraic geometric perspective, we can identify hyperbolic curves as those possessing a positive logarithmic

FIGURE 1. Hyperbolicity from different viewpoints

	$\deg(K_{\overline{C}}+D)$	$\pi_1(C)$	Hyperbolicity
$\mathbb{P}^1,\mathbb{C}$	<0	$\{1\}$	no
$\mathbb{C}^*$, torus	$=0$	Infinite, abelian	no
$\mathbb{P}^1\backslash\{$at least three points$\}$ torus$\backslash\{$at least one point$\}\cdots$	>0	Infinite, non-abelian	yes

canonical bundle. On the other hand, from a topological standpoint, hyperbolic curves are characterized by having infinite and non-abelian fundamental groups.

It's worth noting that the generalized Green-Griffiths-Lang conjecture aligns with the algebraic geometric viewpoint, focusing on the positivity of the logarithmic canonical bundle.

***Conjecture 2.2.1* (Generalized Green-Griffiths-Lang conjecture). —** *Let X be a smooth quasi-projective variety. Then the following properties are equivalent:*

(i) *X is of log general type;*
(ii) *X is pseudo-Picard hyperbolic;*
(iii) *X is pseudo-Brody hyperbolic;*
(iv) *X is* strongly of log general type.

So far Conjecture 2.2.1 remains an open and challenging problem, even in situations where X is a surface. We are fascinated by this conjecture due to its analogy with the Bombieri-Lang conjecture concerning rational points.

***Conjecture 2.2.2* (Bombieri-Lang). —** *Let X be a smooth projective variety defined over a number field k. Then there exists a dense Zariski closed subset $\Xi \subsetneq X$ such that for all number field extensions k' of k, the set of k'-rational points in $X\backslash\Xi$ is finite.*

2.3. Property of Picard hyperbolicity and some examples

It is natural to ask why we are interested in the more general notion Picard hyperbolicity. It indeed enjoys the following algebraicity property.

***Proposition 2.3.1* ([Den23]). —** *Let X be a smooth quasi-projective variety that is pseudo Picard hyperbolic. Then any meromorphic map $f : Y \dashrightarrow X$ from another smooth quasi-projective variety Y to X with $f(Y) \not\subset \mathrm{Sp_p}(X)$ is* rational.

A direct consequence of Proposition 2.3.1 is the following uniqness of algebraic structure of pseudo Picard hyperbolic varieties.

***Corollary 2.3.2* ([Den23]). —** *Let X and Y be smooth quasi-projective varieties such that there exists an analytic isomorphism $\varphi : Y^{\mathrm{an}} \to X^{\mathrm{an}}$ of associated complex spaces. Assume that X is pseudo Picard hyperbolic. Then φ is an algebraic isomorphism.* □

As we will see in Chapter 5, Picard hyperbolicity has more applications in algebraic geometry.

Let us discuss some examples of pseudo Picard hyperbolic varieties. A classical result due to Borel [Bor72] and Kobayashi-Ochiai [KO71] is that quotients of bounded symmetric domains by torsion-free lattices are Picard hyperbolic. In [Den23] we proved a similar result for algebraic varieties that admit a complex variation of Hodge structures.

***Theorem 2.3.3* ([Den23, Theorem A]). —** *Let X be a quasi-projective manifold. Assume that there is a complex variation of Hodge structures on X whose period mapping is injective at one point. Then X is pseudo Picard hyperbolic.* □

Theorem 2.3.3 will be used in the proof of Theorem 2.A.

In [Nad89], Nadel proved the nonexistence of certain level structures on abelian varieties over complex function fields, which was refined by Rousseau in [Rou16]. Precisely, they proved the following theorem:

***Theorem 2.3.4* ([Nad89, Rou16]). —** *Let X be a smooth quasi-projective variety such that $X = \Omega/\Gamma$ where Ω is a bounded symmetric domain and Γ is an arithmetic torsion free lattice acting on Ω. Then there exists a finite index subgroup $\Gamma' \subset \Gamma$, such that for the quasi-projective variety $X' := \Omega/\Gamma'$, its projective compactification $\overline{X'}$ is Brody (moreover Kobayashi) hyperbolic modulo the boundary $\overline{X'}\backslash X'$.*

In [Den23], we obtained the following result which incorporates previous results by Nadel and Rousseau.

***Theorem 2.3.5* ([Den23, Theorem B]). —** *Let X be a quasi-projective manifold. Assume that there is a complex variation of Hodge structures on X whose period mapping is injective at one point. Then there exists a finite étale cover X' of X such that its projective compactification $\overline{X'}$ is pseudo Picard hyperbolic.*

The proofs of Theorems 2.3.3 and 2.3.5 in [Den23] are involved and rely heavily on the analytic aspects of Hodge theories. Later, in [CD21], we present a simplified proof. Nonetheless, Nevanlinna theory plays an essential role in both works.

2.4. How fundamental groups determine hyperbolicity

It is natural to inquire into the relationship between the (topological) fundamental groups, denoted as $\pi_1(X)$, of complex algebraic varieties X and their hyperbolicity property. From a topological perspective, as illustrated in section 2.2, a characterization of hyperbolicity necessitates that the fundamental group $\pi_1(X)$ be both infinite and non-abelian. Moreover, the following example shows that we need more assumptions.

***Example 2.4.1.* —** Let C be a projective curve of genus at least two. Then $\pi_1(C \times \mathbb{P}^1) \simeq \pi_1(C)$ is infinite and non-abelian. It is not pseudo Brody hyperbolic.

To exclude the cases of the above example, we need to introduce the definition of big representation of fundamental groups.

***Definition 2.4.2* (Big representation). —** Let X be a quasi-projective normal variety, and let G be any group. A representation $\varrho : \pi_1(X) \to G$ is said to be *big*, or *generically large* in [Kol95], if for any positive dimensional closed subvariety $Z \subset X$ containing a *very general* point of X, $\varrho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)])$ is infinite, where Z^{norm} denotes the normalization of Z. Moreover, ϱ is called *large* if $\varrho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)])$ is infinite for any positive dimensional closed subvariety Z of X.

Therefore, if $\pi_1(X)$ is big, then it is not $\mathbb{A}^1$-uniruled, i.e., for a very general point $x \in X$ there exists a morphism $\mathbb{A}^1 \to X$ whose image passes to x. Hence a variety with big representation will exclude the non-hyperbolic examples in Example 2.4.1. However, it is easy to construct non-hyperbolic varieties with large fundamental groups.

***Example 2.4.3*. —** Let C be a projective curve of genus at least two and let E be an elliptic curve. Then $\pi_1(C \times E) \simeq \pi_1(C) \times \pi_1(E)$ is *large* and non-abelian. It is not pseudo Brody hyperbolic.

Example 2.4.3 shows that the the product of varieties with elliptic curves, or more generally elliptic surfaces, though their fundamental groups are large and non-abelian, they are not pseudo Brody hyperbolic. To exclude these examples, we introduce the definition of semisimple fundamental groups.

***Definition 2.4.4* (Semisimple group). —** Let G be a finitely generated group. G is also semisimple if it has no non-trivial normal infinite abelian subgroups.

We propose the conjecture that varieties endowed with substantial and semisimple fundamental groups are necessarily hyperbolic.

***Conjecture 2.4.5*. —** *Let X be a quasi-projective normal variety. If there exists a representation $\varrho : \pi_1(X) \to G$ which is big and $\varrho(\pi_1(X))$ is semisimple, then X is strongly of log general type and pseudo Picard hyperbolic.*

2.5. Hyperbolicity of varieties with big fundamental groups

Concerning Conjecture 2.4.5, when G is a linear complex algebraic group and X is projective, this conjecture has been proven in [Yam10, CCE15]. Recall that a linear algebraic group G over a field K is called semisimple if it has no non-trivial connected normal solvable algebraic subgroups defined over the algebraic closure of K, and has positive dimension. Specifically, Campana-Claudon-Eyssidieux [CCE15, Theorem 1] proved that a smooth complex projective variety X with a Zariski dense representation $\varrho : \pi_1(X) \to G(\mathbb{C})$, where G is a semisimple linear algebraic group over $\mathbb{C}$, is of general type when ϱ is big [a]. At almost the same time, Yamanoi [Yam10, Proposition 2.1] proved that X does not admit Zariski dense entire curves $f : \mathbb{C} \to X$.

A representation $\varrho : \pi_1(X) \to G(\mathbb{C})$ is said to be *big*, or *generically large* in [Kol95], if for any closed irreducible subvariety $Z \subset X$ containing a *very general* point of X, $\varrho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)])$ is infinite, where Z^{norm} denotes the normalization of Z. It is worth noting that a stronger notion of largeness exists, where ϱ is called *large* if $\varrho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)])$ is infinite for any closed subvariety Z of X. In [CDY22], with Cadorel and Yamanoi, we generalize and strengthen the above theorems to complex quasi-projective varieties. In [DY24], we establish analogous result for representation in positive characteristic. Our main result is the following:

***Theorem 2.A*. —** *Let X be a complex quasi-projective normal variety and let G be a semisimple algebraic group over an infinite field K. Assume that $\varrho : \pi_1(X) \to G(K)$ is a big and Zariski dense representation.*

(i) [CDY22, Theorem A] *If $\mathrm{char}\, K = 0$, then for any automorphism $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, denoting by X^σ the Galois conjugate variety of X under σ, X^σ is strongly of log general type and pseudo Picard hyperbolic.*

(ii) [DY24, Theorem E] *If $\mathrm{char}\, K > 0$, then X is strongly of log general type and pseudo Picard hyperbolic.*

We also mention that the two conditions for the representation ϱ in Theorem 2.A are essential to conclude the two statements in Theorem 2.A, as we have seen in Examples 2.4.1 and 2.4.3.

It is noteworthy that the condition of bigness for the representations ϱ in Theorem 2.A is not particularly restrictive, unlike the requirement for a large representation. In fact, in [CDY22, DY24] we demonstrate that any linear representation of $\pi_1(X)$ can be factored through a big representation after taking a finite étale cover. This result, combined with Theorem 2.A, yields a factorization theorem for linear representations of $\pi_1(X)$.

***Corollary 2.B*. —** *Let X be a complex quasi-projective normal variety and let G be a semisimple linear algebraic group over any field K. If $\varrho : \pi_1(X) \to G(K)$ is a Zariski dense representation, then there exist a finite étale cover $\nu : \widehat{X} \to X$, a birational and proper morphism $\mu : \widehat{X}' \to \widehat{X}$, a dominant morphism $f : \widehat{X}' \to Y$ with connected general fibers, and a big and Zariski dense representation $\tau : \pi_1(Y) \to G(K)$ such that*

— $f^*\tau = (\nu \circ \mu)^*\varrho$.
— *Y is strongly of log general type.*
— *Y is pseudo Picard hyperbolic.*

In particular, X is not weakly special and does not contain Zariski-dense entire curves.

a. This result was previously claimed by Zuo [Zuo96]. However, it seems to me that some arguments in [Zuo96] need amplification. See [CDY22, Remarks 8.5 & 8.9] on the remarks on Zuo's proof.

Note that by Campana [Cam11], a quasi-projective variety X is *weakly special* if for any finite étale cover $\widehat{X} \to X$ and any proper birational modification $\widehat{X}' \to \widehat{X}$, there exists no dominant morphism $\widehat{X}' \to Y$ with Y a positive-dimensional quasi-projective normal variety of log general type.

Corollary 2.B generalizes the previous work by Mok [Mok92], Corlette-Simpson [CS08], and Campana-Claudon-Eyssidieux [CCE15], in which they proved similar factorisation results.

We sketch the main idea behind the proof of Theorem 2.A.(i). The proof of Theorem 2.A.(ii) will be presented in Chapter 4.

Proof of Theorem 2.A.(i) (sketch). — We can assume that $K = \mathbb{C}$. For simplicity, we assume that the Zariski closure G of $\varrho(\pi_1(X))$ is *almost simple*. There are several cases that occurs.

Case 1. ϱ is rigid. It means that for any continuous deformation ϱ_t of ϱ, we have $[\varrho_t] = [\varrho]$, where $[\varrho]$ denotes the image of ϱ in the character variety $M_{\rm B}(X, N)(\mathbb{C}) := M_{\rm B}(\pi_1(X), {\rm GL}_N)(\mathbb{C})$. By the work of Mochizuki [Moc06], ϱ underlies a $\mathbb{C}$-variation of Hodge structures (VHS for short). Moreover, after replacing ϱ by a conjugation, we may assume that there exists a number field $k \subset \overline{\mathbb{Q}}$ such that

— G is defined over k;
— we have the factorization $\varrho : \pi_1(X) \to G(k)$;
— $\varrho(\pi_1(X))$ is Zariski dense in G.

Case 1.1. Assume that for each non-archimedean place v of k, the composite $\varrho_v : \pi_1(X) \to {\rm GL}_N(k_v)$ of ϱ and $k \hookrightarrow k_v$, is bounded. Here k_v denotes the non-archimedean completion of k with respect to v.

If this case occurs, we have a factorization $\varrho : \pi_1(X) \to {\rm GL}_N(O_k)$. Let us denote by ${\rm Ar}(k)$ the set of archimedean places of k. Note that ${\rm GL}_N(O_k) \to \prod_{w\in {\rm Ar}(k)} {\rm GL}_N(\mathbb{C})$ is a discrete subgroup by [Zim84, Proposition 6.1.3]. We denote by $\varrho_w : \pi_1(X) \to {\rm GL}_N(\mathbb{C})$ the composite of ϱ and $w : k \hookrightarrow \mathbb{C}$. Then ϱ_w is also rigid and thus underlies a $\mathbb{C}$-VHS. It follows that for the product representation

$$\prod_{w\in {\rm Ar}(k)} \varrho_w : \pi_1(X) \to \prod_{w\in {\rm Ar}(k)} {\rm GL}_N(\mathbb{C}),$$

its image Γ is discrete.

Let $\mathscr{D}$ be the period domain corresponding to the $\mathbb{C}$-VHS of $\sigma := \prod_{w\in {\rm Ar}(k)} \varrho_w$. Then $\mathscr{D}/\Gamma$ is a complex space since Γ acts on $\mathscr{D}$ discretely. Let $p : X \to \mathscr{D}/\Gamma$ be the period map. Since we assume that ϱ is big, σ is also big. One can thus show that $\dim X = \dim p(X)$. We apply Theorem 2.3.3 to conclude that X is pseudo Picard hyperbolic, and is strongly of log general type by [BC20].

Case 1.2 Assume that there exists some non-archimedean place v of k, the composite $\varrho_v : \pi_1(X) \to G(k_v)$ of ϱ and $k \hookrightarrow k_v$, is unbounded. Note that $\varrho_v(\pi_1(X))$ is Zariski dense in G. Since ϱ is big, ϱ_v is also big. Then conditions in Theorem 2.C are fulfilled. We conclude our theorem.

Case 2: ϱ is non-rigid. In the previous work like [CS08, Eys04, Zuo96], the authors constructed unbounded representations using curves in character varieties in positive characteristic (after taking reduction mod p). However, these unbounded representations might not be Zariski dense in G nor big (hence we cannot apply Theorem 2.C). In [CDY22] we introduce a completely new method to construct unbounded representations and avoid reduction mod p.

Claim 2.5.1. — *If $\varrho : \pi_1(X) \to G(\mathbb{C})$ is non-rigid, then there exists a big, Zariski dense, and unbounded representation $\varrho' : \pi_1(X) \to G(K)$, where K is a finite extension of some $\mathbb{Q}_p$ with p prime.*

The idea of the proof of Claim 2.5.1 is roughly that, for the set of bounded representations R in the representation variety $R_{\rm B}(\pi_1(X), G)(K)$, its image in the character variety $M_{\rm B}(\pi_1(X), G)(K)$ is compact. Since ϱ is non-rigid and $M_{\rm B}(\pi_1(X), G)$ is affine, the geometric connected component of $M_{\rm B}(\pi_1(X), G)$ containing ϱ is non-compact. Hence there exists some unbounded representation. Moreover, since the Zariski dense representation is a Zariski open condition, we can make such unbounded representation Zariski dense. To ensure that it is big, we need some extra work and refer the readers to [CDY22, Proposition 8.1] for more details.

Since ϱ is non-rigid, by Claim 2.5.1 we can construct a big, Zariski dense, and unbounded representation $\varrho' : \pi_1(X) \to G(K)$, where K is some non-archimedean local field. We then apply Theorem 2.C below to conclude the theorem. □

Theorem 2.C **([CDY22, Theorem I]).** — *Let X be a quasi-projective normal variety and let G be an absolutely almost simple algebraic group defined over a non-archimedean local field K. If $\varrho : \pi_1(X) \to G(K)$ is a big, Zariski dense, and unbounded representation, then X is strongly of log general type, and pseudo Picard hyperbolic.*

We would like to sketch the idea of the proof of Theorem 2.C since the methods are new even if X is projective (compared with [CCE15]).

Proof of Theorem 2.C (sketch). — We will use the same notation as in the proof of Theorem 1.E. Let $\pi : X^{\rm sp} \to X$ be the spectral covering induced by ϱ. It is a finite Galois covering with the Galois group H. Let $\overline{X}$ be a smooth projective compactification of X with $D := \overline{X}\backslash X$ a simple normal crossing divisor and π extends to a finite Galois cover $\overline{X^{\rm sp}} \to \overline{X}$. The spectral covering satisfies the following properties:

— there exists forms $\{\eta_1, \ldots, \eta_\ell\} \subset H^0(\overline{X^{\mathrm{sp}}}, \pi^*\Omega^1_{\overline{X}}(\log D))$ such that $\{\eta_1, \ldots, \eta_\ell\}$ coincides with the mutivalued one-forms $\pi^*\{\omega_1, \ldots, \omega_\ell\}$ induced by the ϱ-equivariant pluriharmonic map u with logarithmic energy at infinity constructed in Theorem 1.A.
— Let us denote by $\mathrm{Ram}(\pi)$ the ramification locus of $\pi : X^{\mathrm{sp}} \to X$. Then we have

$$\mathrm{Ram}(\pi) \subset \bigcup_{\eta_i \neq \eta_j} (\eta_i - \eta_j = 0). \tag{2.5.1}$$

— $\{\eta_1, \ldots, \eta_\ell\}$ is invariant under the Galois group H.

Claim 2.5.2. — *The partial quasi-Albanese map $a : X^{\mathrm{sp}} \to A$ induced by $\{\eta_1, \ldots, \eta_\ell\}$ satisfies* $\dim X^{\mathrm{sp}} = \dim a(X^{\mathrm{sp}})$.
This is indeed a crucial fact in the proof of the theorem. It relies on the following lemma whose proof uses the techniques in the Bruhat-Tits buildings.
Lemma 2.5.3 **([CDY22, Lemma 5.3]).** — *Let G be an almost simple algebraic group defined on a non-archimedean local field K. Assume that $\Gamma \subset G(K)$ is a Zariski dense subgroup which is unbounded. If $N \triangleleft \Gamma$ is a normal subgroup which is bounded, then N is finite.*

Let us explain the proof of Claim 2.5.2. Assume by contradiction that $\dim a(X^{\mathrm{sp}}) < \dim X^{\mathrm{sp}}$. Let F be a connected component of a general fiber of a. By Claim 1.5.1, we know that $\eta_i|_F \equiv 0$ for each η_i.

Let $\pi' : \widetilde{X^{\mathrm{sp}}} \to X^{\mathrm{sp}}$ be the universal covering and denote by $\tilde{\pi} : \widetilde{X^{\mathrm{sp}}} \to \widetilde{X}$ be the map between universal covering lifting $\pi : X^{\mathrm{sp}} \to X$. Then $u \circ \tilde{\pi} : \widetilde{X^{\mathrm{sp}}} \to \Delta(G)$ is $\pi^*\varrho$-equivariant harmonic map with logarithmic energy at infinity. Let F' be a connected component of $\pi'^{-1}(F)$. Since $\{\eta_1, \ldots, \eta_\ell\}$ is generically the $(1,0)$-part of complexified differentials of $u \circ \tilde{\pi}$, it follows that $u \circ \tilde{\pi}(F')$ is a point. This implies that $\pi^*\varrho(\mathrm{Im}[\pi_1(F) \to \pi_1(X^{\mathrm{sp}})])$ fixes a point in $\Delta(G)$, hence is bounded.

On the other hand, note that $\pi^*\varrho(\mathrm{Im}[\pi_1(F) \to \pi_1(X^{\mathrm{sp}})])$ is a normal subgroup of $\pi^*\varrho(\pi_1(X^{\mathrm{sp}}))$, which is unbounded as ϱ is unbounded. By Lemma 2.5.3, we conclude that $\pi^*\varrho(\mathrm{Im}[\pi_1(F) \to \pi_1(X^{\mathrm{sp}})])$ is finite.

Since we assume that ϱ is big, $\pi^*\varrho$ is also big. We obtain a contradiction. Hence $\dim a(X^{\mathrm{sp}}) = \dim X^{\mathrm{sp}}$.

Therefore, the logarithmic Kodaira dimension $\bar{\kappa}(X^{\mathrm{sp}}) \geq 0$. Assume that it is not maximal, then for the logarithmic Iitaka fibration $j : X^{\mathrm{sp}} \to J$ has general fibers positive dimensional. Let F be a general fiber of F. Then $a|_F : F \to A$ is generically finite into the image and $\bar{\kappa}(F) = 0$. By Lemma 2.5.4, we conclude that $\pi_1(F)$ is abelian.

Since $\pi^*\varrho(\mathrm{Im}[\pi_1(F) \to \pi_1(X^{\mathrm{sp}})])$ is a normal subgroup of $\pi^*\varrho(\pi_1(X^{\mathrm{sp}}))$ and ϱ is Zariski dense, we conclude that the Zariski closure of $\pi^*\varrho(\mathrm{Im}[\pi_1(F) \to \pi_1(X^{\mathrm{sp}})])$, denoted by N, is a normal subgroup of G^o (the identify connected component of G). Since G is almost simple, it follows that N is finite. Hence $\pi^*\varrho(\mathrm{Im}[\pi_1(F) \to \pi_1(X^{\mathrm{sp}})])$ is finite, contradicting with the assumption that ϱ is big. Therefore, j is birational, and we conclude that X^{sp} is of log general type.

We will spread the positivity from X^{sp} to X to show that X is of log general type. This step is innovative and as far as I know, the method has never appeared before.

Define a section

$$\sigma := \prod_{h \in H} \prod_{\eta_i \neq \eta_j} h^*(\eta_i - \eta_j) \in H^0(\overline{X^{\mathrm{sp}}}, \mathrm{Sym}^M \pi^*\Omega_{\overline{X}}(\log D)),$$

which is non-zero. By eq. (2.5.1), σ vanishes at $\mathrm{Ram}(\pi)$. Since it is invariant under the H-action, it descends to a section

$$\sigma^H \in H^0(\overline{X}, \mathrm{Sym}^M \Omega_{\overline{X}}(\log D))$$

so that $\pi^*\sigma^H = \sigma$. Let $R \subset X$ be the ramification locus of $\pi : X^{\mathrm{sp}} \to X$ and let $\bar{R}$ be its closure in $\overline{X}$. By the purity we know that $\overline{R}$ is a divisor on X. Note that σ^H vanishes at $\overline{R}$. Therefore, it induces a non-trivial morphism

$$\mathcal{O}_{\overline{X}}(R) \to \mathrm{Sym}^M \Omega_{\overline{X}}(\log D). \tag{2.5.2}$$

Since X^{sp} is of log general type, and π is unramified over $X - R$, it follows that $K_{\overline{X}} + D + \overline{R}$ is big. eq. (2.5.2) together with the criterion in [CP19, Corollary 8.7], implies that $K_{\overline{X}} + D$ is big. Therefore, X is of log general type.

We will prove that X is pseudo Picard hyperbolic after we introduce some notions of Nevanlinna theory in Section 2.6. □

Lemma 2.5.4 **([CDY22, Lemma 3.3]).** — *Let $\alpha : X \to \mathcal{A}$ be a (possibly non-proper) morphism from a quasi-projective manifold X to a semi-abelian variety $\mathcal{A}$ with $\bar{\kappa}(X) = 0$. Assume that* $\dim X = \dim \alpha(X)$. *Then $\pi_1(X)$ is abelian and α is proper in codimension one.*

2.6. A theorem of Nevanlinna theory on semiabelian variety

In this section, A is a semi-abelian variety and Y is a Riemann surface with a proper surjective holomorphic map $\pi : Y \to \mathbb{C}_{>\delta}$, where $\mathbb{C}_{>\delta} := \{z \in \mathbb{C} \mid \delta < |z|\}$ with some fixed positive constant $\delta > 0$.

For $r > 2\delta$, define $Y(r) = \pi^{-1}(\mathbb{C}_{>2\delta}(r))$ where $\mathbb{C}_{>2\delta}(r) = \{z \in \mathbb{C} \mid 2\delta < |z| < r\}$. In the following, we assume that $r > 2\delta$. The *ramification counting function* of the covering $\pi : Y \to \mathbb{C}_{>\delta}$ is defined by

$$N_{\mathrm{ram}\,\pi}(r) := \frac{1}{\deg \pi} \int_{2\delta}^{r} \left[\sum_{y \in Y(t)} \mathrm{ord}_y \mathrm{ram}\, \pi \right] \frac{dt}{t},$$

where $\mathrm{ram}\,\pi \subset Y$ is the ramification divisor of $\pi : Y \to \mathbb{C}_{>\delta}$.

Let L be a line bundle on X. Let $f : Y \to X$ be a holomorphic map. We define the order function $T_f(r, L)$ as follows. First suppose that X is smooth. We equip with a smooth hermitian metric h_L, and let $c_1(L, h_L)$ be the curvature form of (L, h_L).

$$T_f(r, L) := \frac{1}{\deg \pi} \int_{2\delta}^{r} \left[\int_{Y(t)} f^* c_1(L, h_L) \right] \frac{dt}{t}.$$

This definition is independent of the choice of the hermitian metric up to a function $O(\log r)$.

***Theorem 2.D* ([CDY22, Theorem 4.2]). —** *Let X be a smooth quasi-projective variety which is of log general type. Assume that there is a morphism $a : X \to A$ such that $\dim X = \dim a(X)$. Then there exists a proper Zariski closed set $\Xi \subsetneq X$ with the following property: Let $f : Y \to X$ be a holomorphic map such that $N_{\mathrm{ram}\,\pi}(r) = O(\log r) + o(T_f(r))||$ and that $f(Y) \not\subset \Xi$. Then f does not have essential singularity over ∞, i.e., there exists an extension $\overline{f} : \overline{Y} \to \overline{X}$ of f, where $\overline{Y}$ is a Riemann surface such that $\pi : Y \to \mathbb{C}_{>\delta}$ extends to a proper map $\overline{\pi} : \overline{Y} \to \mathbb{C}_{>\delta} \cup \{\infty\}$ and $\overline{X}$ is a compactification of X.*

Note that Theorem 2.D is proven by Yamanoi in [Yam15] when X is compact. The proof of Theorem 2.D is based on techniques in Nevanlinna theories in [Yam15]. Compared with the compact case treated in [Yam15], the lack of Poincaré reducibility theorem is a major difficulty to treat the non-compact case. We use a more general "cover" than étale cover to overcome this problem. We refer the readers to [CDY22, Remark 4.35] for the main difficulty and novelty in the non-compact cases. Since the proof of Theorem 2.D is highly involved and unrelated to other aspects of the paper, we choose to omit it. Instead, we present a fundamental result from Nevanlinna theory.

***Claim 2.6.1.* —** *Let $f : Y \to X$ be as above. If the order function $T_f(r, L) = O(\log r)$ as $r \to \infty$, then f does not have essential singularity at infinity.*

In a nutshell, the ultimate goal in proving Theorem 2.D is to estimate the order function $T_f(r, L)$ utilizing Nevanlinna theory tools, such as the logarithmic derivative lemma, the Second Main Theorem, jet differentials, and other related techniques.

In the context of Nevanlinna theory if [CDY22, §4], another crucial result is obtained.

***Theorem 2.6.2* ([CDY22, Corollary 4.32]). —** *Let X be a smooth quasi-projective variety and let $a : X \to A \times S$ be a morphism such that $\dim X = \dim a(X)$, where S is a smooth quasi-projective variety (S can be a point). Write $b : X \to S$ as the composition of a with the projection map $A \times S \to S$. Assume that b is dominant.*

(i) *Suppose S is pseudo Picard hyperbolic. If X is of log general type, then X is pseudo Picard hyperbolic.*
(ii) *Suppose $\mathrm{Sp}_{\mathrm{alg}}(S) \subsetneq S$. If $\mathrm{Sp}_{\mathrm{sab}}(X) \subsetneq X$, then $\mathrm{Sp}_{\mathrm{alg}}(X) \subsetneq X$.*

We will continue the proof of Theorem 2.C using Theorem 2.D.

Proof of Theorem 2.C (continued). — Let $g : \mathbb{D}^* \to X$ be non-constant holomorphic map that is not contained in $\mathrm{Ram}(\pi)$. Then there exists a Riemann surface Y, a proper surjective holomorphic map $p : Y \to \mathbb{D}^*$ and a holomorphic map $f : Y \to X^{\mathrm{sp}}$ such that we have the following commutative diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "p"] & X^{\mathrm{sp}} \arrow[d, "\pi"] \\ \mathbb{D}^* \arrow[r, "g"] & X \end{tikzcd}$$

We first estimate the ramification counting function of $p : Y \to \mathbb{D}^*$.

***Claim 2.6.3* ([CDY22, Proposition 6.9]). —** *There exists a proper Zariski closed subset $\Xi_1 \subsetneq X$ such that if $g(\mathbb{D}^*) \not\subset \Xi_1$, then we have*

$$N_{\mathrm{ram}\,\pi}(r) = O(\log r) + o(T_f(r)).$$

Recall that X^{sp} is of log general type and we have a morphism $a : X^{\mathrm{sp}} \to A$ such that $\dim X^{\mathrm{sp}} = \dim a(X^{\mathrm{sp}})$. Therefore, we apply Theorem 2.D to conclude that there exists a proper Zariski closed subset $\Xi_2 \subsetneq X$ g does not have essential singularity at the origin provided that $g(\mathbb{D}^*) \not\subset \Xi_1 \cup \Xi_2$. This proves that X is pseudo Picard hyperbolic. □

2.7. On the generalized Green-Griffiths-Lang conjecture

Building upon Theorem 2.A, we further investigate Conjecture 2.2.1 and its relation to the non-hyperbolicity locus of a smooth quasi-projective variety X, under the weaker assumption that $\pi_1(X)$ admits a big and reductive representation. Specifically, we introduce four special subsets of X that measure the non-hyperbolicity locus from different perspectives, as defined in Definition 2.7.1. Our main result, given in Theorem 2.E, establishes the equivalence of several properties of the conjugate variety X^σ under the assumption that $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ is a big and reductive representation, and for any automorphism $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. Additionally, we provide a further result regarding the special subsets, as stated in Theorem 2.F.

***Definition 2.7.1* (Special subsets).** — Let X be a smooth quasi-projective variety.

(i) $\mathrm{Sp}_{\mathrm{sab}}(X) := \overline{\bigcup_f f(A_0)}^{\mathrm{Zar}}$, where f ranges over all non-constant rational maps $f : A \dashrightarrow X$ from all semi-abelian varieties A to X such that f is regular on a Zariski open subset $A_0 \subset A$ whose complement $A \backslash A_0$ has codimension at least two;

(ii) $\mathrm{Sp}_{\mathrm{h}}(X) := \overline{\bigcup_f f(\mathbb{C})}^{\mathrm{Zar}}$, where f ranges over all non-constant holomorphic maps from $\mathbb{C}$ to X;

(iii) $\mathrm{Sp}_{\mathrm{alg}}(X) := \overline{\bigcup_V V}^{\mathrm{Zar}}$, where V ranges over all positive-dimensional closed subvarieties of X which are not of log general type;

(iv) $\mathrm{Sp}_{\mathrm{p}}(X) := \overline{\bigcup_f f(\mathbb{D}^*)}^{\mathrm{Zar}}$, where f ranges over all holomorphic maps from the punctured disk $\mathbb{D}^*$ to X with essential singularity at the origin, i.e., f has no holomorphic extension $\bar{f} : \mathbb{D} \to \overline{X}$ to a projective compactification $\overline{X}$.

The first two sets $\mathrm{Sp}_{\mathrm{sab}}(X)$ and $\mathrm{Sp}_{\mathrm{h}}(X)$ are introduced by Lang for the compact case. He made the following two conjectures:

— $\mathrm{Sp}_{\mathrm{sab}}(X) \subsetneq X$ if and only if X is of general type.

— $\mathrm{Sp}_{\mathrm{sab}}(X) = \mathrm{Sp}_{\mathrm{h}}(X)$.

The first assertion implicitly includes the following third conjecture:

— $\mathrm{Sp}_{\mathrm{sab}}(X) = \mathrm{Sp}_{\mathrm{alg}}(X)$.

The original two conjectures imply the famous strong Green-Griffiths conjecture that varieties of (log) general type are pseudo Brody hyperbolic. Here we note that, by definition, X is pseudo Brody hyperbolic if and only if $\mathrm{Sp}_{\mathrm{h}}(X) \subsetneq X$. Similarly, X is pseudo Picard hyperbolic if and only if $\mathrm{Sp}_{\mathrm{p}}(X) \subsetneq X$.

Theorem 2.E. — *Let X be a complex smooth quasi-projective variety and let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a big representation where K is any infinite field.*

(i) [CDY22, Theorem C] *If* $\operatorname{char} K = 0$ *and ϱ is reductive, then for any automorphism $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, Conjecture 2.2.1 holds for the conjugate variety $X^\sigma := X \times_\sigma \operatorname{Spec} \mathbb{C}$, where $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$.*

(ii) [DY24, Theorem C] *If* $\operatorname{char} K > 0$*, then Conjecture 2.2.1 holds for X.*

Let us explain how to obtain Theorem 2.E.(i) via Theorem 2.6.2 and Theorem 2.A.

Proof of Theorem 2.E.(i) (sketch). — We may assume that $K = \mathbb{C}$. Let G be the Zariski closure of ϱ, which is a complex reductive group as we assume that ϱ is reductive. We may assume that G is connected after we replace X by a finite étale cover. Let $\mathcal{D}G$ be the derived group of G and let $R(G)$ be the radical of G. Define $G_1 := G/R(G)$ which is semisimple and $G_2 := G/\mathcal{D}G$ which is a torus. Then $G \to \mathcal{D}G \times T$ is an isogeny.

Consider the representation $\sigma : \pi_1(X) \to G_1(\mathbb{C})$ by composing ϱ with the quotient $G \to G_1$. Then σ is Zariski dense. By Proposition 2.7.2, after replacing X by a finite étale cover and a birational proper modification, there exists a dominant morphism $f : X \to Y$ with connected general fibers, and a big representation $\tau : \pi_1(Y) \to G(K)$ such that $f^*\tau = \sigma$. Since σ is Zariski dense and G is connected, it follows that τ is also Zariski dense. Therefore, by Theorem 2.A, we conclude that Y is pseudo Picard hyperbolic and strongly of log general type, if it is not a point.

Consider the morphism $(f, \mathrm{alb}_X) : X \to Y \times A$, where $\mathrm{alb}_X : X \to A$ is the quasi-Albanese map of X. Since ϱ is big, we can show that $g := (f, \mathrm{alb}_X)$ is generically finite into its image. Hence we apply Theorem 2.E to conclude that Conjecture 2.2.1 holds for X. □

The following factorization was proved in [CDY22] in the case where $\operatorname{char} K = 0$ and in general in [DY24, Proposition 5.8]. It will be used throughout this memoir.

***Proposition 2.7.2* ([CDY22, DY24]).** — *Let X be a quasi-projective normal variety. Let $\varrho : \pi_1(X) \to G(K)$ be a representation, where G is a linear algebraic group defined over any field K. Then there is a diagram*

$$\begin{array}{ccccc} \widetilde{X} & \xrightarrow{\mu} & \widehat{X} & \xrightarrow{\nu} & X \\ \big\downarrow{\scriptstyle f} & & & & \\ Y & & & & \end{array}$$

where Y and $\widetilde{X}$ are quasi-projective manifolds, and

(a) $\nu : \widehat{X} \to X$ is a finite étale cover;

(b) $\mu : \widetilde{X} \to \widehat{X}$ is a birational proper morphism;

(c) $f : \widetilde{X} \to Y$ *is a dominant morphism with connected general fibers;*

such that there exists a big representation $\tau : \pi_1(Y) \to G(K)$ *with* $f^*\tau = (\nu \circ \mu)^*\varrho$.

In Theorems 2.A and 2.E, the hyperbolicity of conjugate varieties X^σ for any $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ is claimed. It is based on the following result.

***Proposition 2.7.3* ([CDY22, Proposition 8.6]). —** *Let X be a smooth quasi-projective variety and let* $\rho : \pi_1(X) \to \mathrm{GL}_n(\mathbb{C})$ *be a representation. Let* $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. *Then there exists a representation* $\tau : \pi_1(X^\sigma) \to \mathrm{GL}_n(\mathbb{C})$ *such that the Zariski closures satisfy*

$$\overline{\rho(\pi_1(X))}^{\mathrm{Zar}} = \overline{\tau(\pi_1(X^\sigma))}^{\mathrm{Zar}}. \tag{2.7.1}$$

More precisely, τ satisfies the following property: If $Y \to X$ is a morphism from a smooth quasi-projective variety Y, we have

$$\overline{\rho(\mathrm{Im}[\pi_1(Y) \to \pi_1(X)])}^{\mathrm{Zar}} = \overline{\tau(\mathrm{Im}[\pi_1(Y^\sigma) \to \pi_1(X^\sigma)])}^{\mathrm{Zar}}. \tag{2.7.2}$$

In particular, if ρ is big (resp. large), then τ is big (resp. large).

Proposition 2.7.3 is not obvious since in general, the fundamental groups of the complex variety X and X^σ may be quite different, as demonstrated by the famous examples of Serre [Ser64]. Despite this, their algebraic fundamental groups, which are the profinite completions of the topological fundamental groups, are canonically isomorphic. Our proof of Proposition 2.7.3 is based on this fact.

Proof of Proposition 2.7.3 (sketch). — Since $\pi_1(X)$ is finitely generated, we can find a finite subset $S \subset \mathbb{C}$ such that for each element $\gamma \in \pi_1(X)$, each entry of the matrix $\varrho(\gamma)$ lies on the smallest subring of $\mathbb{C}$ generated by S. Let $\mathbb{Q}(S)$ be the smallest subfield of $\mathbb{C}$ containing S. We now apply Cassels' p-adic embedding theorem (cf. [Cas76]) to show that there exist a prime number $p \in \mathbb{N}$ and embeddings $\iota : \mathbb{Q}(S) \hookrightarrow \mathbb{Q}_p$ and $\mu : \mathbb{Q}_p \hookrightarrow \mathbb{C}$ of fields such that

$$|\iota(x)|_p = 1 \tag{2.7.3}$$

for all $x \in S$ and $\mu \circ \iota$ is the identity map. Via such embedding, (2.7.3) yields that e $\mathrm{GL}_n(\mathbb{Z}_p)$. Since $\mathrm{GL}_n(\mathbb{Z}_p)$ is a profinite group, ρ extends to $\widehat{\rho} : \widehat{\pi_1(X)} \to \mathrm{GL}_n(\mathbb{Z}_p)$, where $\widehat{\pi_1(X)}$ is the profinite completion of $\pi_1(X)$. Since $\widehat{\pi_1(X)}$ is isomorphic to $\widehat{\pi_1(X^\sigma)}$, we can find the desired τ from the above construction. □

As for the second and third conjectures of Lang, we obtain the following theorem under the stronger assumption when $\pi_1(X)$ admits a *large* and reductive representation.

***Theorem 2.F* ([CDY22, Theorem D]). —** *Let X be a smooth quasi-projective variety and let* $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ *be a large and reductive representation. Then for any automorphism* $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$,

(i) *the four special subsets defined in Definition 2.7.1 are the same, i.e.,*

$$\mathrm{Sp}_{\mathrm{alg}}(X^\sigma) = \mathrm{Sp}_{\mathrm{sab}}(X^\sigma) = \mathrm{Sp}_{\mathrm{h}}(X^\sigma) = \mathrm{Sp}_{\mathrm{p}}(X^\sigma).$$

(ii) *These special subsets are conjugate under automorphism σ, i.e.,*

$$\mathrm{Sp}_\bullet(X^\sigma) = \mathrm{Sp}_\bullet(X)^\sigma,$$

where $\mathrm{Sp}_\bullet$ *denotes any of* $\mathrm{Sp}_{\mathrm{alg}}$, $\mathrm{Sp}_{\mathrm{sab}}$, Sp_{h} *or* Sp_{p}.

For representation into algebraic groups in positive characteristic, we have a stronger result. We first introduce a notion of special loci $\mathrm{Sp}(\varrho)$ for any big representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ which measure the "non-large locus" of ϱ.

***Definition 2.7.4.* —** Let X be a smooth quasi-projective variety. Let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a representation where K is a field. We define

$$\mathrm{Sp}(\varrho) := \overline{\bigcup_{\iota : Z \hookrightarrow X} Z},$$

where $\iota : Z \hookrightarrow X$ ranges over all positive dimensional closed subvarieties of X such that $\iota^*\varrho(\pi_1(Z))$ is finite.

Subsequently, we establish a theorem concerning these special subsets, thereby refining Theorem 2.E.

***Theorem 2.G* ([DY24, Theorem D]). —** *Let X be a quasi-projective normal variety. Let* $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ *be a big representation where K is a field of positive characteristic. Then* $\mathrm{Sp}(\varrho)$ *is a proper Zariski closed subset of X, and we have*

$$\mathrm{Sp}_{\mathrm{alg}}(X) \setminus \mathrm{Sp}(\varrho) = \mathrm{Sp}_{\mathrm{alg}}(X) \setminus \mathrm{Sp}(\varrho) = \mathrm{Sp}_{\mathrm{p}}(X) \setminus \mathrm{Sp}(\varrho) = \mathrm{Sp}_{\mathrm{h}}(X) \setminus \mathrm{Sp}(\varrho).$$

We have $\mathrm{Sp}_\bullet(X) \subsetneq X$ *if and only if X is of log general type, where* $\mathrm{Sp}_\bullet$ *denotes any of* $\mathrm{Sp}_{\mathrm{sab}}$, $\mathrm{Sp}_{\mathrm{alg}}$, Sp_{h} *or* Sp_{p}.

2.8. On positive characteristic case

In previous subsections, we provided an outline of the proofs for Theorems 2.A.(i) and 2.E.(i). We defer the discussion of the proof for Theorems 2.A and 2.E in the case where $\operatorname{char} K > 0$ until Chapter 4. Indeed, apart from the key result Theorem 2.C, another crucial aspect of the proof involves understanding the structure of the Shafarevich morphism for representations in positive characteristic, as we will discuss in Chapter 4.

CHAPTER 3

ON THE SHAFAREVICH CONJECTURE (I)

3.1. Shafarevich conjecture: some histories

In his famous textbook "Basic Algebraic Geometry" [Sha77, p 407], Shafarevich raised the following tantalizing conjecture.

***Conjecture 3.1.1* (Shafarevich).** — *Let X be a complex projective variety. Then its universal covering is holomorphically convex.*

Recall that a complex normal space X is *holomorphically convex* if it satisfies the following condition: for each compact $K \subset X$, its *holomorphic hull*

$$\left\{ x \in X \mid |f(x)| \leq \sup_K |f|, \forall f \in \mathcal{O}(X) \right\},$$

is compact. X is *Stein* if it is holomorphically convex and holomorphically separable, i.e. for distinct x and y in X, there exists $f \in \mathcal{O}(X)$ such that $f(x) \neq f(y)$. By the Cartan-Remmert theorem, a complex space X is holomorphically convex if and only if it admits a proper surjective holomorphic mapping onto some Stein space.

The study of Conjecture 3.1.1 for smooth projective surfaces has been a subject of extensive studies since the mid-1980s. Gurjar-Shastri [GS85] and Napier [Nap90] initiated this investigation, while Kollár [Kol93] and Campana [Cam94] independently explored the conjecture in the 1990s, utilizing the tools of Hilbert schemes and Barlet cycle spaces. In 1994, Katzarkov discovered that non-abelian Hodge theories developed by Simpson [Sim92] and Gromov-Schoen [GS92] can be utilized to prove Conjecture 3.1.1. His initial work [Kat97] demonstrated Conjecture 3.1.1 for projective varieties with nilpotent fundamental groups. Shortly thereafter, he and Ramachandran [KR98] successfully established Conjecture 3.1.1 for smooth projective surfaces whose fundamental groups admit a faithful Zariski-dense representation in a reductive complex algebraic group.

In [Eys04], Eyssidieux made a significant breakthrough on the Shafarevich conjecture. He explored the previous work [KR98, Kat97] and fully developed the non-abelian Hodge theories [Sim92, Sim93a, Sim93b, GS92] to study the Shafarevich conjecture. He also discovered that the celebrated work of Demailly-Păun [DP04] on the characterization of the Kähler cone can be applied crucially in the proof of Shafarevich conjecture. In [Eys04] Eyssidieux proved that Conjecture 3.1.1 holds for any *smooth* projective variety whose fundamental group possesses a faithful representation that is Zariski dense in a reductive complex algebraic group. This result is commonly referred to as the "*Reductive Shafarevich conjecture*".

It is worth emphasizing that the work of Eyssidieux [Eys04] is not only ingenious but also highly significant in subsequent research. It serves as a foundational basis for advancements in the linear Shafarevich conjecture [EKPR12] and the exploration of compact Kähler cases [CCE15]. In [EKPR12], the authors raised the question of whether their theorem could be extended to projective normal varieties. In our work [DYK23], we address this question, following the general strategy outlined in [Eys04] to study the Shafarevich conjecture.

Currently, all known works have only considered *complex* linear representations of fundamental groups. It is natural to study representations of fundamental groups of algebraic varieties into general linear groups in positive characteristic. In [DY24], we consider this problem along with exploring hyperbolicity and various algebro-geometric properties of these algebraic varieties. In this chapter and Chapter 4, I will recall the main results proven by myself, Yamanoi and Katzarkov in [DYK23, DY24].

3.2. Shafarevich morphism and Shafarevich map

Let X be a projective normal variety. If its universal covering is holomorphically convex, it is not difficult to show that there exists a fibration (i.e. surjective proper morphism with connected fibers) $\mathrm{sh}_X : X \to \mathrm{sh}(X)$ satisfying the following property: for any closed subvariety $Z \subset X$, $\mathrm{Im}[\pi_1(Z) \to \pi_1(X)]$ is finite if and only if $\mathrm{sh}_X(Z)$ is a point. Such morphism sh_X is called the Shafarevich morphism by Kollár in [Kol93, Kol95]. More generally, we introduce the following definition.

***Definition 3.2.1* (Shafarevich morphism). —** Let X be a quasi-projective normal variety, and let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a linear representation where K is any field. A dominant holomorphic map $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$ to a complex normal space $\mathrm{Sh}_\varrho(X)$ whose general fibers are connected is called the *Shafarevich morphism* of ϱ if for any closed subvariety $Z \subset X$, $\mathrm{sh}_\varrho(Z)$ is a point if and only if $\varrho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)])$ is finite. Here Z^{norm} denotes the normalization of Z.

Note that the Shafarevich morphism is not unique unless X is projective.

While it is still an open question about the existence of Shafarevich morphism, Campana [Cam94] and Kollár [Kol93] independently proved the existence of *Shafarevich maps*, which are birational to Shafarevich morphisms. Their theorem can be summarized as follows.

***Theorem 3.2.2* ([Kol93]). —** *Let X be any quasi-projective normal variety X, and let H be any normal subgroup $H \triangleleft \pi_1(X)$. Then there exists a dominant rational map $\mathrm{sh}_X^H : X \dashrightarrow \mathrm{Sh}^H(X)$ such that*

(i) *the indeterminacy locus of sh_X^H does not dominate $\mathrm{Sh}^H(X)$;*
(ii) *the general fibers of sh_X^H are connected;*
(iii) *for any closed subvariety Z of X containing a very general point of X, $\mathrm{sh}_X^H(Z)$ is a point if and only if $\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)/H]$ is finite.*

The above rational map $\mathrm{sh}^H : X \dashrightarrow \mathrm{Sh}^H(X)$ is called the *Shafarevich map* of (X, H). If $H = \{1\}$, we simply write $\mathrm{sh}_X : X \dashrightarrow \mathrm{Sh}(X)$ for the Shafarevich map. In [Cam94], Campana also constructed the Shafarevich map for compact Kähler manifolds (which is also called Γ-reduction). The proofs of their theorems are based on cycle theoretic methods. Therefore, the work [Cam94, Kol93] did not provide the precise structure of the Shafarevich morphism.

In [Cam94], Campana introduced the following definition.

***Definition 3.2.3* (Γ-dimension). —** Let X be a projective normal variety. The Γ-dimension of X is defined to be $\dim \mathrm{Sh}(X)$, where $\mathrm{sh}_X : X \dashrightarrow \mathrm{Sh}(X)$ is the Shafarevich map constructed in Theorem 3.2.2.

3.3. Existence of the Shafarevich morphism and its property

3.3.1. Main results. —

***Theorem 3.A* ([DYK23, Theorem A], [DY24, Theorem A]). —** *Let X be a quasi-projective normal variety and $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a linear representation, where K is any field. When $\mathrm{char}\, K = 0$, we assume additionally that ϱ is reductive (i.e. the Zariski closure of $\varrho(\pi_1(X))$ is a reductive group).*

Then there exists a dominant holomorphic map $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$ over a normal complex space $\mathrm{Sh}_\varrho(X)$ with connected general fibers such that for any connected Zariski closed subset $Z \subset X$, the following properties are equivalent:

(a) *$\mathrm{sh}_\varrho(Z)$ is a point;*
(b) *$\varrho(\mathrm{Im}[\pi_1(Z) \to \pi_1(X)])$ is finite;*
(c) *for each irreducible component Z_o of Z, $\varrho^{ss}(\mathrm{Im}[\pi_1(Z_o^{\mathrm{norm}}) \to \pi_1(X)])$ is finite, where $\varrho^{ss} : \pi_1(X) \to \mathrm{GL}_N(\bar{K})$ is the semisimplification of ϱ and Z_o^{norm} denotes the normalization of Z_o.*

Moreover, when $\mathrm{char}\, K > 0$, $\mathrm{Sh}_\varrho(X)$ is quasi-projective and sh_ϱ is an algebraic morphism.

Note that when $\mathrm{char}\, K = 0$, a representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ is reductive if and only if it is semisimple. This fails in positive characteristic.

We ask whether the Shafarevich morphism in Theorem 3.A is algebraic in the case where $\mathrm{char}\, K = 0$. This seems quite difficult problem. Nevertheless, we can prove a weaker version that $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$ is algebraic in the function field level.

***Theorem 3.B*. —** *Let X be a quasi-projective normal variety and $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ be a reductive representation. For the Shafarevich morphism $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$ constructed in Theorem 3.A, there exists*

(a) *a proper bimeromorphic morphism $\sigma : S \to \mathrm{Sh}_\varrho(X)$ from a smooth quasi-projective variety S;*
(b) *a proper birational morphism $\mu : Y \to X$ from a smooth quasi-projective variety Y;*
(c) *an algebraic morphism $f : Y \to S$ with general fibers connected;*

such that we have the following commutative diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\mu} & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{sh}_\varrho} \\ S & \xrightarrow{\sigma} & \mathrm{Sh}_\varrho(X) \end{array}$$

Furthermore, if X is smooth, then there exists a smooth partial compactification X' of X such that sh_ϱ *extends to a proper surjective holomorphic fibration* $\overline{\mathrm{sh}}_\varrho : X' \to \mathrm{Sh}_{\varrho_0}(X)$*:*

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "\mathrm{sh}_\varrho"'] & X' \arrow[d, "\overline{\mathrm{sh}}_\varrho"] \\ \mathrm{Sh}_\varrho(X) \arrow[r, equal] & \mathrm{Sh}_{\varrho_0}(X) \end{tikzcd}$$

Let us sketch the idea of proof of Theorem 3.B. By Theorem 3.2.2, we know that the Shafarevich map $\mathrm{sh}^{\ker\varrho} : X \dashrightarrow \mathrm{Sh}^{\ker\varrho}(X)$ exists, which is a rational dominant map over a quasi-projective normal variety. We can show that it is birational to the Shafarevich morphism sh_ϱ of ϱ.

We further conjecture that $\mathrm{Sh}_\varrho(X)$ is quasi-projective and sh_ϱ is an algebraic morphism. Our conjecture is motivated by Griffiths' conjecture, which predicted the same result when ϱ underlies a $\mathbb{Z}$-VHS. We show this in what follows.

Let $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{Z})$ be a representation underlying a $\mathbb{Z}$ variations of Hodge structure (VHS for short) which has infinite monodromy at infinity (see Definition 3.4.5). By a theorem of Griffiths [Gri70], its period map $p : X \to \mathscr{D}/\Gamma$ is proper, where $\mathscr{D}$ is the period domain of this VHS and $\Gamma := \varrho(\pi_1(X))$ is the monodromy group. Hence by the Remmert theorem, $p(X)$ is a closed subvariety of $\mathscr{D}$. The Griffiths conjecture predicted that the image $p(X)$ is algebraic and the period map is algebraic.

We can show that the Shafarevich morphism $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$ is the Stein factorization of the period map p. Therefore, the algebraicity of $\mathrm{Sh}_\varrho(X)$ and sh_ϱ is a consequence of the Griffiths conjecture. Let us mention that Griffiths' conjecture was recently proved by Baker-Brunebarbe-Tsimerman [BBT23] using o-minimality theory.

Based on Theorem 3.A, in [DYK23] we construct the Shafarevich morphism for families of representations.

Corollary 3.C. — *Let X be a quasi-projective normal variety. Let Σ be a (non-empty) set of reductive representations $\varrho : \pi_1(X) \to \mathrm{GL}_{N_\varrho}(\mathbb{C})$. Then there is a dominant holomorphic map $\mathrm{sh}_\Sigma : X \to \mathrm{Sh}_\Sigma(X)$ with general fibers connected onto a complex normal space such that for closed subvariety $Z \subset X$, $\mathrm{sh}_\Sigma(Z)$ is a point if and only if $\varrho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)])$ is finite for every $\varrho \in \Sigma$.*

3.3.2. $\mathbb{R}^*$-action on character varieties and Mochizuki's ubiquity. — We recall some techniques in the non-abelian Hodge theory by Simpson and Mochizuki. Consider a smooth projective variety $\overline{X}$ equipped with a simple normal crossing divisor D. We define X as the complement of D in $\overline{X}$. Additionally, we fix an ample line bundle L on $\overline{X}$. Let $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ be a reductive representation.

By [Moc07b], $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ is a reductive representation if and only if there exists a tame pure imaginary harmonic bundle (E, θ, h) on X such that $(E, \nabla_h + \theta + \theta_h^\dagger)$ is flat, with the monodromy representation being precisely ϱ. Here ∇_h is the Chern connection of (E, h) and $\theta_h^\dagger$ is the adjoint of θ with respect to h. Let $(\boldsymbol{E}_*, \theta)$ be the prolongation of (E, θ) on $\overline{X}$ defined in [Moc07a]. By [Moc06, Theorem 1.4], $(\boldsymbol{E}_*, \theta)$ is a μ_L-polystable regular filtered Higgs bundle on $(\overline{X}, D)$ with trivial characteristic numbers. Therefore, for any $t \in \mathbb{C}^*$, $(\boldsymbol{E}_*, t\theta)$ be also a μ_L-polystable regular filtered Higgs bundle on $(\overline{X}, D)$ with trivial characteristic numbers. By [Moc06, Theorem 9.4], there is a pluriharmonic metric h_t for $(E, t\theta)$ adapted to the parabolic structures of $(\boldsymbol{E}_*, t\theta)$. Then $(E, t\theta, h_t)$ is a harmonic bundle and thus the connection $\nabla_{h_t} + t\theta + \bar{t}\theta_{h_t}^\dagger$ is flat. Here ∇_{h_t} is the Chern connection for (E, h_t) and $\theta_{h_t}^\dagger$ is the adjoint ot θ with respect to h_t. Let us denote by $\varrho_t : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ the monodromy representation of $\nabla_{h_t} + t\theta + \bar{t}\theta_{h_t}^\dagger$. It should be noted that the representation ϱ_t is well-defined up to conjugation. As a result, the $\mathbb{C}^*$-action is only well-defined over $M_{\mathrm{B}}(X, N)$ and we shall denote it by

$$t.[\varrho] := [\varrho_t] \quad \text{for any } t \in \mathbb{C}^*.$$

It is important to observe that unlike the compact case, ϱ_t is not necessarily reductive in general, even if the original representation ϱ is reductive. However, if $t \in \mathbb{R}^*$, $(E, t\theta)$ is also pure imaginary and by Mochizuki's theorem [Moc07b], ϱ_t is reductive. Nonetheless, we can obtain a family of (might not be semisimple) representations $\{\varrho_t : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})\}_{t \in \mathbb{C}^*}$. By [Moc06, Proofs of Theorem 10.1 and Lemma 10.2] we have

Lemma 3.3.1. — *The map*

$$\Phi : \mathbb{R}^* \to M_{\mathrm{B}}(\pi_1(X), N)$$
$$t \mapsto [\varrho_t]$$

is continuous. $\Phi(\{t \in \mathbb{R}^* \mid |t| < 1\})$ *is relatively compact in* $M_{\mathrm{B}}(\pi_1(X), N)$. □

Note that Lemma 3.3.1 can not be seen directly from [Moc06, Lemma 10.2] as he did not treat the character variety in his paper. Indeed, based on Uhlenbeck's compactness in Gauge theory, Mochizuki's proof can be read as follows: for any $t_n \in \mathbb{R}^*$ converging to 0, after subtracting to a subsequence, there exists some $\varrho_0 : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ and $g_n \in \mathrm{GL}_N(\mathbb{C})$ such that $\lim_{n\to\infty} g_n^*\varrho_{t_n} = \varrho_0$ in the representation variety $R(\pi_1(X), \mathrm{GL}_N)(\mathbb{C})$. Moreover, one can check that ϱ_0 corresponds to some tame pure imaginary harmonic bundle, and thus it is reductive (cf. [BDDM22] for a more detailed study). For this reason, we can see that it will be more practical to work with $\mathbb{R}^*$-action instead of $\mathbb{C}^*$-action as the representations we encounter are all reductive.

When X is compact, Simpson proved that $\lim_{t\to 0}\Phi(t)$ exists and underlies a $\mathbb{C}$-VHS. However, this result is current unknown in the quasi-projective setting. Instead, Mochizuki proved that, we achieve a $\mathbb{C}$-VHS after finite steps of deformations. Let us recall it briefly and the readers can refer to [Moc06, §10.1] for more details.

Let $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ be a reductive representation. Then there exists a tame and pure imaginary harmonic bundle (E,θ,h) corresponding to ϱ. Then the induced regular filtered Higgs bundle $(\boldsymbol{E}_*,\theta)$ on $(\overline{X},D)$ is μ_L-polystable with trivial characteristic numbers. Hence we have a decomposition

$$(\boldsymbol{E}_*,\theta) = \oplus_{j\in\Lambda}(\boldsymbol{E}_{j*},\theta_j)\otimes\mathbb{C}^{m_j}$$

where $(\boldsymbol{E}_{j*},\theta_j)$ is μ_L-stable regular filtered Higgs bundle with trivial characteristic numbers. Put $r(\varrho) := \sum_{j\in\Lambda} m_j$. Then $r(\varrho) \le \operatorname{rank} E$. For any $t\in\mathbb{R}^*$, we know that $(E,t\theta)$ is still tame and pure imaginary and thus ϱ_t is also reductive. Since $\varrho(\{t\in\mathbb{R}^* \mid |t|<1\})$ is relatively compact, then there exists some $t_n\in\mathbb{R}^*$ which converges to zero such that $\lim_{t_n\to 0}[\varrho_{t_n}]$ exists, denoting by $[\varrho_0]$. Moreover, ϱ_0 corresponds to some tame harmonic bundle. There are two possibilities:

- For each $j\in\Lambda$, $(\boldsymbol{E}_{j*},t_n\theta_j)$ converges to some μ_L-stable regular filtered Higgs sheaf (cf. [Moc06, p. 96] for the definition of convergence). Then by [Moc06, Proposition 10.3], ϱ_0 underlies a $\mathbb{C}$-VHS.
- For some $i\in\Lambda$, $(\boldsymbol{E}_{i*},t_n\theta_i)$ converges to some μ_L-semistable regular filtered Higgs sheaf, but not μ_L-stable. Then by [Moc06, Lemma 10.4], we have $r(\varrho) < r(\varrho_0)$. In other words, letting ϱ_i be the representation corresponding to $(\boldsymbol{E}_{j*},\theta_j)$ and $\varrho_{i,t}$ be the deformation under $\mathbb{C}^*$-action. Then $\lim_{n\to\infty}\varrho_{i,t_n}$ exists, denoted by $\varrho_{i,0}$. Then $\varrho_{i,0}$ corresponds to some tame harmonic bundle, and thus also a μ_L-polystable regular filtered Higgs bundle which is not stable. In this case, we further deform ϱ_0 until we achieve Case 1.

In summary, Mochizuki's result implies the following, which we shall refer to as *Mochizuki's ubiquity*, analogous to the term *Simpson's ubiquity* for the compact case (cf. [Sim91]).

Theorem 3.3.2. — *Let X be a smooth quasi-projective variety. Consider $\mathfrak{C}$, a Zariski closed subset of $M_{\mathrm{B}}(X,G)(\mathbb{C})$, where G denotes a complex reductive group. If $\mathfrak{C}$ is invariant under the action of $\mathbb{R}^*$ defined above, then each geometrically connected component of $\mathfrak{C}(\mathbb{C})$ contains a $\mathbb{C}$-point $[\varrho]$ such that $\varrho : \pi_1(X)\to\mathrm{GL}_N(\mathbb{C})$ is a reductive representation that underlies a $\mathbb{C}$-variation of Hodge structure.* □

3.4. Constructing the Shafarevich morphism

In this section we shall outline the idea of the proof of Theorem 3.A. For simplicity, we shall assume that the absolutely constructible subset $\mathfrak{C}$ is $M_{\mathrm{B}}(X,N) := M_{\mathrm{B}}(\pi_1(X),\mathrm{GL}_N)$.

3.4.1. Simultaneous Stein factorization. —

Lemma 3.4.1. — *Let V be a quasi-projective normal variety and let $(f_\lambda : V\to S_\lambda)_{\lambda\in\Lambda}$ be a family of morphisms into quasi-projective varieties S_λ. Then there exist a normal projective variety S_∞ and a morphism $f_\infty : V\to S_\infty$ such that*

- *for every subvariety $Z\subset V$, $f_\infty(Z)$ is a point if and only if $f_\lambda(Z)$ is a point for every $\lambda\in\Lambda$, and*
- *there exist $\lambda_1,\ldots,\lambda_n\in\Lambda$ such that $f_\infty : V\to S_\infty$ is the quasi-Stein factorization of $(f_1,\ldots,f_n) : V\to S_{\lambda_1}\times\cdots S_{\lambda_n}$.*

3.4.2. Factorizing through non-rigidity. — In this subsection, X is assumed to be a smooth quasi-projective variety. Note that the character variety $M_{\mathrm{B}}(X,N) := M_{\mathrm{B}}(\pi_1(X),\mathrm{GL}_N)$ is a finite type affine scheme defined over $\mathbb{Z}$.

Let us utilize Lemma 3.4.1 and Theorem 1.E to construct a reduction map $s_{\mathrm{fac}} : X\to S_{\mathrm{fac}}$, which allows us to factorize non-rigid representations into those underlying $\mathbb{C}$-VHS with discrete monodromy.

Definition 3.4.2. — The *reduction map* $s_{\mathrm{fac}} : X\to S_{\mathrm{fac}}$ is obtained through the simultaneous Stein factorization of the reductions $\{s_\tau : X\to S_\tau\}_{[\tau]\in M_{\mathrm{B}}(X,N)}$, employing Lemma 3.4.1. Here $\tau : \pi_1(X)\to\mathrm{GL}_N(K)$ ranges over all reductive representations with K a non-archimedean local field of characteristic zero and $s_\tau : X\to S_\tau$ is the reduction map constructed in Theorem 1.E.

Note that $s_{\mathrm{fac}} : X\to S_{\mathrm{fac}}$ is a dominant morphism with connected general fibers. For every subvariety $Z\subset X$, $s_{\mathrm{fac}}(Z)$ is a point if and only if $s_\tau(Z)$ is a point for any reductive representation $\tau : \pi_1(X)\to\mathrm{GL}_N(K)$ with K a non-archimedean local field.

***Proposition 3.4.3* ([DYK23, Proposition 3.9]).** — *Let X be a smooth quasi-projective variety. Let $f : F\to X$ be a morphism from a quasi-projective* normal *variety F such that $s_{\mathrm{fac}}\circ f(F)$ is a point. Let $\{\tau_i : \pi_1(X)\to\mathrm{GL}_N(\mathbb{C})\}_{i=1,2}$ be reductive representations such that $[\tau_1]$ and $[\tau_2]$ are in the same geometric connected component of $M_{\mathrm{B}}(X,N)(\mathbb{C})$. Then $\tau_1\circ\iota$ is conjugate to $\tau_2\circ\iota$, where $\iota : \pi_1(F)\to\pi_1(X)$ is the homomorphism of fundamental groups induced by f. In other words, $j(M_{\mathrm{B}}(X,N))$ is zero-dimensional, where $j : M_{\mathrm{B}}(X,N)\to M_{\mathrm{B}}(F,N)$ is the natural morphism of character varieties induced by $\iota : \pi_1(F)\to\pi_1(X)$.*

The proof of Proposition 3.4.3 is based on Lemma 1.4.3 and Claim 1.4.1.

By Proposition 3.4.3, we note that for any reductive representation $\tau : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$, $[\tau \circ \iota] \in M_{\mathrm{B}}(F, N)$ is invariant under $\mathbb{R}^*$-action for any reductive $\tau : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$. Hence it underlies a $\mathbb{C}$-VHS by Theorem 3.3.2. This means that, the restriction of τ on the normalization of each fiber of s_{fac} underlies a $\mathbb{C}$-VHS. In next proposition, we will see that our construction also incorporates the *integrality* (i.e. $\tau \circ \iota$ is moreover a complex direct factor of a $\mathbb{Z}$-VHS).

***Proposition 3.4.4* ([DYK23, Proposition 3.12]). —** *Let X be a smooth quasi-projective variety. Then there exist reductive representations $\{\sigma_i^{VHS} : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})\}_{i=1,\ldots,m}$ such that each σ_i^{VHS} underlies a $\mathbb{C}$-VHS, and for a morphism $\iota : Z \to X$ from any quasi-projective* normal *variety Z with $s_{\mathrm{fac}} \circ \iota(Z)$ being a point, the following properties hold:*

(i) *For $\sigma := \oplus_{i=1}^m \sigma_i^{VHS}$, $\iota^*\sigma(\pi_1(Z))$ is discrete in $\prod_{i=1}^m \mathrm{GL}_N(\mathbb{C})$.*
(ii) *For each reductive representation $\tau : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$, $\iota^*\tau$ is conjugate to some $\iota^*\sigma_i^{VHS}$.*
(iii) *For each σ_i^{VHS}, there exists a reductive representation $\tau : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ such that $\iota^*\tau$ is conjugate to $\iota^*\sigma_i^{VHS}$.*

This proposition is crucial in the proof of Theorems 3.A and 3.D. The construction of factorization map $s_{\mathrm{fac}} : X \to S_{\mathrm{fac}}$ is new compared with that in [Eys04].

3.4.3. Infinite monodromy at infinity. — When considering a non-compact quasi-projective variety X, it is important to note that the Shafarevich conjecture fails in simple examples. For instance, take $X := A\backslash\{0\}$, where A is an abelian surface. Its universal covering $\widetilde{X}$ is $\mathbb{C}^2 - \Gamma$, where Γ is a lattice in $\mathbb{C}^2$. Then $\widetilde{X}$ is not holomorphically convex. Therefore, additional conditions on the fundamental groups at infinity are necessary to address this issue.

***Definition 3.4.5* (Infinity monodromy at infinity). —** Let X be a quasi-projective normal variety and let $\overline{X}$ be a projective compactification of X. We say a subset $M \subset M_{\mathrm{B}}(X, N)(\mathbb{C})$ *has infinite monodromy at infinity* if for any holomorphic map $\gamma : \mathbb{D} \to \overline{X}$ with $\gamma^{-1}(\overline{X} \setminus X) = \{0\}$, there exists a reductive $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ such that $[\varrho] \in M$ and $\gamma^*\varrho : \pi_1(\mathbb{D}^*) \to \mathrm{GL}_N(\mathbb{C})$ has infinite image.

***Proposition 3.4.6* ([DYK23, Proposition 3.18]). —** *Let X be a smooth quasi-projective variety. Assume that $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ is a torsion free representation. Then there exists a smooth partial compactification X' of X such that ϱ extends to a representation $\varrho' : \pi_1(X') \to \mathrm{GL}_N(\mathbb{C})$ with infinite monodromy at infinity.*

A more general result is proved in [DYK23, Proposition 3.18] without assuming the torsion freeness of ϱ.

3.4.4. Outline of the proof of Theorem 3.A. — The proof of Theorem 3.A in the case where char $K = 0$ can be reduced to the following theorem based on Proposition 3.4.6.

***Theorem 3.4.7* ([DYK23, Theorem 3.28]). —** *Let X be a smooth quasi-projective variety. Assume that $M := M_{\mathrm{B}}(X, N)(\mathbb{C})$ has infinite monodromy at infinity (cf. Definition 3.4.5). Then there exists a proper surjective holomorphic fibration $\mathrm{sh}_M : X \to \mathrm{Sh}_M(X)$ over a* normal *complex space $\mathrm{Sh}_M(X)$ such that for any connected Zariski closed subset Z of X, the following properties are equivalent:*

(i) *$\mathrm{sh}_M(Z)$ is a point;*
(ii) *$\varrho(\mathrm{Im}[\pi_1(Z) \to \pi_1(X)])$ is finite for any reductive representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$;*
(iii) *for any irreducible component Z_1 of Z, $\varrho(\mathrm{Im}[\pi_1(Z_1^{\mathrm{norm}}) \to \pi_1(X)])$ is finite for any reductive representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$.*

When X is compact, $\mathrm{Sh}_M(X)$ is projective.

Proof (sketch). — We use the same notations in Theorem 3.4.7. Define $H := \cap_\varrho \ker \varrho$, where $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ ranges over all reductive representation. Denote by $\widetilde{X}_H := \widetilde{X}/H$. Let $\mathscr{D}$ be the period domain associated with the $\mathbb{C}$-VHS σ and let $p : \widetilde{X}_H \to \mathscr{D}$ be the period mapping. We define a holomorphic map

$$\begin{aligned} \Psi : \widetilde{X}_H &\to S_{\mathrm{fac}} \times \mathscr{D}, \\ z &\mapsto (s_{\mathrm{fac}} \circ \pi_H(z), p(z)) \end{aligned} \tag{3.4.1}$$

where $\pi_H : \widetilde{X}_H \to X$ denotes the covering map and $s_{\mathrm{fac}} : X \to S_{\mathrm{fac}}$ is the reduction map defined in Definition 3.4.2.

***Claim 3.4.8.* —** *Each connected component of any fiber of Ψ is compact.*

Proof of Claim 3.4.8 (outline). — Let F be a fiber of $s_{\mathrm{fac}} : X \to S_{\mathrm{fac}}$. For simplicity we assume that F is smooth and connected. Denote by $\iota : F \to X$ the inclusive map. Note that $\iota^*\sigma$ has discrete monodromy Γ. We consider its period map $p : F \to \mathscr{D}/\Gamma$. Let us show that it is proper.

For any reductive $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$, we note that $\iota^*\varrho : \pi_1(F) \to \mathrm{GL}_N(\mathbb{C})$ is a complex direct factor of $\iota^*\sigma$. Since we assume that M has infinite monodromy at infinity, it follows that $\iota^*\sigma$ has infinite monodromy at infinity. By a theorem of Griffiths [Gri70], p is proper.

Let Z be any fiber of p which is compact since p is proper. It follows that $\iota^*\varrho(\mathrm{Im}[\pi_1(Z) \to \pi_1(F)])$ is finite. From this fact together with Proposition 3.4.4.(ii) it is not hard to show that $[\pi_1(Z) \to \pi_1(X)] \cap H$ is a finite index subgroup of $[\pi_1(Z) \to \pi_1(X)]$. Therefore, the inverse image $\pi_H^{-1}(Z)$ is a (possibly infinite) disjoint union of compact subvarieties which are finite étale covers of Z. The fibers of Ψ are precisely these subvarieties. □

By a theorem of Henri Cartan [Car60], the set $\widetilde{S}_H$ of connected components of fibers of Ψ can be endowed with the structure of a complex normal space such that $\Psi = g \circ \mathrm{sh}_H$ where $\mathrm{sh}_H : \widetilde{X}_H \to \widetilde{S}_H$ is a proper holomorphic fibration and $g : \widetilde{S}_H \to \widetilde{S}_{\mathrm{fac}} \times \mathscr{D}$ is a holomorphic map. We can show that each fiber of g is discrete.

Claim 3.4.9. — *$\widetilde{S}_H$ does not contain any compact subvariety.*

Proof. — If $\Sigma \subset \widetilde{S}_H$ is a compact subvariety, then $Z := \mathrm{sh}_H^{-1}(\Sigma)$ is also compact since sh_H is proper. Then, $W := \pi_H(Z)$ is also a compact irreducible subvariety in X with $\dim Z = \dim W$. Hence $\mathrm{Im}\,[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(W^{\mathrm{norm}})]$ is a finite index subgroup of $\pi_1(W^{\mathrm{norm}})$. Note that W can be endowed with an algebraic structure induced by X. As the natural map $Z \to W$ is finite, Z can be equipped with an algebraic structure such that the natural map $Z \to X$ is algebraic.

For any reductive representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ where K is a non archimedean local field of characteristic zero, we have $\varrho(\mathrm{Im}\,[\pi_1(Z) \to \pi_1(X)]) \subset \varrho(\mathrm{Im}\,[\pi_1(\widetilde{X}_H) \to \pi_1(X)]) = \{1\}$. Hence, $\varrho(\mathrm{Im}\,[\pi_1(W^{\mathrm{norm}}) \to \pi_1(X)])$ is finite which is thus bounded. By the construction of s_{fac}, W is contained in a fiber F of s_{fac}. Note that $\sigma(\mathrm{Im}[\pi_1(Z) \to \pi_1(X)])$ is trivial. It follows from Proposition 3.4.4.(iii) that the variation of Hodge structure induced by $\sigma|_{\pi_1(Z)}$ is trivial. Therefore, $p \circ \pi_H(Z)$ is a point where $p : F \to \mathscr{D}/\Gamma$ is the period map defined above. Hence Z is contracted by Ψ. The claim follows. □

One can show that the Galois action $\pi_1(X)/H$ on $\widetilde{X}_H$ induces an action on $\widetilde{S}_H$ such that sh_H is equivariant with respect to such an action. We take the quotient by $\pi_1(X)/H$ and it gives rise to a proper holomorphic fibration $\mathrm{sh}_M : X \to \mathrm{Sh}_M(X)$. The readers can easily check from Claim 3.4.9 that sh_M is the desired Shafarevich morphism in the theorem. □

3.5. On the holomorphic convexity

Let X be a smooth projective variety. Let H be a normal subgroup of $\pi_1(X)$. Denote by $\widetilde{X}_H \to X$ be the Galois covering of X with the Galois group $\pi_1(X)/H$. It is natural to ask whether $\widetilde{X}_H$ is holomorphically convex for any $H \triangleleft \pi_1(X)$. Indeed this depends on the group H in view of the Cousin example:

Example 3.5.1. — Let X be a simple abelian variety and $\rho : \pi_1(X) = H_1(X) \to \mathbb{Z}$ be a surjective homomorphism. Then $\ker(\rho)\backslash\widetilde{X^{un}}$ has no positive dimensional compact complex subvariety but does not carry any non constant holomorphic function either.

Hence the Shafarevich problem is an instance of the problem of determining the pairs $(X, H \triangleleft \pi_1(X))$ with $\widetilde{X}_H$ holomorphically convex. An ingenious discovery by Eyssidieux [Eys04] is that, $\widetilde{X}_H$ will be holomorphically convex if it is intersection of the kernel of reductive representations lying in *absolutely constructible subsets* of character varieties $M_{\mathrm{B}}(X, N)$ defined by Simpson [Sim93b].

In this subsection, we present the second main result in [DYK23] which generalizes Eyssidieux' theorem to projective *normal* varieties. We will sketch the proof in the case of projective surfaces.

3.5.1. Holomorphic convexity (I). — The second main result in [CDY22] is on the holomorphic convexity of certain topological Galois coverings corresponding to the intersections of the kernels of all reductive representations of projective normal varieties.

Theorem 3.D ($\subset$ **[DYK23, Theorem C]**). — *Let X be a projective normal variety and let N be a fixed positive integer. Set $H := \cap_\varrho \ker \varrho$, where $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ ranges over all reductive representations. Let $\widetilde{X}$ be the universal covering of X, and denote $\widetilde{X}_H := \widetilde{X}/H$. Then the complex space $\widetilde{X}_H$ is holomorphically convex. In particular, if $\pi_1(X)$ is a subgroup of $\mathrm{GL}_N(\mathbb{C})$ whose Zariski closure is reductive, then the universal covering $\widetilde{X}$ of X is holomorphically convex.*

In [DYK23], a more general theorem is proved concerning the *absolutely constructible subsets* in the character variety introduced by Simpson [Sim93b]. We list a special version here for simplicity. We will outline the proof of Theorem 3.D when $\dim X = 2$.

Let us remark that Theorem 3.D is proven by Eyssidieux [Eys04] when X is smooth. In [DYK23] we make an effort to simplify Eyssidieux' original difficult proof. Nevertheless, the general strategy follows closely Eyssidieux' one.

3.5.2. Some analytic tools. — Let X be a smooth projective variety. Let $\tau : \pi_1(X) \to \mathrm{GL}_N(K)$ be a reductive representation where K is a non-archimedean local field. According to Theorem 1.E, the reduction map $s_\tau : X \to S_\tau$ of τ exists, fulfilling the properties outlined therein. We will outline the construction of certain *canonical* positive closed $(1,1)$-currents over S_τ.

By Theorem 1.A, there exists a τ-equivariant harmonic map $u : \widetilde{X} \to \Delta(G)$, where $\Delta(G)$ is the Bruhat-Tits building of G. We use the same notation as in the proof of Theorem 1.B.

Recall that we constructed some multivalued 1-forms $\{\omega_1,\ldots,\omega_\ell\}$ over X° that are $(1,0)$-part of the complexified differential of u. These multivalued 1-forms correspond to the $(1,0)$-part of the complexified differential of the harmonic map u. Based on Theorem 1.B, we can prove that there exists a finite Galois covering $\pi : X^{\mathrm{sp}} \to X$ with Galois group H such that $\pi^*\{\omega_1,\ldots,\omega_\ell\}$ becomes single valued; i.e. there exists forms $\{\eta_1,\ldots,\eta_\ell\} \subset H^0(X^{\mathrm{sp}}, \pi^*\Omega^1_X)\}$ such that $\{\eta_1,\ldots,\eta_\ell\}$ coincides with $\pi^*\{\omega_1,\ldots,\omega_\ell\}$ over X°. Such $X^{\mathrm{sp}} \to X$ is called *spectral covering* associated with the representation τ and these 1-forms $\{\eta_1,\ldots,\eta_\ell\}$ are called *spectral forms*. By Claim 1.5.3, they are invariant under the action of the Galois group H.

Note that the Stein factorization of the *partial Albanese morphism* $a : X^{\mathrm{sp}} \to A$ of $\{\eta_1,\ldots,\eta_k\}$ leads to the reduction map $s_{\pi^*\tau} : X^{\mathrm{sp}} \to S_{\pi^*\tau}$ of $\pi^*\tau$. This map $s_{\pi^*\tau}$ is H-equivariant and its quotient by H gives rise to the reduction map $s_\tau : X \to S_\tau$ by τ. More precisely, we have the following commutative diagram:

$$\begin{tikzcd}
X^{\mathrm{sp}} \arrow[r, "\pi"] \arrow[d, "s_{\pi^*\tau}"] \arrow[dd, bend right=60, "a"'] & X \arrow[d, "s_\tau"] \\
S_{\pi^*\tau} \arrow[r, "\sigma_\pi"] \arrow[d, "b"] & S_\tau \\
A &
\end{tikzcd}$$

Here σ_π is also a finite ramified Galois cover with Galois group H. Note that there are 1-forms $\{\eta'_1,\ldots,\eta'_m\} \subset H^0(A,\Omega^1_A)$ such that $a^*\eta'_i = \eta_i$. We define a positive smooth $(1,1)$-form $T_{\pi^*\tau} := b^*\sum_{i=1}^m i\eta'_i \wedge \overline{\eta_i}'$ on $S_{\pi^*\tau}$. Note that $T_{\pi^*\tau}$ is invariant under the Galois action H. Therefore, there is a positive closed $(1,1)$-current T_τ defined on S_τ with continuous potential such that $\sigma_\pi^* T_\tau = T_{\pi^*\tau}$.

Definition 3.5.2 **(Canonical current).** — The closed positive $(1,1)$-current T_τ on S_τ is called the *canonical current* of τ.

Lemma 3.5.3. — *$\{T_\tau\}$ is strictly nef. Namely, for any irreducible curve $C \subset S_\tau$, we have $\{T_\tau\}\cdot C > 0$.*

Proof. — Let $C' \subset \sigma_\pi^{-1}(C)$ be an irreducible component which is dominant over C. Consider its image $b(C')$. By the property of the partial Albanese morphism, there exists some $\eta'_i \in H^0(A,\Omega^1_A)$ such that $\eta'_i|_{b(C')} \neq 0$. Hence $i\eta'_i \wedge \overline{\eta'_i}|_{b(C')}$ is strictly positive at general points. Consequently, $\{T_\tau\}\cdot C > 0$. □

More generally, let $\{\varrho_i : \pi_1(X) \to \mathrm{GL}_N(K_i)\}_{i=1,\ldots,k}$ be reductive representations where K_i is a non-archimedean local field. We shall denote by the bolded letter $\boldsymbol{\varrho} := \{\varrho_i\}_{i=1,\ldots,k}$ be such family of representations. Let $s_{\boldsymbol{\varrho}} : X \to S_{\boldsymbol{\varrho}}$ be the Stein factorization of $(s_{\varrho_1},\ldots,s_{\varrho_k}) : X \to S_{\varrho_1}\times\cdots\times S_{\varrho_k}$ where $s_{\varrho_i} : X \to S_{\varrho_i}$ denotes the reduction map associated with ϱ_i and $p_i : S_{\boldsymbol{\varrho}} \to S_{\varrho_i}$ is the induced finite morphism. $s_{\boldsymbol{\varrho}} : X \to S_{\boldsymbol{\varrho}}$ is called the *reduction map* for the family $\boldsymbol{\varrho}$ of representations.

Definition 3.5.4 **(Canonical current II).** — The closed positive $(1,1)$-current $T_{\boldsymbol{\varrho}} := \sum_{i=1}^k p_i^* T_{\varrho_i}$ on $S_{\boldsymbol{\varrho}}$ is called the canonical current of $\boldsymbol{\varrho}$.

The canonical current T_ϱ will serve as a lower bound for the complex hessian of plurisubharmonic functions constructed by the method of harmonic mappings.

Proposition 3.5.5 **([Eys04, Proposition 3.3.6, Lemme 3.3.12]).** — *Let X be a projective normal variety and let $\varrho : \pi_1(X) \to G(K)$ be a Zariski dense representation where K is a non archimedean local field and G is a reductive group. Let $x_0 \in \Delta(G)$ be an arbitrary point. Let $u : \widetilde{X} \to \Delta(G)$ be the associated harmonic mapping, where $\widetilde{X}$ is the universal covering of X. The function $\phi : \widetilde{X} \to \mathbb{R}_{\geq 0}$ defined by*

$$\phi(x) = 2d^2\big(u(x), u(x_0)\big)$$

satisfies the following properties:

(a) *ϕ descends to a function ϕ_ϱ on $\widetilde{X_\varrho} = \widetilde{X}/\ker(\varrho)$.*

(b) *Let Σ be a normal complex space and $r : \widetilde{X}_\varrho \to \Sigma$ a proper holomorphic fibration such that $s_\varrho \circ \pi : \widetilde{X}_\varrho \to S_\varrho$ factorizes via a morphism $\nu : \Sigma \to S_\varrho$. The function ϕ_ϱ is of the form $\phi_\varrho = \phi_\varrho^\Sigma \circ r$ with ϕ_ϱ^Σ being a continuous plurisubharmonic function on Σ;*

(c) $\mathrm{dd^c}\phi_\varrho^\Sigma \geq \nu^* T_\varrho$. □

3.5.3. Outline of the proof of Theorem 3.D. —

We first start with the following criterion for the Steiness of a topological Galois covering of a compact complex normal space.

Proposition 3.5.6 **([Eys04, Proposition 4.1.1]).** — *Let X be a compact complex normal space and let $\pi : \widetilde{X}' \to X$ be an infinite topological Galois covering. Let T be a positive current on X with continuous potential such that $\{T\}$ is a Kähler class. Assume that there exists a continuous plurisubharmonic function $\phi : \widetilde{X}' \to \mathbb{R}_{\geq 0}$ such that $\mathrm{dd^c}\phi \geq \pi^* T$. Then $\widetilde{X}'$ is a Stein space.*

For simplicity, we will prove the following baby version of Theorem 3.D. It nevertheless gives the rough strategy of the proof of Theorem 3.D.

Theorem 3.5.7. — *Let X be a smooth projective surface. Assume that $M := M_{\mathrm{B}}(X, N)$ is large in the sense that, i.e. for any closed positive-dimensional subvariety $Z \subset X$, there exists reductive representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ such that $\varrho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)])$ is infinite. Then the universal covering of X is Stein.*

Proof (sketch). — We will use the same notations as in Proposition 3.4.4. Let $s_{\mathrm{fac}} : X \to S_{\mathrm{fac}}$ be the reduction map defined in Definition 3.4.2. There are three cases.

Case 1: $\dim S_{\mathrm{fac}} = 0$. This means that there exists no unbounded representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ with K a non-archimedean local field of characteristic zero. Applying Lemma 1.4.3 in conjunction with the subsequent discussion yields the conclusion that $M_{\mathrm{B}}(X, N)(\mathbb{C})$ contains only finitely many points $[\varrho_1], \ldots, [\varrho_m]$, and each representation $\varrho_i : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ is integral and is a direct factor of a $\mathbb{Z}$-VHS σ_i. Set $\sigma := \oplus_{i=1}^m \sigma_i$. Since $M_{\mathrm{B}}(X, N)$ is large, it follows that σ is a large representation. Consider the period mapping $p : X \to \mathscr{D}/\Gamma$ of σ. It is a finite map. Let (E, θ, h) be the Hodge bundle associated with this $\mathbb{Z}$-VHS. Then $\omega := i\mathrm{tr}(\theta \wedge \theta_h^*)$ is a semi-Kähler form on X, which is strictly positive at the points where $dp : T_D \to T_{\mathscr{D}/\Gamma}$ is immersive. Since p is finite, it follows that for any positive-dimensional subvariety Z of X, we have $\int_Z \omega^{\dim Z} > 0$. By Demailly-Păun's theorem in [DP04], $\{\omega\}$ is a Kähler class.

Let $\mathscr{S}$ be the symmetric space associated with $\mathscr{D}$ endowed with the natural metric $d_{\mathscr{S}}$. Then there exists natural quotient map $\mathscr{D} \to \mathscr{S}$. It induces a σ-equivariant pluri-harmonic mapping $u : \widetilde{X}_H \to \mathscr{S}$. Define $\phi := 2d_{\mathscr{S}}^2(u(x), u(x_0))$. By [Eys04, Proposition 3.3.2], we have

$$\mathrm{dd^c}\phi \geq \pi_H^* \omega. \tag{3.5.1}$$

We apply Proposition 3.5.6 to conclude that $\widetilde{X}_H$ is Stein.

Case 2: $\dim S_{\mathrm{fac}} = 1$. By the construction of $s_{\mathrm{fac}} : X \to S_{\mathrm{fac}}$, there exists an unbounded reductive representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ with K a non-archimedean local field of characteristic zero such that $s_\varrho : X \to S_\varrho$ factors through s_{fac}. Since $\dim S_{\mathrm{fac}} = 1$, it follows that $\dim S_\varrho = 1$ and thus s_{fac} coincides with s_ϱ. Let T_ϱ be the canonical current on S_ϱ associated with ϱ defined in Definition 3.5.2. Then $\{T_\varrho\}$ is a Kähler class on S_ϱ by Lemma 3.5.3. We apply Proposition 3.5.5 to conclude that there exists continuous psh function $\phi_\varrho : \widetilde{X}_H \to \mathbb{R}_{\geq 0}$ such that $\mathrm{dd^c}\phi_\varrho \geq \pi_H^* s_\varrho^* T_\varrho$.

Consider the $\mathbb{C}$-VHS σ in Proposition 3.4.4. Since $M_{\mathrm{B}}(X, N)$ is large, by Proposition 3.4.4.(iii), σ is also large. Then for each fiber F of s_{fac}, the period map $F \to \mathscr{D}/\Gamma$ of $\sigma|_{\pi_1(F)}$ is finite. Here $\mathscr{D}$ is the period domain and Γ is the monodromy group of $\sigma|_{\pi_1(F)}$. Let (E, θ, h) be the Hodge bundle associated with this $\mathbb{Z}$-VHS. Then $\omega := i\mathrm{tr}(\theta \wedge \theta_h^*)$ is a semi-Kähler form on X. Therefore, $\{\omega\}|_F$ is Kähler. Let $\phi : X \to \mathbb{R}_{\geq 0}$ be the continuous function defined in case 1 such that $\mathrm{dd^c}\phi \geq \pi_H^*\omega$ by (3.5.1).

One can see that $\{\omega + s_{\mathrm{fac}}^* T_\varrho\}$ is Kähler. Therefore, we have

$$\mathrm{dd^c}(\phi_\varrho + \phi) \geq \pi_H^*(s_\varrho^* T_\varrho + \omega).$$

We apply Proposition 3.5.6 once again to conclude that $\widetilde{X}_H$ is Stein.

Case 3: $\dim S_{\mathrm{fac}} = 2$. This is the most difficult case.

Let $\boldsymbol{\tau} := \{\tau_i : \pi_1(X) \to \mathrm{GL}_N(K_i)\}_{i=1,\ldots,\ell}$ be a family of unbounded reductive representations with K_i non-archimedean local fields of characteristic zero. Let $T_{\boldsymbol{\tau}} = \sum_{i=1}^{\ell} p_i^* T_{\tau_i}$ be the canonical current on $S_{\boldsymbol{\tau}}$ defined in Definition 3.5.4. Let $X^{\mathrm{sp}} \to X$ be the spectral covering with respect to $\boldsymbol{\tau}$. The collective spectral forms on X^{sp} induced by $\tau_1, \ldots, \tau_\ell$ will be called the *spectral forms for* $\boldsymbol{\tau}$.

Note that we have

$$\begin{tikzcd} X \arrow[d, "s_{\mathrm{fac}}"'] \arrow[dr, "s_{\boldsymbol{\tau}}"] \arrow[drr, bend left, "s_{\tau_i}"] & & \\ S_{\mathrm{fac}} \arrow[r] & S_{\boldsymbol{\tau}} \arrow[r, "p_i"] & S_{\tau_i} \end{tikzcd}$$

Case 3.1: The spectral 1-forms have rank 2. Assume that there exist $\boldsymbol{\tau}$ such that, their spectral forms have rank 2. By the construction of s_{fac}, we can add more unbounded reductive representations $\{\tau_i : \pi_1(X) \to \mathrm{GL}_N(K_i)\}_{i=\ell+1,\ldots,m}$ with K_i non-archimedean local fields of characteristic zero, to assume additionally that $s_{\mathrm{fac}} : X \to S_{\mathrm{fac}}$ coincides with $s_{\boldsymbol{\tau}'} : X \to S_{\boldsymbol{\tau}'}$, where $\boldsymbol{\tau}' := \{\tau_i\}_{i=1,\ldots,m}$.

By the construction of the canonical current $T_{\boldsymbol{\tau}'}$ on S_{fac} is strictly positive at general points since the spectral forms associated with $T_1, \ldots, T_m$ has rank 2. Since $T_{\boldsymbol{\tau}'}$ has continuous potentials, it follows that $\{T_{\boldsymbol{\tau}'}\}$ is a big and nef class.

On the other hand, by Lemma 3.5.3, we conclude that $\{T_{\boldsymbol{\tau}'}\}$ is strictly nef. We now apply a theorem by Demailly-Păun [DP04] to conclude that $\{T_{\boldsymbol{\tau}'}\}$ is a Kähler class.

According to Proposition 3.5.5, there exist continuous plurisubharmonic functions $\{\phi_i : \widetilde{X}_H \to \mathbb{R}_{\geq 0}\}_{i=1,\ldots,m}$ such that

$$\mathrm{dd^c}\phi_i \geq \pi_H^* s_{\tau_i}^* T_{\tau_i}$$

for $i = 1, \ldots, m$. Let $\phi : X \to \mathbb{R}_{\geq 0}$ be the continuous function and ω be the semi-Kähler form defined in Case 2 such that $\mathrm{dd^c}\phi \geq \pi_H^*\omega$. Moreover, $\{\omega\}|_F$ is Kähler for each fiber F of s_{fac}. Hence $\{s_{\mathrm{fac}}^* T_{\boldsymbol{\tau}'} + \omega\}$ is a Kähler class on X. Therefore, we have

$$\mathrm{dd^c}(\sum_{i=1}^{m} \phi_i + \phi) \geq \pi_H^*(s_{\mathrm{fac}}^* T_{\boldsymbol{\tau}'} + \omega).$$

We apply Proposition 3.5.6 once again to conclude that $\widetilde{X}_H$ is Stein.

Case 3.2: The spectral 1-forms always have rank 1. Assume that for any family of unbounded reductive representations any family of $\{\tau_i : \pi_1(X) \to \mathrm{GL}_N(K_i)\}_{i=1,\ldots,\ell}$ with K_i non-archimedean local fields of characteristic zero, their spectral forms $\{\eta_1, \ldots, \eta_n\} \subset H^0(X^{\mathrm{sp}}, \pi^*\Omega_X^1)$ always have rank 1, i.e., $\eta_i \wedge \eta_j \equiv 0$ for every η_i and η_j.

Case 3.2.1: The dimension of spectral 1-forms is at least two. Assume that there exists a family of unbounded reductive representations $\{\tau_i : \pi_1(X) \to \mathrm{GL}_N(K_i)\}_{i=1,\ldots,\ell}$ with K_i non-archimedean local fields of characteristic zero, such that their spectral forms $\{\eta_1, \ldots, \eta_n\} \subset H^0(X^{\mathrm{sp}}, \pi^*\Omega_X^1)$ have rank 1, and

$$\dim_{\mathbb{C}}\mathrm{Span}\{\eta_1, \ldots, \eta_n\} \geq 2.$$

Here $\pi : X^{\mathrm{sp}} \to X$ is the spectral covering with respect to $T_1, \ldots, T_k$. Therefore, without loss of generality, we may assume that $\eta_1 \wedge \eta_2 \equiv 0$ and $\eta_1 \notin \{\mathbb{C}\eta_2\}$. According to the Castelnuovo-De Franchis theorem (cf. [ABC+96, Theorem 2.7]), there exists a proper fibration $h : X^{\mathrm{sp}} \to C$ over a smooth projective curve C such that $\{\eta_1, \eta_2\} \subset h^* H^0(C, \Omega_C^1)$. Since s_{fac} is birational, we can choose a general fiber F of h, which is irreducible and such that $s_{\mathrm{fac}} \circ \pi(F)$ is not a point. By the construction of s_{fac}, there exists an unbounded reductive representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ with K a non-archimedean local fields of characteristic zero such that $s_\varrho \circ \pi(F)$ is not a point. We replace X^{sp} by some Galois covering such that it is also the spectral covering for $\{\varrho\} \cup \{\tau_i\}_{i=1,\ldots,\ell}$. For the spectral forms $\{\omega_1, \ldots, \omega_m\} \subset H^0(X^{\mathrm{sp}}, \pi^*\Omega_X^1)$ induced by ϱ, there exists some i such that $\omega_i|_F \neq 0$. Given that $\eta_1|_F \equiv 0$, this implies that $\omega_i \wedge \eta_1 \neq 0$. It contradicts with our assumption that the spectral 1-forms always have rank 1. Therefore, this case cannot occur.

Case 3.2.2: The dimension of spectral 1-forms is always *1.* Assume that for any family of unbounded reductive representations any family of $\{\tau_i : \pi_1(X) \to \mathrm{GL}_N(K_i)\}_{i=1,\ldots,\ell}$ with K_i non-archimedean local fields of characteristic zero, the set of their spectral forms $\{\eta_1, \ldots, \eta_n\} \subset H^0(X^{\mathrm{sp}}, \pi^*\Omega_X^1)$ have rank 1, and they are all $\mathbb{C}$-linear.

By the construction of s_{fac}, we can find $\boldsymbol{\tau} := \{\tau_i : \pi_1(X) \to \mathrm{GL}_N(K_i)\}_{i=1,\ldots,\ell}$ such that s_{fac} coincides with $s_{\boldsymbol{\tau}} : X \to S_{\boldsymbol{\tau}}$.

Let G_i be the Zariski closure of $\tau_i(\pi_1(X))$, which is reductive. Consider the isogeny

$$g : G_i \to G_i/Z_i \times G_i/\mathcal{D}G_i$$

where Z_i is the central torus of G_i and $\mathcal{D}G_i$ is the derived group of G_i. As a result, $G_i' := G_i/Z_i$ is semisimple and $G_i'' := G_i/\mathcal{D}G_i$ is a torus. Let $\tau_i' : \pi_1(X) \to G_i'(\overline{K_i})$ be the composite of τ_i with the projection $G_i \to G_i'$, and $\tau_i'' : \pi_1(X) \to G_i''(\overline{K_i})$ be the composite of τ_i with the projection $G_i \to G_i''$. Then τ_i' and τ_i'' are both Zariski dense representations.

Notably, both families of representations, $\boldsymbol{\tau}$ and $\{\tau_i', \tau_i''\}_{i=1,\ell}$ define the same reduction map.

Next, we consider the partial Albanese morphism $a : X^{\mathrm{sp}} \to A$ induced by η_1. If $\dim a(Y) = 1$, then the Stein factorization $h : X^{\mathrm{sp}} \to C$ of a is a proper holomorphic fibration over a smooth projective curve C such that $\eta_1 \in h^* H^0(C, \Omega_C^1)$. We are now in a situation akin to Case 3.2.1, and we can apply the same arguments to reach a contradiction.

Let $\nu : Y \to X^{\mathrm{sp}}$ be a desingularization and denote $\eta := \nu^*\eta_1$. Consider the partial Albanese morphism $a : Y \to A$ induced by η. Then there exists a one form $\eta' \in H^0(A, \Omega_A^1)$ such that $a^*\eta' = \eta$. By the above argument, we know that $\dim a(Y) \geq 2$. Let $\pi_A : \widetilde{A} \to A$ denote the universal covering map. We denote by $Y' := Y \times_{\widetilde{A}} A$ a connected component of the fiber product and let $\pi' : Y' \to Y$ be the induced étale cover. It's worth noting that $\pi'^*\eta$ is exact. Consequently, we can define the following holomorphic map:

$$h : Y' \to \mathbb{C}$$
$$y \mapsto \int_{y_0}^{y} \pi'^*\eta.$$

We then have the following commutative diagram:

$$\begin{tikzcd}
\widetilde{Y} \arrow[r, "p"] \arrow[rr, bend left, "\pi_Y"] & Y' \arrow[r, "\pi'"] \arrow[d] \arrow[dd, bend right, "h"'] & Y \arrow[d, "a"] \\
& \widetilde{A} \arrow[r, "\pi_A"] \arrow[d] & A \\
& \mathbb{C} &
\end{tikzcd}$$

The holomorphic map $\widetilde{A} \to \mathbb{C}$ in the above diagram is defined by the linear 1-form $\pi_A^*\eta'$. By Simpson's Lefschetz theorem [Sim93a], for any $t \in \mathbb{C}$, $h^{-1}(t)$ is connected and we have the surjectivity $\pi_1(h^{-1}(t)) \twoheadrightarrow \pi_1(Y')$. By the definition of h, $\pi_Y^*\eta|_Z \equiv 0$ where Z is any connected component of $p^{-1}(h^{-1}(t))$. Here $p : \widetilde{Y} \to Y'$ is the natural covering map.

Consider any Zariski dense representation $\tau_i' : \pi_1(X) \to G_i'\big(\overline{\mathbb{F}_{q_i}((t))}\big)$ as defined previously. Let K_i be a finite extension of $\mathbb{F}_{q_i}((t))$ such that $\tau_i' : \pi_1(X) \to G_i'(K_i)$. We denote by $\sigma_i : \pi_1(Y) \to G_i'\big(K_i\big)$ the pullback of τ_i' via the morphism $Y \to X$. The existence of a σ_i-equivariant harmonic mapping $u : \widetilde{Y} \to \Delta(G_i')$ is guaranteed by [GS92].

We note that $\pi_Y^*\eta$ is the (1,0)-part of the complexified differential of the harmonic mapping u at general points of $\widetilde{Y}$, with $\pi_Y : \widetilde{Y} \to Y$ denoting the universal covering. For any connected component Z of $p^{-1}(h^{-1}(t))$ for a general $t \in \mathbb{C}$, since $\pi_Y^*\eta|_Z \equiv 0$, and all the spectral forms are assumed to be $\mathbb{C}$-linearly equivalent, it follows that $u(Z)$ is constant. Since u is σ_i-equivariant, it follows that $\pi'^*\sigma_i(\mathrm{Im}[\pi_1(h^{-1}(t)) \to \pi_1(Y')])$ is contained in the subgroup of $G_i'(K_i)$ fixing the point $u(Z)$. Recall that $\pi_1(h^{-1}(t)) \to \pi_1(Y')$ is surjective. Hence $\pi'^*\sigma_i(\pi_1(Y'))$ is a bounded subgroup of $G_i'(K_i)$. Additionally, note that $\mathcal{D}\pi_1(Y) \subset \mathrm{Im}[\pi_1(Y') \to \pi_1(Y)]$, and it follows that $\sigma_i(\mathcal{D}\pi_1(Y))$ is bounded. Since τ_i' is Zariski dense, and $\mathrm{Im}[\pi_1(Y) \to \pi_1(X)]$ is a finite index subgroup of $\pi_1(X)$, the Zariski closure of $\sigma_i(\pi_1(Y))$ contains the identity component of G_i', and it is also semisimple. We apply Lemma 2.5.3 to conclude that $\sigma_i(\pi_1(Y))$ is bounded.

Since $\sigma_i(\pi_1(Y))$ is a finite index subgroup of $\tau_i'(\pi_1(X))$, it follows that $\tau_i'(\pi_1(X))$ is also bounded. Then the reduction map $s_{\tau_i'}$ is the constant map. This means that the reduction map of representations $\{\tau_i''\}_{i=1,\ldots,\ell}$ is identified with that of $\{\tau_i', \tau_i''\}_{i=1,\ldots,\ell}$. It follows that s_τ is the Stein factorization of

$$(s_{\tau_1''}, \ldots, s_{\tau_\ell''}) : X \to S_{\tau_1''} \times \cdots \times S_{\tau_\ell''}.$$

Recall that each G_i'' is a tori. By [CDY22, Proof of Theorem 0.10], we know that there exists a morphism $a_i : X \to A_i$ with A_i an abelian variety such that $s_{\tau_i''}$ is the Stein factorization of a_i. Therefore, s_{fac} is the Stein factorization of $(a_1, \ldots, a_\ell) : X \to A_1 \times \cdots \times A_\ell$. Write $\psi : X \to \mathcal{A}$ for $(a_1, \ldots, a_\ell) : X \to A_1 \times \cdots \times A_\ell$.

Let $\eta_1, \ldots, \eta_k \subset H^0(\mathcal{A}, \Omega^1_{\mathcal{A}})$ be a basis. Then $\omega_0 := i\sum_{i=1}^k \eta_i \wedge \bar{\eta}_i$ is a Kähler form on $\mathcal{A}$. Note that the universal covering $\pi_{\mathcal{A}} : \widetilde{\mathcal{A}} \to \mathcal{A}$ of $\mathcal{A}$ is isomorphic to $\mathbb{C}^{\dim \mathcal{A}}$. There exists a smooth function $\psi_0 : \mathcal{A} \to \mathbb{R}_{\geq 0}$ such that

$$\mathrm{dd}^{\mathrm{c}}\psi_0 \geq \pi_{\mathcal{A}}^*\omega_0.$$

Let $\phi : X \to \mathbb{R}_{\geq 0}$ be the continuous function and ω be the semi-Kähler form defined in Case 2 such that $\mathrm{dd}^{\mathrm{c}}\phi \geq \pi_H^*\omega$. Moreover, $\{\omega\}|_F$ is Kähler for each fiber F of s_{fac}. Since s_{fac} is the Stein factorization of ψ, $\{\psi^*\omega_0 + \omega\}$ is a Kähler class on X. Let $f : \widetilde{X} \to \widetilde{\mathcal{A}}$ be the lift of ψ to the universal coverings and let $\pi_X : \widetilde{X} \to X$ be the universal covering. We have

$$\mathrm{dd}^{\mathrm{c}}(f^*\psi_0 + \phi) \geq \pi_X^*(\psi^*\omega_0 + \omega).$$

We apply Proposition 3.5.6 once again to conclude that $\widetilde{X}$ is Stein.

Note that any unramified cover of a Stein space is Stein. In conclusion, the universal covering of X is Stein. □

Remark 3.5.8. — The proof of Theorem 3.D in the general case is much more involved. In addition to the techniques used above, we have to apply Simpson's work on *absolutely constructible subsets* in character varieties (cf. [Sim93b, WB20, DYK23]).

Remark 3.5.9. — While we follow the general strategy in [Eys04], our approach in proving Theorems 3.A and 3.D introduces several novel elements, including:

(a) An avoidance of the reduction mod p method used in [CS08, Eys04]. Roughly speaking, in the treatment of non-rigid representation in [CS08, Eys04], Corlette-Simpson and Eyssidieux constructed unbounded $\varrho : \pi_1(X) \to \mathrm{GL}_N(k((t)))$, where k is some number field. Unfortunately, the field $k((t))$ is not locally compact, which prevents the application of Gromov-Schoen's theory (cf. Theorem 1.A). To address this issue, in [Eys04, CS08] the authors perform reduction modulo p for ϱ and obtain a family of representation $\varrho_q : \pi_1(X) \to \mathrm{GL}_N(\mathbb{F}_q((t)))$. In this chapter, we only work on finite extensions of $\mathbb{Q}_p$. As we have seen in

Proposition 3.4.4, our construction of $s_{\mathrm{fac}} : X \to S_{\mathrm{fac}}$ in section 3.4.2 incorporates both rigid and non-rigid cases, previously treated separately in [Eys04].

(b) We relax the definition of absolutely constructible subsets in [Sim93b, Eys04], enabling us to extend our results to projective normal varieties and establish the reductive Shafarevich conjecture for such varieties. Note that adopting the original definition of absolutely constructible subsets by Simpson and Eyssidieux would pose significant challenges in extending the Shafarevich conjecture to the singular setting.

3.5.4. Holomorphic convexity (II). — It is natural to ask whether Theorem 3.D holds in the case of linear representation in positive characteristic. For surfaces we can prove the following result.

***Theorem 3.E* ([DY24, Theorem B]).** — *Let X be a projective normal variety and let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a faithful representation where K is a field of positive characteristic. If the Γ-dimension (see Definition 3.2.3) of X is at most two (e.g. when* $\dim X \leq 2$*), then the universal covering $\widetilde{X}$ of X is holomorphically convex.*

Let us explain briefly the idea of the proof. One can first show that after replacing X by a suitable finite étale cover, ϱ factors through the Shafarevich morphism $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$: there exists a large representation $\sigma : \pi_1(\mathrm{Sh}_\varrho(X)) \to \mathrm{GL}_N(K)$ such that $(\mathrm{sh}_\varrho)^*\tau = \varrho$. Since we assume that the Γ-dimension of X is at most two, and ϱ is faithful, it follows that $\dim \mathrm{Sh}_\varrho(X) \leq 2$ and $(\mathrm{sh}_\varrho)_* : \pi_1(X) \to \pi_1(\mathrm{Sh}_\varrho(X))$ is an isomorphism. Let S be the universal covering of $\mathrm{Sh}_\varrho(X)$. Therefore, there exists a proper holomorphic fibration $f : \widetilde{X} \to S$ between the universal coverings that lifts sh_ϱ. By Remmert-Cartan's theorem, it suffices to prove that S is Stein. It is obvious if $\dim \mathrm{Sh}_\varrho(X) = 1$. For the case $\dim \mathrm{Sh}_\varrho(X) = 2$, the proof of Steiness of S is very close to that of Theorem 3.5.7 once we know the structure of the Shafarevich morphism in Theorem 4.1.6. We thus omit it.

CHAPTER 4

SHAFAREVICH CONJECTURE AND HYPERBOLICITY: POSITIVE CHARACTERISTIC CASE

In Chapters 2 and 3 we state the theorems for linear representation in arbitrary field, but only sketched the proof in characteristic zero. In this chapter we will discuss the proof of Theorems 2.A.(ii), 2.E.(ii) and 3.A for the case of representation in positive characteristic.

4.1. Constructing Shafarevich morphism (II)

In this section we will sketch the proof of Theorem 3.A for the case $\operatorname{char} K > 0$. As we will see soon, the proof is simpler than the case of characteristic zero. We begin with the following lemma.

***Lemma 4.1.1* ([DY24, Lemma 2.1]).** — *Let K be an algebraically closed field of positive characteristic and let Γ be a finitely generated group. Let $\varrho : \Gamma \to G(K)$ be a representation such that its semisimplification has finite image. Then $\varrho(\Gamma)$ is finite.*

Proof. — Since the semisimplification ϱ^{ss} of ϱ has finite image, we can replace Γ by a finite index subgroup such that $\varrho^{ss}(\Gamma)$ is trivial. Therefore, some conjugation σ of ϱ has image in the subgroup $\mathrm{U}_N(K)$ consisting of all upper-triangular matrices in $\mathrm{GL}_N(K)$ with 1's on the main diagonal.

Note $\mathrm{U}_N(K)$ admits a central normal series whose successive quotients are isomorphic to $\mathbb{G}_{a,K}$. We remark that a finitely generated subgroup of $\mathbb{G}_{a,K}$ is a finite group. By [ST00, Proposition 4.17], any finite index subgroup of a finitely generated group is also finitely generated. Consequently, $\sigma(\Gamma)$ admits a central normal series whose successive quotients are finitely generated subgroups of $\mathbb{G}_{a,K}$, which are all finite groups. It follows that $\sigma(\Gamma)$ is finite. The lemma is proved. □

As an immediate consequence, if X is a quasi-projective normal variety and if $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ is a big representation with $\operatorname{char} K > 0$, then ϱ^{ss} is also big. This explains why in positive characteristic zero, we do not need to assume that the representation is reductive in Theorem 2.E.

In what follows, X is assumed to be a smooth quasi-projective variety. The variety of N-dimensional linear representations of $\pi_1(X)$ in characteristic zero is represented by an affine $\mathbb{Z}$-scheme R of finite type. Namely, given a commutative ring A, the set of A-points of R is:

$$R(A) = \mathrm{Hom}(\pi_1(X), \mathrm{GL}_N(A)).$$

Let p be a prime number. Consider $R_{\mathbb{F}_p} := R \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{F}_p$ and note that the general linear group over $\mathbb{F}_p$, denoted by $\mathrm{GL}(N, \mathbb{F}_p)$, acts on $R_{\mathbb{F}_p}$ by conjugation. Using Seshadri's extension of geometric invariant theory quotients for schemes of arbitrary field [Ses77, Theorem 3], we can take the GIT quotient of $R_{\mathbb{F}_p}$ by $\mathrm{GL}(N, \mathbb{F}_p)$, denoted by $M_{\mathrm{B}}(X, N)_{\mathbb{F}_p}$. Then $M_{\mathrm{B}}(X, N)_{\mathbb{F}_p}$ is also an affine $\mathbb{F}_p$-scheme of finite type. For any algebraically closed field K of characteristic p, the K-points $M_{\mathrm{B}}(X, N)_{\mathbb{F}_p}(K)$ is identified with the conjugacy classes of semi-simple representations $\pi_1(X) \to \mathrm{GL}_N(K)$.

Consider $M \subset M_{\mathrm{B}}(X, N)_{\mathbb{F}_p}$, a Zariski closed subset defined over $\overline{\mathbb{F}_p}$. Let $\pi : R_{\mathbb{F}_p} \to M_{\mathrm{B}}(X, N)_{\mathbb{F}_p}$ be the GIT quotient, which is a surjective $\mathbb{F}_p$-morphism. Let $T \subset \pi^{-1}(M)$ be any irreducible affine curve defined over $\overline{\mathbb{F}_p}$. Take $\bar{C}$ as the compactification of the normalization C of T, and let $\{P_1, \ldots, P_\ell\} = \bar{C} \setminus C$. There exists $q = p^n$ for some $n \in \mathbb{Z}_{>0}$ such that $\bar{C}$ is defined over $\mathbb{F}_q$ and $P_i \in \bar{C}(\mathbb{F}_q)$ for each i.

By the universal property of the representation scheme R, C gives rise to a representation $\varrho_C : \pi_1(X) \to \mathrm{GL}_N(\mathbb{F}_q[C])$, where $\mathbb{F}_q[C]$ is the coordinate ring of C. Consider the discrete valuation $v_i : \mathbb{F}_q(C) \to \mathbb{Z}$ defined by P_i, where $\mathbb{F}_q(C)$ is the function field of C. Let $\widehat{\mathbb{F}_q(C)}_{v_i}$ be the completion of $F_q(C)$ with respect to v_i. Then we have the isomorphism $\big(\widehat{\mathbb{F}_q(C)}_{v_i}, v_i\big) \simeq \big(\mathbb{F}_q((t)), v\big)$, where $\big(\mathbb{F}_q((t)), v\big)$ is the formal Laurent field of $\mathbb{F}_p$ with the valuation v defined by $v(\sum_{i=m}^{+\infty} a_i t^i) = \min\{i \mid a_i \neq 0\}$. Let $\varrho_i : \pi_1(X) \to \mathrm{GL}_N(\mathbb{F}_q((t)))$ be the extension of ϱ_C with respect to $\widehat{\mathbb{F}_q(C)}_{v_i}$.

Lemma 4.1.2. — *If ϱ_i is bounded for each i, then $\pi(T)$ is a point. Moreover $\varrho_C(\pi_1(X))$ is a finite group.*

Proof. — Since ϱ_i is bounded for each i, after we replace ϱ_i by some conjugation, we have $\varrho_i(\pi_1(X)) \subset \mathrm{GL}_N(\mathbb{F}_q[[t]])$, where the $\mathbb{F}_q[[t]]$ is the ring of integers of $\mathbb{F}_q((t))$, i.e.

$$\mathbb{F}_q[[t]] := \{\sum_{i=m}^{+\infty} a_i t^i \mid a_i \in \mathbb{F}_q, m \geq 0\}.$$

For any matrix $A \in \mathrm{GL}_N(K)$, we denote by $\chi(A) = T^N + \sigma_1(A)T^{N-1} + \cdots + \sigma_N(A)$ its characteristic polynomial. Then $\sigma_j(\varrho_C(\gamma)) \in \mathbb{F}_q[C]$ for each $\gamma \in \pi_1(X)$. Since we have assumed that $\varrho_i(\pi_1(X)) \subset \mathrm{GL}_N(\mathbb{F}_q[[t]])$ for each i, it follows that $\sigma_j(\varrho_i(\gamma)) \in \mathbb{F}_q[[t]]$ for each i. Therefore, by the definition of ϱ_i, $v_i\big(\sigma_j(\varrho_C(\gamma))\big) \geq 0$ for each i. It follows that $\sigma_j(\varrho_C(\gamma))$ extends to a regular function on $\overline{C}$, which is thus constant. This implies that for any two representations $\eta_1 : \pi_1(X) \to \mathrm{GL}_N(K_1)$ and $\eta_2 : \pi_1(X) \to \mathrm{GL}_N(K_2)$ such that $\operatorname{char} K_1 = \operatorname{char} K_2 = p$ and $\eta_i \in C(K_i)$, we have $\chi(\eta_1(\gamma)) = \chi(\eta_2(\gamma))$ for each $\gamma \in \pi_1(X)$. In other words, η_1 and η_2 has the same characteristic polynomial. It follows that $[\eta_1] = [\eta_2]$ by the Brauer–Nesbitt theorem. Hence $\pi(T)$ is a point.

Note that $\varrho_C : \pi_1(X) \to \mathrm{GL}_N(\mathbb{F}_q[C])$ corresponds to the generic point of C. It follows that $\varrho_C \in C(\mathbb{F}_q(C))$. Since C is defined over $\overline{\mathbb{F}_p}$, we can find a point $\eta : \pi_1(X) \to \mathrm{GL}_N(\overline{\mathbb{F}_p})$ such that $\eta \in C(\overline{\mathbb{F}_p})$. Then since $\pi(T)$ is a point, we have $[\varrho_C] = [\eta]$. Since $\eta(\pi_1(X))$ is finite, the semisimplification η^{ss} of η has also finite image. As the semisimplification of ϱ_C is isomorphic to η^{ss}, by virtue of Lemma 4.1.1, we conclude that $\varrho_C(\pi_1(X))$ is finite. □

Note that for any $q = p^n$ with $n \in \mathbb{Z}_{>0}$, we have $\overline{\mathbb{F}_q((t))} = \overline{\mathbb{F}_p((t))}$. Let $\tau_i : \pi_1(X) \to \mathrm{GL}_N(\overline{\mathbb{F}_q((t))})$ be the semisimplification of ϱ_i. Then we have

$$[\tau_i] \in M(\overline{\mathbb{F}_q((t))}). \tag{4.1.1}$$

Lemma 4.1.3. — *Let Z be any closed subvariety of X such that $s_\tau(Z)$ is a point. Then $j \circ \pi(T)$ is a point, where $j : M_{\mathrm{B}}(X,N)_{\mathbb{F}_p} \to M_{\mathrm{B}}(Z,N)_{\mathbb{F}_p}$ is the natural morphism induced by the inclusion $\iota : Z \to X$.*

Proof. — By the definition of s_τ, we have $\iota^*\tau_i$ is bounded for each i. By Claim 1.4.1, $\iota^*\varrho_i$ is also bounded for each i. By the definition of ϱ_i and Lemma 4.1.2, we conclude that $j \circ \pi(T)$ is a point. □

Definition 4.1.4. — The *reduction map* $s_M : X \to S_M$ is obtained through the simultaneous Stein factorization of the reductions $\{s_\tau : X \to S_\tau\}_{[\tau] \in M(K)}$. Here $\tau : \pi_1(X) \to \mathrm{GL}_N(K)$ ranges over all reductive representations with K a local field of characteristic p such that $[\tau] \in M(K)$ and $s_\tau : X \to S_\tau$ is the reduction map defined in Theorem 1.E.

The reduction map $s_M : X \to S_M$ enjoys the following crucial property.

Lemma 4.1.5. — *Let $Z \subset X$ be a connected Zariski closed subset such that $s_M(Z)$ is a point in S_M. Then $j(M)$ is zero dimensional, where $j : M_{\mathrm{B}}(X,N)_{\mathbb{F}_p} \to M_{\mathrm{B}}(Z,N)_{\mathbb{F}_p}$ is the natural morphism induced by the inclusion $\iota : Z \to X$.*

Proof. — We may assume that Z and M are all geometrically irreducible. Since $\pi : R_{\mathbb{F}_p} \to M_{\mathrm{B}}(X,N)_{\mathbb{F}_p}$ is a surjective morphism between affine $\mathbb{F}_p$-schemes of finite type and M is defined over $\overline{\mathbb{F}_p}$, for affine curve $T_o \subset M$ defined over $\overline{\mathbb{F}_p}$, there exists an affine curve $T \subset \pi^{-1}(M)$ defined over $\overline{\mathbb{F}_p}$ such that $\pi(T)$ dominates T_o. By Definition 4.1.4 and (4.1.1), $s_{\tau_i}(Z)$ is a point for each i. It thus follows from Lemma 4.1.3 that $j(T_o)$ is a point. Since M is an affine $\overline{\mathbb{F}_p}$-scheme of finite type, any two general closed points in M can be covered by an affine curve in M defined over $\overline{\mathbb{F}_p}$. It follows that $j(M)$ is a point. □

Theorem 4.1.6. — *Let M be a Zariski closed subset of $M_{\mathrm{B}}(X,N)_{\mathbb{F}_p}$ defined over $\overline{\mathbb{F}_p}$ with $p > 0$. The reduction map $s_M : X \to S_M$ is the Shafarevich morphism for M. That is, for any connected Zariski closed subvariety Z of X, the following properties are equivalent:*

(a) *$s_M(Z)$ is a point;*
(b) *for any linear representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ with K a field of characteristic p and $[\varrho] \in M(K)$, we have $\varrho(\mathrm{Im}[\pi_1(Z) \to \pi_1(X)])$ is finite;*
(c) *for any semisimple representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ with K a field of characteristic p such that $[\varrho] \in M(K)$, we have $\varrho(\mathrm{Im}[\pi_1(Z_o^{\mathrm{norm}}) \to \pi_1(X)])$ is finite, where Z_o is any irreducible component of Z.*

Proof. — (a) $\Longrightarrow$ (b): Let $M_1, \ldots, M_k$ be all geometric connected component of M. Since M is defined over $\overline{\mathbb{F}_p}$, we can find semisimple representations $\varrho_i : \pi_1(X) \to \mathrm{GL}_N(\overline{\mathbb{F}_p})$ such that $[\varrho_i] \in M_i(\overline{\mathbb{F}_p})$. Let K be any field of characteristic p and $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a linear representation such that $[\varrho] \in M(K)$. Then $[\varrho] \in M_i(\bar{K})$ for some i. Note that $j(M_i)$ is a point by Lemma 4.1.5, where $j : M_{\mathrm{B}}(X,N)_{\mathbb{F}_p} \to M_{\mathrm{B}}(Z,N)_{\mathbb{F}_p}$ is the natural morphism induced by the inclusion $\iota : Z \to X$. It follows that $[\iota^*\varrho] = [\iota^*\varrho_i] \in M_{\mathrm{B}}(Z,N)_{\mathbb{F}_p}(\bar{K})$, where $\iota : Z \to X$ is the inclusion. Therefore, the semisimplification of $\iota^*\varrho$ is conjugate that of $\iota^*\varrho_i$. Note that $\iota^*\varrho_i(\pi_1(Z))$ is finite, for $\varrho_i(\pi_1(X)) \subset \mathrm{GL}_N(\overline{\mathbb{F}_p})$ is finite. Hence the image of its semisimplification is also finite. By Lemma 4.1.1, we conclude that $\iota^*\varrho(\pi_1(Z))$ is finite.

(b) $\Longrightarrow$ (c): this is obvious.

(c) $\implies$ (a): Let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ with K a field of characteristic p such that $[\varrho] \in M(K)$. Then, the image $\varrho(\mathrm{Im}[\pi_1(Z_o^{\mathrm{norm}}) \to \pi_1(X)])$ is finite by our assumption, and is thus bounded. By the property in Theorem 1.E, $s_\varrho(Z)$ is a point. By Definition 4.1.4, $s_M(Z)$ is also a point. □

Theorem Theorem 4.1.6 yields the positive characteristic part of Theorem 3.A.

Theorem 4.1.7 **($\subset$Theorem 3.A). —** *Let X be a quasi-projective normal variety and $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a linear representation, where K is a field of characteristic $p > 0$. Then the Shafarevich morphism $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$ exists. That is, for any connected Zariski closed subset $Z \subset X$, the following properties are equivalent:*

(a) *$\mathrm{sh}_\varrho(Z)$ is a point;*
(b) *$\varrho(\mathrm{Im}[\pi_1(Z) \to \pi_1(X)])$ is finite;*
(c) *for each irreducible component Z_o of Z, $\varrho^{ss}(\mathrm{Im}[\pi_1(Z_o^{\mathrm{norm}}) \to \pi_1(X)])$ is finite.*

Proof (sketch). — We will assume that X is smooth. Define

$$M := \bigcap_{f : Y \to X} j_f^{-1}\{1\}, \tag{4.1.2}$$

where 1 stands for the trivial representation, and $f : Y \to X$ ranges over all proper morphisms from positive dimensional quasi-projective normal varieties such that $[f^*\varrho] = 1$. Here $j_f : M_{\mathrm{B}}(X, N)_{\mathbb{F}_p} \to M_{\mathrm{B}}(Y, N)_{\mathbb{F}_p}$ is a morphism of affine $\mathbb{F}_p$-scheme induced by f. Then M is a Zariski closed subset defined over $\overline{\mathbb{F}_p}$. We apply Theorem 4.1.6 to construct the Shafarevich morphism $s_M : X \to S_M$ associated with M. It is a dominant morphism with general fibers connected. Let $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$ be $s_M : X \to S_M$ and we will prove that it satisfies the properties in the theorem.

(a) $\Rightarrow$ (b): this follows from the fact that $[\varrho] \in M(K)$ and Theorem 4.1.6.

(b) $\Rightarrow$ (c): obvious.

(c) $\Rightarrow$ (a): We take a finite étale cover $Y \to Z_o^{\mathrm{norm}}$ such that $f^*\varrho^{ss}(\pi_1(Y))$ is trivial, where we denote by $f : Y \to X$ the natural proper morphism. Let $\tau : \pi_1(X) \to \mathrm{GL}_N(L)$ be any linear representation such that $[\tau] \in M(L)$ where L is any field of characteristic p. Then $[f^*\tau] = 1$ by (4.1.2). Thanks to Lemma 4.1.1, $f^*\tau(\pi_1(Y))$ is finite, and it follows that $\tau(\mathrm{Im}[\pi_1(Z_o^{\mathrm{norm}}) \to \pi_1(X)])$ is finite as $\mathrm{Im}[\pi_1(Y) \to \pi_1(Z_o^{\mathrm{norm}})]$ is a finite index subgroup of $\pi_1(Z_o^{\mathrm{norm}})$. According to Theorem 4.1.6, $s_M(Z)$, and thus $\mathrm{sh}_\varrho(Z)$ is a point. □

Remark 4.1.8. **—** If we compare the reduction maps defined in characteristic zero as per Definition 3.4.2 and in characteristic $p > 0$ as per Definition 4.1.4, we observe that they are defined in the same way, and have the same properties as shown in Proposition 3.4.3 and Lemma 4.1.5.

However, the construction of the Shafarevich morphism in Theorems 4.1.6 and 4.1.7 does not involve transcendental aspects such as $\mathbb{C}$-VHS and period maps as we have seen in the proof of Theorem 3.4.7. Instead, we solely rely on the reduction map for unbounded representations in algebraic groups over positive characteristic fields. In characteristic zero, the construction becomes significantly more involved, and the consideration of $\mathbb{C}$-VHS is unavoidable.

4.2. Hyperbolicity and fundamental groups (II)

In this section, we will use the structure of the Shafarevich morphism constructed in Theorem 4.1.6 to sketch the proof of Theorems 2.A.(ii) and 2.E.(ii).

4.2.1. On the generalized Green-Griffiths-Lang conjecture (II). —

Theorem 4.2.1 **(=Theorem 2.E.(i)). —** *Let X be a quasi-projective normal variety. Let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a big representation where K is a field of positive characteristic. Then the following properties are equivalent:*

(i) *X is of log general type;*
(ii) *X is pseudo Picard hyperbolic;*
(iii) *X is pseudo Brody hyperbolic;*
(iv) *there exists a proper Zariski closed subset $\Xi \subsetneq X$ such that any positive dimensional closed subvariety $V \subset X$ is of log general type provided that $V \not\subset \Xi$.*

Proof of Theorem 4.2.1. — By replacing X with a desingularization and ϱ with the pullback on this birational model, we can assume that X is smooth. Let $\overline{X}$ be a smooth projective compactification of X such that $D := \overline{X} \backslash X$ is a simple normal crossing divisor. By Theorem 4.1.7, the Shafarevich morphism $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$ exists and is the quasi-Stein factorization of $(s_{\tau_1}, \ldots, s_{\tau_k}) : X \to S_{\tau_1} \times \cdots \times S_{\tau_k}$, where $s_{\tau_i} : X \to S_{\tau_i}$ is the reduction map associated with some unbounded and semisimple representation $\tau_i : \pi_1(X) \to \mathrm{GL}_N\left(\overline{\mathbb{F}_{q_i}((t))}\right)$.

By the construction of s_{τ_i} outlined in Theorem 1.E, we can take a spectral covering $\overline{X^{\rm sp}} \to \overline{X^{\rm sp}}$ (a Galois cover with Galois group G) there exists a finite (ramified) Galois cover $\pi_i : \overline{X}_i \to \overline{X}$ with Galois group H_i such that

(a) there exists (spectral) forms $\{\eta_1, \ldots, \eta_m\} \subset H^0(\overline{X^{\rm sp}}, \pi^*\Omega_{\overline{X}}(\log D))$ associated with $\tau_1, \ldots, \tau_k$, which are invariant under H;
(b) π is étale outside

$$R := \{x \in \overline{X^{\rm sp}} \mid \exists \eta_i \neq \eta_j \text{ with } (\eta_i - \eta_j)(x) = 0\} \tag{4.2.1}$$

(c) There exists a morphism $a : X^{\rm sp} \to A$ to a semi-abelian variety A with H acting on A such that a is H-equivariant.
(d) $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$ is the quasi-Stein factorization of the quotient $X \to A/H$ of a by H.

Since ϱ is big, it follows that sh_ϱ is birational and thus

Claim 4.2.2. — *We have* $\dim X^{\rm sp} = \dim a(X^{\rm sp})$.

Based on Claim 4.2.2, we can apply techniques in Theorem 2.C to prove the theorem.

(i) $\Rightarrow$ (ii): We will use notions of Nevanlinna theory in section 2.6. For any holomorphic map $f : \mathbb{C}_{>\delta} \to X$ whose image is not contained in $\pi(R)$, there exists a surjective finite holomorphic map $p : Y \to \mathbb{C}_{>\delta}$ from a connected Riemann surface Y to $\mathbb{C}_{>\delta}$ and a holomorphic map $g : Y \to X^{\rm sp}$ satisfying the following diagram:

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "p"] & X^{\rm sp} \arrow[d, "\pi"] \\ \mathbb{C}_{>\delta} \arrow[r, "f"] & X \end{tikzcd} \tag{4.2.2}$$

By Claim 2.6.3, there exists a proper Zariski closed subset $E \subsetneq X$ such that for any holomorphic map $f : \mathbb{C}_{>\delta} \to X$ whose image not contained in E, one has

$$N_{\mathrm{ram}\, p}(r) = o(T_g(r, L)) + O(\log r)||,$$

where $g : Y \to X^{\rm sp}$ is the induced holomorphic map in (4.2.2), L is an ample line bundle on $\overline{X^{\rm sp}}$ equipped with a smooth hermitian metric and $T_g(r, L)$ is the Nevanlinna order function. Note that $X^{\rm sp}$ of log general type as we assume that X is of log general type and $\pi : X^{\rm sp} \to X$ is a Galois cover. We apply [CDY22, Theorem 4.1] to conclude that there exists an extension $\overline{g} : \overline{Y} \to \overline{X^{\rm sp}}$ of g, where $\overline{Y}$ is a Riemann surface such that $p : Y \to \mathbb{C}_{>\delta}$ extends to a proper map $\overline{p} : \overline{Y} \to \mathbb{C}_{>\delta} \cup \{\infty\}$. This induces an extension $\bar{f} : \mathbb{C}_{>\delta} \cup \{\infty\} \to \overline{X}$. Hence, X is pseudo Picard hyperbolic.

(ii) $\Rightarrow$ (iii), (iv) $\Rightarrow$ (i): obvious.

(iii) $\Rightarrow$ (iv): the step is exactly the same as that in the proof of Theorem 2.C. □

4.2.2. Hyperbolicity via fundamental groups. —

***Theorem 4.2.3* (=Theorem 2.A.(ii)).** — *Let X be a smooth quasi-projective variety. Let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a big representation where K is a field of positive characteristic. If the Zariski closure G of $\varrho(\pi_1(X))$ is a semisimple algebraic group, then $\mathrm{Sp}_\bullet(X) \subsetneq X$ where $\mathrm{Sp}_\bullet$ denotes any of $\mathrm{Sp}_{\rm sab}$, $\mathrm{Sp}_{\rm alg}$, $\mathrm{Sp}_{\rm h}$ or $\mathrm{Sp}_{\rm p}$ defined in Definition 2.7.1.*

Proof. — We may assume that K is algebraically closed. Replacing X by a desingularization, we may assume that X is smooth. We will still maintain the same notations as introduced in Theorem 4.2.1. Let $\pi : X^{\rm sp} \to X$ be the Galois covering defined therein. Consider the representation $\pi^*\varrho : \pi_1(X^{\rm sp}) \to G(K)$, which is Zariski dense as $\mathrm{Im}[\pi_1(X^{\rm sp}) \to \pi_1(X)]$ is a finite index subgroup of $\pi_1(X)$. By the proof of Theorem 4.2.1, there exists a morphism $a : X^{\rm sp} \to A$ where A is a semiabelian variety such that $\dim X^{\rm sp} = \dim a(X^{\rm sp})$. Hence we have $\bar{\kappa}(X^{\rm sp}) \geq 0$.

Claim 4.2.4. — *$X^{\rm sp}$ is of log general type.*

Proof. — Let $\mu : Y \to X^{\rm sp}$ be a desingularization such that the logarithmic Iitaka fibration $j : Y \to J(Y)$ is regular. For a very general fiber F of j, we have $\bar{\kappa}(F) = 0$ and $\dim F = \dim a(F)$. By [CDY22, Lemma 3.3], we have $\pi_1(F)$ is abelian.

We write $\tau = (\pi \circ \mu)^*\varrho : \pi_1(Y) \to G(K)$. Notably, $\tau(\pi_1(Y))$ is Zariski dense in G. By [CDY22, Lemma 2.2], $\mathrm{Im}[\pi_1(F) \to \pi_1(Y)]$ is a normal subgroup of $\pi_1(Y)$. Consequently, the Zariski closure N of $\tau(\mathrm{Im}[\pi_1(F) \to \pi_1(Y)])$ is a normal subgroup of G. It's worth noting that the connected component N° of N is a tori since $\tau(\mathrm{Im}[\pi_1(F) \to \pi_1(Y)])$ is commutative. Therefore, N° must be trivial since G is assumed to be semisimple. Consequently, $\tau(\mathrm{Im}[\pi_1(F) \to \pi_1(Y)])$ is finite.

Given our assumption that ϱ is big, we conclude that τ is also big. Therefore, we arrive at the conclusion that $\dim F = 0$, leading us to deduce that both Y and, consequently, $X^{\rm sp}$ are of log general type. □

We can carry out the same proof as the step (i)$\Rightarrow$(ii) in the proof of Theorem 4.2.1 to conclude that X is pseudo Picard hyperbolic. It's essential to emphasize that in that proof, the condition of X being of log general type is only used to show that X^{sp} is of log general type. We now apply Theorem 2.6.2 to conclude that $\mathrm{Sp}_\bullet \subsetneq X$ where $\mathrm{Sp}_\bullet$ denotes any of $\mathrm{Sp}_{\mathrm{sab}}$, $\mathrm{Sp}_{\mathrm{alg}}$, Sp_{h} or Sp_{p}. □

Remark 4.2.5. — It is noteworthy that in characteristic zero, we first establish Theorem 2.A and subsequently deduce Theorem 2.E from it. Conversely, in the positive characteristic case, we reverse this order. Nonetheless, the fundamental result underlying these theorems remains Theorem 2.C.

CHAPTER 5

SOME APPLICATIONS IN ALGEBRAIC GEOMETRY

5.1. Algebraic varieties with compactifiable universal coverings

In the work [CHK13, CH13], Claudon, Höring and Kollár proposed the following intriguing conjecture:
Conjecture 5.1.1. — *Let X be a complex projective manifold with infinite fundamental group $\pi_1(X)$. Suppose that the universal cover $\widetilde{X}$ is quasi-projective. Then after replacing X by a finite étale cover, there exists a locally trivial fibration $X \to A$ with simply connected fiber F onto a complex torus A. In particular we have $\widetilde{X} \simeq F \times \mathbb{C}^{\dim A}$.*

It's worth noting that assuming abundance conjecture, Claudon, Höring and Kollár proved this conjecture in [CHK13]. In [CH13], Claudon-Höring proved Conjecture 5.1.1 in the case where $\pi_1(X)$ is virtually abelian.

As the first application of Theorem 2.A, in this section we establish a linear version of Conjecture 5.1.1 without relying on the abundance conjecture.
***Theorem 5.A* ([DY24, Theorem F]).** — *Let X be a smooth projective variety with an infinite fundamental group $\pi_1(X)$, such that its universal covering $\widetilde{X}$ is a Zariski open subset of some compact Kähler manifold $\overline{X}$. If there exists a faithful representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$, where K is any field, then the Albanese map of X is (up to finite étale cover) locally isotrivial with simply connected fiber F. In particular we have $\widetilde{X} \simeq F \times \mathbb{C}^{q(X)}$ with $q(X)$ the irregularity of X.*
Proof of Theorem 5.A. — We may assume that K is algebraically closed. Let G be the Zariski closure of $\varrho(\pi_1(X))$. After replacing X by a finite étale cover, we may assume that G is connected. Let $R(G)$ be the radical of G.

Step 1: we prove that G is solvable. Let $H := G/R(G)$, which is semisimple. Then ϱ induces a Zariski dense representation $\sigma : \pi_1(X) \to H(K)$. It is noteworthy that the Shafarevich morphism $\mathrm{sh}_\sigma : X \to \mathrm{Sh}_\sigma(X)$ of σ exists by Theorem 3.A. By the property of the Shafarevich morphism, each fiber F of sh_σ is connected and $\sigma(\mathrm{Im}[\pi_1(F) \to \pi_1(X)])$ is finite.

We can prove that after we replace X by a finite étale cover, σ factors through its Shafarevich morphism. Namely, there exists a large representation $\tau : \pi_1(\mathrm{Sh}_\sigma(X)) \to H(K)$ such that $\sigma = \mathrm{sh}_\sigma^* \tau$.
Claim 5.1.2. — *The group H is trivial.*
Proof. — Given that G is connected, it follows that $G/R(G)$ is also connected. Consequently, to prove that H is trivial, it suffices to show that $\dim H = 0$. Assume for the sake of contradiction that $\dim H > 0$.

Since $\sigma(\pi_1(X))$ is Zariski dense in H, $\sigma(\pi_1(X))$ is infinite, and thus $\dim \mathrm{Sh}_\sigma(X) > 0$. Note that $\tau : \pi_1(\mathrm{Sh}_\sigma(X)) \to H(K)$ is a large and Zariski dense representation. We apply Theorem 2.A to conclude that $\mathrm{Sh}_\sigma(X)$ is pseudo Picard hyperbolic.

Consider the surjective holomorphic map $h : \widetilde{X} \to \mathrm{Sh}_\sigma(X)$, which is the composition of $\mathrm{sh}_\sigma : X \to \mathrm{Sh}_\sigma(X)$ with the universal covering $\pi : \widetilde{X} \to X$. Given that $\widetilde{X}$ is a Zariski open subset of a compact Kähler manifold $\overline{\widetilde{X}}$, by Proposition 2.3.1, h can be extended to a meromorphic map $\bar{h} : \overline{\widetilde{X}} \dashrightarrow \mathrm{Sh}_\sigma(X)$. By blowing up the boundary $\overline{\widetilde{X}} \backslash \widetilde{X}$, we can assume that $\bar{h}$ is holomorphic.

Now, consider a general fiber F given by $\mathrm{sh}_\sigma^{-1}(y)$ with $y \in \mathrm{Sh}_\sigma(X)$, which is smooth and connected. As $\sigma(\mathrm{Im}[\pi_1(F) \to \pi_1(X)])$ is trivial, and $\sigma(\pi_1(X))$ is infinite, it implies that $\mathrm{Im}[\pi_1(F) \to \pi_1(X)]$ has infinite index in $\pi_1(X)$. Therefore, $\pi_0(\pi^{-1}(F))$ is infinite.

We note that $\pi^{-1}(F) = \widetilde{X} \cap \bar{h}^{-1}(y)$. Since $\overline{\widetilde{X}}$ is compact, we can deduce that $\pi_0(\pi^{-1}(F))$ is finite, leading to the contradiction. Hence H is trivial. □

Hence $G = R(G)$ is solvable.

Step 2: we prove that $\pi_1(X)$ is virtually abelian in the case where $\mathrm{char}\, K = 0$.
Claim 5.1.3. — *The Albanese map is surjective for every finite étale cover of X.*
Proof. — We replace X by any finite étale cover and would like to prove that its Albanese map $a : X \to A$ is surjective. Assume for the sake of contradiction that a is not surjective. By the universal property of the Albanese map, $a(X)$ is not a translation of abelian subvariety and thus the Kodaira dimension $\kappa(a(X)) > 0$. Let $B \subset A$

be the stabilizer of $a(X)$. We consider the morphism $c : X \to C = A/B$ which is the composition of a and the quotient $A \to A/B$. Then $Y = c(X) \subsetneq C$ is general type. Hence Y is pseudo Picard hyperbolic by the Bloch-Ochiai theorem (see also Theorem 2.D). We then use the same fashion as Step 1 to conclude a contradiction. □

By Claim 5.1.3, we apply [Cam04, Theorem 7.4] by Campana to conclude that every linear solvable quotient of $\pi_1(X)$ in characteristic zero is virtually abelian. Since ϱ is faithful and G is solvable, it follows that, up to some étale cover, the image $\varrho(\pi_1(X))$, hence $\pi_1(X)$ is abelian.

Step 3: we prove that $\pi_1(X)$ is virtually abelian when char $K > 0$. Since G is solvable and ϱ is faithful, it follows that $\pi_1(X)$ is solvable. By a theorem of Delzant [Del10, Théorème 1.4], $\pi_1(X)$ is virtually nilpotent. Thanks to Lemma 5.1.4, we conclude that $\varrho(\pi_1(X))$, hence $\pi_1(X)$ is virtually abelian.
Step 4. Completion of the proof. By Step 2 for char $K = 0$ and Step 3 for char $K > 0$, $\pi_1(X)$ is virtually abelian. By [CH13, Theorem 1.5], replacing X by a suitable finite étale cover, its Albanese map is a locally trivial fibration with simply connected fiber. We accomplish the proof of the theorem. □

Lemma 5.1.4. — *Let $\Gamma \subset \mathrm{GL}_N(K)$ be a finitely generated subgroup where K is an algebraically closed field of positive characteristic. If Γ is virtually nilpotent, then it is virtually abelian.*

5.2. Campana's abelianity conjecture

5.2.1. Special and h-special varieties: properties and conjectures. — We first recall the definition of special varieties by Campana [Cam04, Cam11].
***Definition 5.2.1* (Campana's specialness).** — Let X be a quasi-projective normal variety.

(i) X is *weakly special* if for any finite étale cover $\widehat{X} \to X$ and any proper birational modification $\widehat{X}' \to \widehat{X}$, there exists no dominant morphism $\widehat{X}' \to Y$ with connected general fibers such that Y is a positive-dimensional quasi-projective variety of log general type.
(ii) X is *special* if for any proper birational modification $X' \to X$ there is no dominant morphism $X' \to Y$ to with connected general fibers over a positive-dimensional quasi-projective variety Y such that the *Campana orbifold base* (or simply orbifold base) is of log general type.
(iii) X is *Brody special* if it contains a Zariski dense entire curve.

Campana defined X to be H*-special* if X has vanishing Kobayashi pseudo-distance. Motivated by [Cam11, 11.3 (5)], in [CDY22, Definition 1.11] we introduce the following definition.
***Definition 5.2.2* (h-special).** — Let X be a smooth quasi-projective variety. We define the equivalence relation $x \sim y$ of two points $x, y \in X$ iff there exists a sequence of holomorphic maps $f_1, \ldots, f_l : \mathbb{C} \to X$ such that letting $Z_i \subset X$ to be the Zariski closure of $f_i(\mathbb{C})$, we have

$$x \in Z_1, Z_1 \cap Z_2 \neq \emptyset, \ldots, Z_{l-1} \cap Z_l \neq \emptyset, y \in Z_l.$$

We set $R = \{(x, y) \in X \times X; x \sim y\}$. We define X to be *hyperbolically special* (h-special for short) iff $R \subset X \times X$ is Zariski dense.

By definition, rationally connected projective varieties are h-special without referring to a theorem of Campana and Winkelmann [CW16], who proved that all rationally connected projective varieties contain Zariski dense entire curves. It also has the following properties.
***Lemma 5.2.3* ([CDY22, Lemmas 10.2 & 10.4]).** — (i) *If a smooth quasi-projective variety X is Brody special, then it is h-special.*
(ii) *Let X be an h-special smooth quasi-projective variety, and let $p : X' \to X$ be a finite étale morphism or proper birational morphism from a quasi-projective variety X'. Then X' is h-special.* □
***Proposition 5.2.4* ([CDY22, Proposition 11.11]).** — *If a quasi-projective smooth variety X is special or h-special, the quasi-albanese map $a : X \to A$ of X is dominant with general fibers connected. Moreover, it is π_1-exact, i.e., we have the following exact sequence:*

$$\pi_1(F) \to \pi_1(X) \to \pi_1(A) \to 1,$$

where F is a general fiber of a. □
In [Cam04, Cam11], Campana proposed the following tantalizing abelianity conjecture.
***Conjecture 5.2.5* (Campana).** — *A special smooth quasi-projective variety has virtually abelian fundamental group.*
In [CDY22] we observed that Conjecture 5.2.5 fails for non-proper quasi-projective variety.

5.2.2. Counter-example to Campana's conjecture. — In [CDY22, Example 11.23], we constructed a smooth quasi-projective variety such that it is both special and Brody special, yet it has nilpotent fundamental group that is not virtually abelian.

Example 5.2.6. — Fix $\tau \in \mathbb{H}$ from the upper half plane. Then $\mathbb{C}/<\mathbb{Z}+\mathbb{Z}\tau>$ is an elliptic curve. We define a nilpotent group G as follows.

$$G = \left\{ g(l,m,n) = \begin{pmatrix} 1 & 0 & m & n \\ -m & 1 & -\frac{m^2}{2} & l \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_4(\mathbb{Z}) \mid l,m,n \in \mathbb{Z} \right\}$$

Thus as sets $G \simeq \mathbb{Z}^3$. However, G is non-commutative as direct computation shows:

$$g(l,m,n)\cdot g(l',m',n') = g(-mn'+l+l', m+m', n+n'). \tag{5.2.1}$$

We define $C \subset G$ by letting $m = 0$ and $n = 0$.

$$C = \left\{ g(l,0,0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_4(\mathbb{Z}) \mid l \in \mathbb{Z} \right\}$$

Then C is a free abelian group of rank one, thus $C \simeq \mathbb{Z}$ as groups. By (5.2.1), C is a center of G. We have an exact sequence

$$1 \to C \to G \to L \to 1,$$

where $L \simeq \mathbb{Z}^2$ is a free abelian group of rank two. This is a central extension. The quotient map $G \to L$ is defined by $g(l,m,n) \mapsto (m,n)$.

Claim 5.2.7. — *G is not almost abelian.*

Now we define an action $G \curvearrowright \mathbb{C}^2$ as follows: For $(z,w) \in \mathbb{C}^2$, we set

$$\begin{pmatrix} z \\ w \\ \tau \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & m & n \\ -m & 1 & -\frac{m^2}{2} & l \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ w \\ \tau \\ 1 \end{pmatrix}$$

Hence

$$g(l,m,n)\cdot(z,w) = \left(z + m\tau + n, -mz + w - \frac{m^2}{2}\tau + l \right).$$

This action is properly discontinuous. We set $X = \mathbb{C}^2/G$. Hence $\pi_1(X) = G$. Then X is a smooth complex manifold. We have $\mathbb{C}^2/C \simeq \mathbb{C}\times\mathbb{C}^*$. The action $L \curvearrowright \mathbb{C}\times\mathbb{C}^*$ is written as

$$(z,\xi) \mapsto (z+m\tau+n, e^{-2\pi imz-\pi im^2\tau}\xi), \tag{5.2.2}$$

where $\xi = e^{2\pi iw}$. The first projection $\mathbb{C}\times\mathbb{C}^* \to \mathbb{C}$ is equivariant $L \curvearrowright \mathbb{C}\times\mathbb{C}^* \to \mathbb{C} \curvearrowleft L$. By this, we have $X \to E$, where $E = \mathbb{C}/<\mathbb{Z}+\mathbb{Z}\tau>$ is an elliptic curve. The action (5.2.2) gives the action on $\mathbb{C}\times\mathbb{C}$ by the natural inclusion $\mathbb{C}\times\mathbb{C}^* \subset \mathbb{C}\times\mathbb{C}$. We consider this as a trivial line bundle by the first projection $\mathbb{C}\times\mathbb{C} \to \mathbb{C}$. We set $Y = (\mathbb{C}\times\mathbb{C})/L$. This gives a holomorphic line bundle $Y \to E$. By Serre's GAGA, Y is algebraic. Hence $X = Y - Z$ is quasi-projective, where Z is the zero section of Y.

Claim 5.2.8. — *The quasi-projective surface X is special and contains a Zariski dense entire curve. In particular, it is h-special.*

Consequently, within the quasi-projective context, we revised Conjecture 5.2.5 as follows.

Conjecture 5.2.9 **([CDY22, Conjecture 1.14]).** — *A special or h-special smooth quasi-projective variety has virtually nilpotent fundamental group.*

5.2.3. Nilpotency conjecture in the linear case. — In [CDY22, DY24], we confirm Conjecture 5.2.9 for quasi-projective varieties with linear fundamental groups.

Theorem 5.B. — *Let X be a special or h-special smooth quasiprojective variety. Let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a linear representation where K is any field.*

(i) [CDY22, Theorem 11.2] *If* char $K = 0$, *then $\varrho(\pi_1(X))$ is nilpotent.*
(ii) [DY24, Theorem G] *If* char $K > 0$, *then $\varrho(\pi_1(X))$ is virtually abelian.*

□

By Example 5.2.6, Theorem 5.B is shown to be sharp. Surprisingly, in the context of representations in positive characteristic, we obtain a stronger result.

Proof of Theorem 5.B. — *Step 1. We prove that $\varrho(\pi_1(X))$ is solvable.* The idea is quite close to Step 1 in the proof of Theorem 5.A and we sketch it. We may assume that K is algebraically closed. Let G be the Zariski closure of $\varrho(\pi_1(X))$. Note that any finite étale cover of a special (resp. h-special) variety is still special (resp. h-special). After replacing X by a finite étale cover, we may assume that G is connected. Let $R(G)$ be the radical of G. Let $H := G/R(G)$, which is semisimple. If $\dim H > 0$, then ϱ induces a Zariski dense representation $\sigma : \pi_1(X) \to H(K)$. Let $\mathrm{sh}_\sigma : X \to \mathrm{Sh}_\sigma(X)$ be the Shafarevich morphism of σ. By the property of the

Shafarevich morphism in Theorem 3.A, a general fiber F of sh_σ is connected and $\sigma(\mathrm{Im}[\pi_1(F) \to \pi_1(X)])$ is finite. We can prove that after replacing X by a composition of birational modifications and finite étale Galois covers, there exists a a dominant morphism $f : X \to Y$ over a smooth quasi-projective variety Y with connected general fibers, and a big representation $\tau : \pi_1(Y) \to H(K)$ such that $\sigma = f^*\tau$. Hence τ is Zariski dense in H. By Theorem 2.A, Y is of log general type and pseudo Picard hyperbolic. This leads to a contradiction since X is special (thus weakly special by [Cam11]) or h-special. Hence $G = R(G)$.

Step 2. We prove that $\varrho(\pi_1(X))$ is virtually abelian if $\mathrm{char}\, K > 0$. Note that any finite étale cover of a special (resp. h-special) variety is still special (resp. h-special) by [Cam04] and Lemma 5.2.3. Replacing X by a finite étale cover, we may assume that $\pi_1(X)^{ab} \to \pi_1(A)$ is an isomorphism, where $\pi_1(X)^{ab} := \pi_1(X)/[\pi_1(X), \pi_1(X)]$. Since X is special or h-special, by Proposition 5.2.4, the quasi-albanese map $a : X \to A$ of X is π_1-exact, i.e., we have the following exact sequence:

$$\pi_1(F) \to \pi_1(X) \to \pi_1(A) \to 1,$$

where F is a general fiber of a. Hence $[\pi_1(X), \pi_1(X)]$ is the image of $\pi_1(F) \to \pi_1(X)$, which is thus finitely generated. It implies that $[\varrho(\pi_1(X)), \varrho(\pi_1(X))] = \varrho([\pi_1(X), \pi_1(X)])$ is also finitely generated. By Step 1, G is solvable. Hence we have $\mathcal{D}G \subset R_u(G)$, where $R_u(G)$ is the unipotent radical of G and $\mathcal{D}G$ is the the derived group of G. Consequently, we have

$$[\varrho(\pi_1(X)), \varrho(\pi_1(X))] \subset [G(K), G(K)] \subset R_u(G)(K).$$

Note that every subgroup of finite index in $[\pi_1(X), \pi_1(X)]$ is also finitely generated (cf. [ST00, Proposition 4.17]. By the same arguments in Lemma 5.1.4, we conclude that $[\varrho(\pi_1(X)), \varrho(\pi_1(X))]$ is finite. Hence $\varrho(\pi_1(X))$ is virtually abelian.

Step 3. We prove that $\varrho(\pi_1(X))$ is virtually nilpotent if $\mathrm{char}\, K = 0$. The proof is non-trivial and based on 5.2.10 below. □

Theorem 5.2.10 **([CDY22, Theorem 11.3]). —** *Let X be a special or h-special quasi-projective manifold. Let G be a connected, solvable algebraic group defined over $\mathbb{C}$. Assume that there exists a Zariski dense representation $\varphi : \pi_1(X) \to G$. Then G is nilpotent. In particular, $\varphi(\pi_1(X))$ is nilpotent.*

The proof of Theorem 5.2.10 is involved. It is inspired by [Cam01] and is based on Proposition 5.2.4 together with Deligne's theorem: the radical of the algebraic monodromy group of an admissible variation of mixed Hodge structures is *unipotent*. Let us explain the rough idea.

Proof of Theorem 5.2.10 (sketch). — We might assume that G is connected. We have an exact sequence

$$1 \to U \to G \to T \to 1, \tag{5.2.3}$$

where $U = R_u(G)$ is the unipotent radical and $T \subset G$ is a maximal torus. Then T acts on U/U' by the conjugate, where U' is the commutator subgroup of U.

Since T is commutative, we have $G' \subset U$, where $G' = [G, G]$ is the commutator subgroup. Hence we have

$$1 \to U/G' \to G/G' \to T \to 1.$$

Since G/G' is commutative and U/G' is unipotent, we have $G/G' = (U/G') \times T$. By

$$1 \to G'/G'' \to G/G'' \to G/G' \to 1,$$

G/G' acts on G'/G'' by the conjugate. By $T \subset (U/G') \times T = G/G'$, we get T-action on G'/G''. Then we have the following criterion for nilpotency of a solvable group.

***Lemma 5.2.11.* —** *Assume T acts trivially on G'/G''. Then G is nilpotent.*

Consider the quasi-Albanese map $a : X \to A$ and let F be a general fiber. Let $\Phi \subset \pi_1(X)$ be the image of $\pi_1(F) \to \pi_1(X)$. Let $\Pi \subset G'/G''$ be the image of $\pi_1(F) \to G'/G''$. Since $\pi_1(F)$ is finitely generated, Π is a finitely generated, abelian group. By Proposition 5.2.4, we have the following exact sequence:

$$1 \to \Phi \to \pi_1(X) \to \pi_1(A) \to 1.$$

Note that $\Phi' \subset \pi_1(X)$ is a normal subgroup. Hence we get

$$\begin{tikzcd}
1 \arrow[r] & \Phi^{ab} \arrow[r] \arrow[d] & \pi_1(X)/\Phi' \arrow[r] \arrow[d] & \pi_1(A) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & G'/G'' \arrow[r] & G/G'' \arrow[r] & G/G' \arrow[r] & 1
\end{tikzcd}$$

By the conjugation, we get $\pi_1(A) \to \mathrm{Aut}(\Phi^{ab})$. This induces $\rho : \pi_1(A) \to \mathrm{Aut}(\Pi)$. Note that G'/G'' is a commutative unipotent group. Hence $G'/G'' \simeq (\mathbb{G}_a)^n$, where $\mathbb{G}_a$ is the additive group. The exponential map $\mathrm{Lie}(G'/G'') \to G'/G''$ is an isomorphism. Note that $(\mathbb{G}_a)^n = \mathbb{C}^n$ as additive group. We have

$$\mathrm{Aut}((\mathbb{G}_a)^n) = \mathrm{GL}(\mathrm{Lie}(G'/G'')) = \mathrm{GL}(\mathbb{C}^n).$$

Hence by the conjugate, we have

$$\mu : G/G' \to \mathrm{Aut}(G'/G'') = \mathrm{GL}(\mathbb{C}^n).$$

Let $1 \to U \to G \to T \to 1$ be the sequence as in (5.2.3). We have $G/G' = (U/G') \times T$, from which we obtain $\mu|_T : T \to \mathrm{GL}(\mathbb{C}^n)$. In the following, we are going to prove that $\mu|_T$ is trivial and thus by Lemma 5.2.11 we can prove that G is unipotent.

Now we define a subgroup

$$\Sigma = \{\sigma \in \mathrm{Aut}(G'/G''); \ \sigma\Pi = \Pi\} \subset \mathrm{GL}(\mathbb{C}^n).$$

Note that $\rho : \pi_1(A) \to \mathrm{Aut}(\Pi)$ factors through $\mu : G/G' \to \mathrm{Aut}(G'/G'')$. This induces the following commutative diagram:

$$\begin{tikzcd} \pi_1(A) \arrow[r, "\rho"] \arrow[d] & \Sigma \arrow[d] \\ G/G' \arrow[r, "\mu"'] & \mathrm{GL}(\mathbb{C}^n) \end{tikzcd} \tag{5.2.4}$$

Since $\Pi \subset \mathbb{C}^n$ is finitely generated, Π is a free abelian group of finite rank. Since $\Pi \subset \mathbb{C}^n$ is Zariski dense, the linear subspace spanned by Π is $\mathbb{C}^n$. Hence we may embed $\Sigma \subset \mathrm{GL}(\Pi \otimes_{\mathbb{Z}} \bar{\mathbb{Q}})$. Let $E \subset \mathrm{GL}(\Pi \otimes_{\mathbb{Z}} \bar{\mathbb{Q}})$ be the Zariski closure of $\rho(\pi_1(A)) \subset \mathrm{GL}(\Pi \otimes_{\mathbb{Z}} \bar{\mathbb{Q}})$. Then E is commutative. Let $E^o \subset E$ be the identity component.

***Claim 5.2.12.* —** *E^o is unipotent.*

This is the place where we apply Deligne's theorem. Roughly speaking, for the algebraic monodromy group Z of an admissible variation of mixed Hodge structures induced by the smooth family $X^\circ \to A^\circ$ (after we shrinking $a : X \to A$ to a Zariski dense open subset A^o), we can prove that there is a surjection $Z \to E$. Since Z^o is unipotent, it follows that E^o is unipotent.

Let $Y \subset \mathrm{GL}(\mathbb{C}^n)$ be the Zariski closure of $\rho(\pi_1(A)) \subset \mathrm{GL}(\mathbb{C}^n)$. Let $Y^o \subset Y$ be the identity component. Since $Y \subset E$ and E^o is unipotent, Y^o is unipotent. Since the image of $\pi_1(A) \to G/G'$ is Zariski dense, the commutativity of (5.2.4) implies $\mu(G/G') \subset Y$. This is Zariski dense, in particular $Y^o = Y$. By $G/G' = (U/G') \times T$, μ induces $\mu|_T : T \to Y$. Since Y is unipotent, this is trivial as there exists no non-trivial morphism from algebraic torus to unipotent groups. Hence the action of T onto G'/G'' is trivial. By Lemma 5.2.11, G is nilpotent. □

5.3. A structure theorem: on a conjecture by Kollár

In [Kol95, Conjecture 4.18], Kollár raised the following conjecture on the structure of varieties with big fundamental group.

***Conjecture 5.3.1.* —** *Let X be a smooth projective variety with big fundamental group such that $0 < \kappa(X) < \dim X$. Then X has a finite étale cover $p : X' \to X$ such that X' is birational to a smooth family of abelian varieties over a projective variety of general type Z which has big fundamental group.*

In this section we address Conjecture 5.3.1. Our theorem is the following:

***Theorem 5.C* ([CDY22, Theorem G], [DY24, Theorem 6.2]). —** *Let X be a smooth quasi-projective variety and let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a big representation where K is a field of positive characteristic. When $\operatorname{char} K = 0$, we assume additionally that ϱ is reductive.*

(a) *The logarithmic Kodaira dimension $\bar{\kappa}(X) \geq 0$.*
(b) *If the logarithmic Kodaira dimension $\bar{\kappa}(X) = 0$, then up to a finite étale cover, the Albanese map $\alpha : X \to A$ of X is birational and proper in codimension one, i.e. there exists a Zariski closed subset $Z \subset A$ of codimension at least two such that α is proper over $A \backslash Z$. In particular, $\pi_1(X)$ is virtually abelian.*
(c) *More generally, after replacing X by a suitable finite étale cover and a birational modification, there are a semiabelian variety A, a quasi-projective manifold V, and a birational morphism $a : X \to V$ such that we have the following commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "a"] \arrow[dr, "j"'] & & V \arrow[dl, "h"] \\ & J(X) & \end{tikzcd}$$

where j is the logarithmic Iitaka fibration and $h : V \to J(X)$ is a locally trivial fibration with fibers isomorphic to A. Moreover, for a general fiber F of j, $a|_F : F \to A$ is proper in codimension one.
(d) *If Y is special or h-special, then there exists a finite étale cover X of Y, such that its Albanese map $\alpha : X \to A$ is birational and $\alpha_* : \pi_1(X) \to \pi_1(A)$ is an isomorphism.*

Proof. — We may assume that K is algebraically closed. To prove the theorem we are free to replace X by a birational modification and by a finite étale cover since the logarithmic Kodaira dimension will remain unchanged. If $\operatorname{char} K > 0$, we will replace ϱ by its semisimplification, which is still big by Lemma 4.1.1. Hence we might assume that ϱ is big and semisimple. Consequently, after replacing X by a finite étale cover, the Zariski closure G of ϱ is reductive and connected. Let $\mathcal{D}G$ be the derived group of G, which is semisimple. Then $T := G/\mathcal{D}G$ is a torus and the natural morphism $G \to \mathcal{D}G \times T$ is a central isogeny. The induced representation $\varrho' : \pi_1(X) \to \mathcal{D}G(K) \times T(K)$ by ϱ is also big. Consider the representation $\sigma : \pi_1(X) \to \mathcal{D}G(K)$, obtained by composing ϱ' with the projection $\mathcal{D}G \times T \to \mathcal{D}G$. Then $\sigma(\pi_1(X))$ is Zariski dense. Let $\mathrm{sh}_\sigma : X \to \mathrm{Sh}_\sigma(X)$ be the Shafarevich morphism of σ.

We can show that there exist

(i) a generically finite proper surjective morphism $\mu : X_1 \to X$ from a smooth quasi-projective variety obtained by the composition of birational modifications and finite étale Galois covers;
(ii) a generically finite dominant morphism $\nu : Y_1 \to \mathrm{Sh}_\sigma(X)$;
(iii) a dominant morphism $f_1 : X_1 \to Y_1$ with Y_1 a smooth quasi-projective variety with connected general fibers;
(iv) a big representation $\tau : \pi_1(Y_1) \to \mathcal{D}G(K)$;

such that we have following commutative diagram

$$\begin{tikzcd} X_1 \arrow[r, "\mu"] \arrow[d, "f_1"] & X \arrow[d, "\mathrm{sh}_\sigma"] \\ Y_1 \arrow[r, "\nu"] & \mathrm{Sh}_\sigma(X) \end{tikzcd}$$

and $\mu^*\sigma = f_1^*\tau$. It is straightforward to show that τ is a big representation. Thanks to Theorem 4.2.3, the special loci $\mathrm{Sp}_{\mathrm{alg}}(Y_1)$ and $\mathrm{Sp}_{\mathrm{p}}(Y_1)$ are both proper Zariski closed subset of Y_1. In particular, Y_1 is of log general type. Note that Y_1 can be a point.

Consider the morphism

$$\begin{aligned} g : X_1 &\to A \times Y_1 \\ x &\mapsto (\alpha(x), f_1(x)). \end{aligned}$$

where $\alpha : X_1 \to A$ is the quasi Albanese map of X_1.

Claim 5.3.2. — *We have* $\dim X_1 = \dim g(X_1)$.

Proof. — For a general smooth fiber F of f_1, $\mu^*\sigma(\mathrm{Im}[\pi_1(F) \to \pi_1(X)])$ is trivial. Since $\mu^*\varrho' : \pi_1(X_1) \to \mathcal{D}G(K) \times T(K)$ is big, by the construction of σ, we conclude that the representation $\eta : \pi_1(F) \to T(K)$ obtained by

$$\pi_1(F) \to \pi_1(X_1) \xrightarrow{\mu^*\varrho'} \mathcal{D}G(K) \times T(K) \to T(K)$$

is big. Since $T(K)$ is commutative, η factors through $\pi_1(F) \to \pi_1(A) \to T(K)$. This implies that $\dim F = \dim \alpha(F)$. Hence $\dim X_1 = \dim g(X_1)$. □

Let us prove Item (a). Thanks to Claim 5.3.2, for a general smooth fiber F of f_1, we have $\dim F = \dim \alpha(F)$. Hence $\bar{\kappa}(F) \geq 0$. Since Y_1 is of log general type, by the subadditivity of the logarithmic Kodaira dimension proven in [Fuj17, Theorem 1.9], we obtain

$$\bar{\kappa}(X_1) \geq \bar{\kappa}(Y_1) + \bar{\kappa}(F) \geq \bar{\kappa}(Y_1) = \dim Y_1 \geq 0.$$

Hence $\bar{\kappa}(X) = \bar{\kappa}(X_1) \geq 0$. The first claim is proved.

Let us prove item (b). Note that $\bar{\kappa}(X_1) = \bar{\kappa}(X) = 0$. Assume that $\dim Y_1 > 0$. Let F be a very general fiber of f_1. Then $\varrho|_{\pi_1(F)}$ is big, and by item (a), $\bar{\kappa}(F) \geq 0$. Since Y_1 is of log general type, by the above-mentioned Fujino's theorem, it follows that $\bar{\kappa}(X_1) \geq \dim Y_1 + \bar{\kappa}(F) > 0$. We obtain a contradiction. Hence Y_1 is a point. Therefore, the quasi-Albanese map $\alpha : X \to A$ satisfies that $\dim X = \dim \alpha(X)$. By Lemma 2.5.4, we conclude item (b).

The proof of item (c) is a bit involved, but the idea is straightforward. Let F be a very general fiber of the logarithmic Iitaka fibration $j : X \to J(X)$. By Claim 5.3.2, the closure $\overline{\alpha(F)}$ of $\alpha(F)$ is a translate of semi-abelian subvariety in A. One can show that after replacing X by a finite étale cover, we may assume that $\alpha : F \to \overline{\alpha(F)}$ is birational and proper in codimension one. Note that there are at most countable many semi-abelian subvarieties in A. Hence $\overline{\alpha(F)}$ is rigid. It follows that two general fibers of j is birationally isomorphic.

Finally let us show item (d). Note that X_1 is h-special or special. It follows that Y_1 is a point, or else, Y_1 is of log general type and pseudo Brody hyperbolic, leading to a contradiction. Hence the quasi-Albanese map $\alpha : X \to A$ satisfies that $\dim X = \dim \alpha(X)$. The claim follows from Proposition 5.2.4. □

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