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A characterization of complex quasi-projective manifolds uniformized by unit balls

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Abstract

In 1988 Simpson extended the Donaldson–Uhlenbeck–Yau theorem to the context of Higgs bundles, and as an application he proved a uniformization theorem which characterizes complex projective manifolds and quasi-projective curves whose universal coverings are complex unit balls. In this paper we give a necessary and sufficient condition for quasi-projective manifolds to be uniformized by complex unit balls. This generalizes the uniformization theorem by Simpson. Several byproducts are also obtained in this paper.

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1 Introduction

1.1 Main result

The main goal of this paper is to characterize complex quasi-projective manifolds whose universal coverings are complex unit balls.

Theorem A (=Theorem 5.7.(i)) *Let X be an n-dimensional complex projective manifold and let D be a* smooth *divisor on X (which might contain several disjoint components). Let L be an ample polarization on X. For the log Higgs bundle $(\Omega_X^1(\log D) \oplus \mathcal{O}_X, \theta)$ on (X, D) with the Higgs field θ defined by*

$$\begin{aligned} \theta : \Omega_X^1(\log D) \oplus \mathcal{O}_X &\to (\Omega_X^1(\log D) \oplus \mathcal{O}_X) \otimes \Omega_X^1(\log D) \\ (a, b) &\mapsto (0, a), \end{aligned} \tag{1.1.1}$$

if it is μ_L-polystable (see Sect. 2.3 *for the definition), then one has the following inequality*

$$\Big(2c_2(\Omega_X^1(\log D)) - \frac{n}{n+1} c_1(\Omega_X^1(\log D))^2\Big) \cdot c_1(L)^{n-2} \geq 0. \tag{1.1.2}$$

When the equality holds, then $X - D \simeq \mathbb{B}^n/\Gamma$ for some torsion free lattice $\Gamma \subset PU(n, 1)$ acting on $\mathbb{B}^n$. Moreover, X is the (unique) toroidal compactification of $\mathbb{B}^n/\Gamma$, and each connected component of D is the smooth *quotient of an Abelian variety A by a finite group acting freely on A.*

Let us stress here that the *smoothness* of D in Theorem A is indeed necessary if one would like to characterize non-compact ball quotients: in Theorem 5.7.(ii) we prove that the universal cover of $X - D$ is not the complex unit ball $\mathbb{B}^n$ if D is assumed to be simple normal crossing but not smooth, leaving other conditions in Theorem A unchanged. Thus, it might be more appropriate to say that in this paper we give a characterization of *smooth toroidal compactification* of non-compact ball quotients.

Note that when D is empty or when $\dim X = 1$, Theorem A has already been proved by Simpson [51, Proposition 9.8]. As we will see later, we follow his strategy closely to prove the above theorem. Let us also mention that the inequality (1.1.2) is a direct consequence of Mochizuki's deep work on the Bogomolov-Gieseker inequality for parabolic Higgs bundles [40, Theorem 6.5]. Our main contribution is the uniformization result when the equality in (1.1.2) is achieved. The proof builds on Simpson's ingenious ideas [51] on characterizations of complete varieties uniformized by Hermitian symmetric spaces, as well as Mochizuki's celebrated work on Simpson correspondence for tame harmonic bundles [40]. Since the Kobayashi-Hitchin correspondence for general slope polystable parabolic Higgs bundles is still unproven, we need some additional methods to prove the above uniformization result (see Sect. 1.3 for rough ideas).

We will show that the conditions in Theorem A is indeed necessary, by proving the following slope stability (with respect to a more general polarization) result for the natural log Higgs bundles associated to toroidal compactification of non-compact ball quotient by torsion free lattice.

Theorem B (=Sect. 6.4) *Let $\Gamma \subset PU(n,1)$ be a torsion free lattice with only unipotent parabolic elements. Let X be the (smooth) toroidal compactification of the ball quotient $\mathbb{B}^n/\Gamma$. Write $D := X - \mathbb{B}^n/\Gamma$ for the boundary divisor, which is a disjoint union of Abelian varieties. Let $\alpha \in H^{1,1}(X,\mathbb{R})$ be a big and nef cohomology $(1,1)$-class on X containing a positive closed $(1,1)$-current $T \in \alpha$ so that $T|_{X-D}$ is a smooth Kähler form and has at most* Poincaré growth *near D (for example, $\alpha = c_1(K_X + D)$ or α contains a Kähler form ω). Then one has the following equality for Chern classes*

$$2c_2(\Omega_X^1(\log D)) - \frac{n}{n+1}c_1(\Omega_X^1(\log D))^2 = 0. \tag{1.1.3}$$

The log Higgs bundle $(\Omega_X^1(\log D) \oplus \mathcal{O}_X, \theta)$ defined in (1.1.1) is μ_α-polystable for the above big and nef polarization α. In particular, it is slope polystable with respect to any Kähler polarization and the polarization by the big and nef class $c_1(K_X + D)$.

Since both stability of log Higgs bundles and Chern equality (1.1.3) are invariant under taking conjugates with respect to the Galois action, a direct consequence of Theorems A and B is the following rigidity result of ball quotient under the automorphism of complex number field $\mathbb{C}$ to its coefficients of defining equations.

Corollary C *Let $\Gamma \subset PU(n,1)$ be a torsion free lattice, and let $X := \mathbb{B}^n/\Gamma$ be the ball quotient, which carries a unique algebraic structure, denoted by X_{alg}. For any automorphism $\sigma \in \text{Aut}(\mathbb{C})$, let $X_{\text{alg}}^\sigma := X_{\text{alg}} \times_\sigma \text{Spec}(\mathbb{C})$ be the conjugate variety of X_{alg} under the automorphism σ, and denote by X^σ the analytification of X_{alg}^σ. Then X^σ is also a ball quotient, namely there is another torsion free lattice $\Gamma^\sigma \subset PU(n,1)$ so that $X^\sigma = \mathbb{B}^n/\Gamma^\sigma$.*

When Γ is arithmetic, Corollary C has been proved by Kazhdan [30]. When Γ is non-arithmetic, it was proved by Mok–Yeung [46, Theorem 1] and by Baldi-Ullmo [10, Theorem 8.4.2].

In this paper we obtain some byproducts, and let us mention a few. We prove the Simpson–Mochizuki correspondence for principal system of log Hodge bundles over projective log pairs (see Theorem 4.1). We give a characterization of slope stability with respect to big and nef classes for log Higgs bundles on Kähler log pairs (see Theorem 6.7). We also give a very simple proof of the negativity of kernels of Higgs fields of tame harmonic bundles by Brunebarbe [9] (originally by Zuo [60] for system of log Hodge bundles), using some extension theorems of plurisubharmonic functions in complex analysis (see Theorem 5.6). In the appendix written jointly with Benoît Cadorel, we prove a metric rigidity result for toroidal compactification of non-compact ball quotients (see Theorem A.7).

1.2 A few histories

Since the main purpose of this paper is to prove the uniformization result rather than the Miyaoka–Yau type inequality (1.1.2), we shall only recall some earlier work related to the characterization of ball quotient, and we refer the readers to [24,25] for more references on the Miyaoka–Yau type inequalities.

Based on his proof of the Calabi conjecture [57], Yau established the inequality (1.1.2) when X is a projective manifold and $D = \varnothing$ with K_X ample. He proved that X is uniformized by the complex unit ball in case of equality. Miyaoka–Yau inequality and uniformization result were extended to the context of compact Kähler varieties with quotient singularities by Cheng–Yau [16] using orbifold Kahler–Einstein metrics. A partial uniformization result for smooth minimal models of general type have been obtained by Zhang [59]. More recently, uniformization result has been extended to projective varieties with klt singularities in the series of work [22,23] by Greb–Kebekus–Peternell–Taji.

All the above works dealt with *compact* varieties. A strong uniformization result was established by Kobayashi [33,34] in the case of *open orbifold surfaces* (see also [16]). In [16] Cheng–Yau also gave a differential geometric characterization of quasi-projective ball quotients of any dimensions using the method of bounded geometry in [15]. At almost the same time, based on [16], Tian–Yau [56] and Tsuji [55] independently established similar *algebraic geometric* characterizations of non-compact ball quotient of any dimension. See also [32,35,58] for more related works on uniformization results.

All these aforementioned uniformization results are built on the positivity of the (log) canonical sheaf of the varieties together with existence of Kähler–Einstein metrics. In [51], Simpson established a remarkable uniformization result in terms of stability of Higgs bundles. We essentially follow his approaches in this paper. In next subsection, we shall recall his ideas and discuss main difficulties in generalizing his methods to the context of *non-compact varieties*.

1.3 Main strategy

We mainly follow Simpson's strategy [51] to prove Theorem A. Let us explain our rough ideas in the proof of Theorem A when the equality in (1.1.2) holds.

Step 1. Following Simpson in the compact setting, we first define systems of log Hodge bundles over log pairs. We prove that, a system of log Hodge bundles on a projective log pair with vanishing first and second Chern classes admits an *adapted* Hodge metric. The proof is based on Mochizuki's celebrated theorem [40, Theorem 9.4] on the existence of harmonic metric, and $\mathbb{C}^*$-action invariant property of log Hodge bundles.

Step 2. We generalize the result in Step 1 to the context of *principal bundles*. Fix a Hodge group G_0. Following Simpson again, we define a principal system of log Hodge bundles (P, τ) on log pairs (X, D) with the structure group $K \subset G$, where G is the complexification of G_0. Based on the result in Step 1 together with some similar Tannakian arguments in [52], in Theorem 4.1 we prove that if there is a faithful Hodge representation $\rho : G \to GL(V)$ for some polarized Hodge structure $(V = \oplus_{i+j=w} V^{i,j}, h_V)$ so that the system of log Hodge bundles $(P \times_K V, d\rho(\tau))$ is μ_L-polystable with $\int_X ch_2(P \times_K V) \cdot c_1(L)^{\dim X - 2} = 0$, then there is a metric reduction P_H for $P|_{X-D}$ so that the triple $(P|_{X-D}, \tau|_{X-D}, P_H)$ gives rise to a *principal variation of Hodge structures* on $X - D$.

Step 3. For the system of log Hodge bundles $(E := \Omega^1_X(\log D) \oplus \mathcal{O}_X, \theta)$ in Theorem A, we first associate it a principal system of log Hodge bundles (P, τ) in Proposition 3.11, whose Hodge group $G_0 = PU(n, 1)$ is of Hermitian type (see Definition 3.5). One can easily show that $c_2(P \times_K \mathfrak{g}) = c_2(\mathrm{End}(E)^{\perp}) = 0$ when the equality in (1.1.2) holds, where $\mathrm{End}(E)^{\perp}$ denotes the trace free part of $\mathrm{End}(E)$. By Theorem 2.9, the system of log Hodge bundles $(P \times_K \mathfrak{g}, d(Ad)(\tau)) = (\mathrm{End}(E)^{\perp}, \theta_{End(E)^{\perp}})$ is also slope polystable if (E, θ) is slope polystable. Since the adjoint representation $Ad : G \to GL(\mathfrak{g})$ is a faithful Hodge representation, by the result in Step 2, there is a metric reduction P_H for $P|_{X-D}$ so that the triple $(P|_{X-D}, \tau|_{X-D}, P_H)$ gives rise to a *principal variation of Hodge structures* on $X - D$. Since $\tau : T_X(-\log D) \to P \times_K \mathfrak{g}^{-1,1}$ is an isomorphism, this implies that the period map $p : \widetilde{X - D} \to PU(n, 1)/U(n)$ associated to $(P|_{X-D}, \tau|_{X-D}, P_H)$ from the universal cover $\widetilde{X - D}$ of $X - D$ to the period domain $G_0/K_0 = PU(n, 1)/U(n)$ is *locally biholomorphic*. For more details, see Step one of the proof of Theorem 5.7.

Step 4. We have to prove that the period map p in Step 3 is moreover a biholomorphism. Note that when $D = \varnothing$, this step is quite easy. In Remark 3.7 we show that it suffices to prove that the hermitian metric $\tau^* h_H$ on $X - D$ is *complete*, where h_H is the hermitian metric on $P \times_K \mathfrak{g}^{-1,1}|_{X-D}$ induced by the metric reduction P_H together with the Killing form of $\mathfrak{g}$. This step is slightly involved and the readers can find it in Step two of the proof of Theorem 5.7. To be brief, we establish a precise model metric (ansatz) for $(E, \theta) \otimes (E^*, \theta^*)$ locally around D with at most log growth, and we prove that this local metric and h_H are mutually bounded by one another using similar ideas in [52, § 4]. Based on this model metric, we obtain a precise norm estimates for h_H near D, so that we can prove that $\tau^* h_H$ is a complete metric on $X - D$. This concludes that the universal cover of $X - D$ is the unit ball $PU(n, 1)/U(n)$.

2 Log Higgs bundles and system of log Hodge bundles

2.1 Higgs bundles and tame harmonic bundles

In this section we recall the definition of Higgs bundles and tame harmonic bundles. We refer the readers to [39,41,51–53] for further details.

Definition 2.1 Let X be a complex manifold. A *Higgs bundle* on X is a pair (E, θ) where E is a holomorphic vector bundle with $\bar{\partial}_E$ its complex structure, and $\theta : E \to E \otimes \Omega^1_X$ is a holomorphic one form with value in $\mathrm{End}(E)$, say *Higgs field*, satisfying $\theta \wedge \theta = 0$.

Let (E, θ) be a Higgs bundle over a complex manifold X. A smooth hermitian metric h of E is called *harmonic* if $D_h := d_h + \theta + \overline{\theta}_h$ is flat. Here d_h is the Chern connection of (E, h), and $\overline{\theta}_h$ is the adjoint of θ with respect to h.

Definition 2.2 (*Harmonic bundle*) A harmonic bundle on a complex manifold X is triple (E, θ, h) where (E, θ) is a Higgs bundle and h is a harmonic metric for (E, θ).

A *log pair* consists of an n-dimensional complex manifold X, and a simple normal crossing divisor D on X.

Definition 2.3 (*Admissible coordinate*) Let p be a point of X, and assume that $\{D_j\}_{j=1,\ldots,\ell}$ be components of D containing p. An *admissible coordinate* around p is the tuple $(U; z_1, \ldots, z_n; \varphi)$ (or simply $(U; z_1, \ldots, z_n)$ if no confusion arises) where

- U is an open subset of X containing p.
- there is a holomorphic isomorphism $\varphi : U \to \Delta^n$ so that $\varphi(D_j) = (z_j = 0)$ for any $j = 1, \ldots, \ell$.

We shall write $U^* := U - D$, $U(r) := \{z \in U \mid |z_i| < r, \forall i = 1, \ldots, n\}$ and $U^*(r) := U(r) \cap U^*$.

Recall that the Poincaré metric ω_P on $(\Delta^*)^\ell \times \Delta^{n-\ell}$ is described as

$$\omega_P = \sum_{j=1}^{\ell} \frac{\sqrt{-1}dz_j \wedge d\bar{z}_j}{|z_j|^2(\log|z_j|^2)^2} + \sum_{k=\ell+1}^{n} \frac{\sqrt{-1}dz_k \wedge d\bar{z}_k}{(1-|z_k|^2)^2}.$$

Definition 2.4 (*Poincaré growth*) Let (X, D) be a log pair. A hermitian metric ω on $X - D$ has at most (resp. the same) *Poincaré growth near* D if for any point $x \in D$, there is an admissible coordinate $(U; z_1, \ldots, z_n)$ centered at x and a constant $C_U > 0$ so that $\omega \le C_U \omega_P$ (resp. $\omega \sim \omega_P$) holds over $U^*(r)$ for some $0 < r < 1$.

Remark 2.5 (Global Kähler metric with Poincaré growth) Let (X, ω) be a compact Kähler manifold and $D = \sum_{i=1}^{\ell} D_i$ is a simple normal crossing divisor on X. By Cornalba–Griffiths [12], one can construct a *Kähler current* T over X, whose restriction on $X - D$ is a complete Kähler form, which has the same Poincaré growth near D as follows.

Let σ_i be the section $H^0(X, \mathcal{O}_X(D_i))$ defining D_i, and we pick any smooth metric h_i for the line bundle $\mathcal{O}_X(D_i)$. One can prove that the closed (1, , 1)-current

$$T := \omega - \sqrt{-1}\partial\bar{\partial}\log(-\prod_{i=1}^{\ell}\log|\varepsilon\cdot\sigma_i|^2_{h_i}), \tag{2.1.1}$$

the desired Kähler current when $0 < \varepsilon \ll 1$.

2.2 Log Higgs bundle and adapted harmonic metrics

Throughout this paper, we mainly consider *log Higgs bundles* (E, θ) over log pairs.

Definition 2.6 (*Log Higgs bundles*) Let (X, D) be a log pair. A log Higgs bundle consists of a pair (E, θ) with E a holomorphic vector bundle on X and $\theta : E \to E \otimes \Omega^1_X(\log D)$ with $\theta \wedge \theta = 0$.

Definition 2.7 (*Adapted harmonic metric*) Let (X, D) be a log pair, and let (E, θ) be a log Higgs bundle on (X, D). Suppose that h is a harmonic metric for the Higgs bundle $(E, \theta)|_{X-D}$. It is called *adapted to* the log Higgs bundle if for any admissible coordinate $(U; z_1, \ldots, z_n)$ and any $(a_1, \ldots, a_\ell) \in [0, 1)^\ell$, one has

$$E(U) = \{\sigma \in E(U - D) \mid |\sigma|_h = \mathcal{O}(\prod_{i=1}^{\ell} |z_i|^{-a_i - \varepsilon}) \text{ for any } \varepsilon > 0\}$$

In the terminology of [40,41], the above definitions are equivalent that (E, θ) is a parabolic Higgs bundle with trivial parabolic structures over (X, D) of weight $(0, \ldots, 0)$, and the harmonic bundle h for $(E, \theta)|_{X-D}$ is adapted to its parabolic structures.

2.3 Slope stability

Let (X, ω) be a compact Kähler manifold of dimension n and let D be a simple normal crossing divisor on X. Let (E, θ) be a log Higgs bundle on (X, D). Let α be a big and nef cohomology $(1, 1)$-class on X. For any torsion free coherent sheaf F, its *degree with respect to* α is defined by $\deg_\alpha(F) := c_1(F) \cdot \alpha^{n-1}$, and its *slope with respect to* α is defined by $\mu_\alpha(F) := \frac{\deg_\alpha(F)}{\operatorname{rank} F}$. Consider a log Higgs bundle (E, θ) on (X, D). A *Higgs sub-sheaf* is a saturated coherent torsion free subsheaf $E' \subset E$ so that $\theta(E') \subset E' \otimes \Omega^1_X(\log D)$. We say (E, θ) is μ_α*-stable* if for Higgs sub-sheaf E' of E, with $0 < \operatorname{rank} E' < \operatorname{rank} E$, the condition $\mu_\alpha(E') < \mu_\alpha(E)$ is satisfied. (E, θ) is μ_α*-polystable* if it is a direct sum of μ_α-stable log Higgs bundles with the same slope.

When $\alpha = \{\omega\}$ where ω is a Kähler form on X, we write μ_ω instead of μ_α. When $\alpha = c_1(L)$ for some ample line bundle L on X, we use the notation μ_L instead of μ_α.

By Simpson [52], there is a $\mathbb{C}^*$-action on log Higgs bundles (E, θ) defined by $(E, t\theta)$ for any $t \in \mathbb{C}^*$. It follows from the definition that, if (E, θ) is μ_α-stable, then $(E, t\theta)$ is also μ_α-stable for any $t \in \mathbb{C}^*$.

The following celebrated *Simpson correspondence for tame harmonic bundles* proved by Mochizuki [40] is a crucial ingredient in this paper.

Theorem 2.8 (Mochizuki) *Let (X, D) be a projective log pair endowed with an ample polarization L. A log Higgs bundle (E, θ) on (X, D) is μ_L-polystable with $\int_X c_1(E) \cdot c_1(L)^{\dim X - 1} = \int_X ch_2(E) \cdot c_1(L)^{\dim X - 2} = 0$ if and only if there is a harmonic metric h for $(E|_{X-D}, \theta|_{X-D})$ adapted to (E, θ). When (E, θ) is moreover stable, such a harmonic metric h is unique up to some positive constant multiplication.*

Let us mention that in [5] Biquard has proved a stronger theorem when the divisor D in Theorem 2.8 is smooth.

The poly-stability is also preserved under tensor product and dual by Mochizuki [42, Proposition 4.10].

Theorem 2.9 (Mochizuki) *Let (X, D) be a projective log pair endowed with an ample polarization L. Let (E, θ) be a μ_L-polystable log Higgs bundle on (X, D). Then the* *tensor product* $T^{a,b}(E,\theta)$ *is still a* μ_L*-polystable log Higgs bundle for* $a,b\in\mathbb{Z}_{\geq 0}$. *Here* $T^{a,b}(E,\theta):=\big(Hom(E^{\otimes a},E^{\otimes b}),\theta_{a,b}\big)$ *is the induced log Higgs bundle by taking the tensor product.*

Since [42, Proposition 4.10] worked with the much more general case than what we need, we shall provide a quick proof for Theorem 2.9 for completeness sake. The idea essentially follows [53, Corollary 3.8] in the compact setting.

Proof of Theorem 2.9 By the Mehta–Ramanathan type theorem proved by Mochizuki [40, Proposition 3.29], $T^{a,b}(E,\theta)$ is μ_L-polystable if and only if $T^{a,b}(E,\theta)|_Y$ is μ_L-polystable, where Y denotes a complete intersection of sufficiently ample general hypersurfaces in X. This enables us to reduce the desired statement to the case of curves. Assume now that $\dim X=1$. By [52] or [5, Théorème 8.1], $(E,\theta)|_{X-D}$ admits a Hermitian–Yang–Mills metric h:

$$\Lambda_\omega F_h(E)=\lambda\otimes \mathbb{1}_E,$$

where ω is some Kähler form in $c_1(L)$, and λ is some topological constant. Moreover, h is adapted to (E,θ), and is adapted to log order in the sense of Definition 5.1. Hence $(h^*)^{\otimes a}\otimes h^{\otimes b}$ is the Hermitian–Yang–Mills metric for $T^{a,b}(E,\theta)|_{X-D}$, which is also adapted to log order. It follows from Theorem 6.7 below that $T^{a,b}(E,\theta)$ is also μ_L-polystable. □

2.4 Simpson–Mochizuki correspondence for systems of log Hodge bundles

A typical and important class of log Higgs bundle is the *system of log Hodge bundles*. In this subsection, we shall apply Theorem 2.8 to prove the *Simpson–Mochizuki correspondence for systems of log Hodge bundles*.

Definition 2.10 (*System of log Hodge bundles*) Let (E,θ) be a log Higgs bundle on a log pair (X,D). We say that (E,θ) is a *system of log Hodge bundles* if there is a decomposition of E into holomorphic vector bundles $E:=\oplus_{p+q=w}E^{p,q}$ such that

$$\theta: E^{p,q}\to E^{p-1,q+1}\otimes\Omega^1_X(\log D).$$

When $D=\varnothing$, such (E,θ) is called a *system of Hodge bundles*. A system of log Hodge bundles is μ_α-(poly)stable if it is μ_α-(poly)stable in the sense of log Higgs bundles.

Definition 2.11 (*Hodge metric*) Let $(E:=\oplus_{p+q=w}E^{p,q},\theta)$ be a system of Hodge bundles on a complex manifold X. A hermitian metric h for E is called a *Hodge metric* if h is harmonic, and it is a direct sum of metrics on the bundles $E^{p,q}$.

By Simpson [51], a system of Hodge bundles equipped with a Hodge metric is equivalent to a *complex variation of Hodge structures*. He then established his correspondence for Hodge bundles over compact Kähler manifolds in [51, Proposition 8.1]. In the rest of this subsection, we will extend his result to the log setting.

Let us state and prove the main result in this subsection.

Proposition 2.12 *Let (X, D) be a projective log pair. Let $(E, \theta) = (\oplus_{p+q=w} E^{p,q}, \theta)$ be a system of log Hodge bundles on (X, D) which is μ_L-polystable with $\int_X c_1(E) \cdot c_1(L)^{\dim X-1} = \int_X ch_2(E) \cdot c_1(L)^{\dim X-2} = 0$. Then there is a decomposition $(E, \theta) = \oplus_{i\in I}(E_i, \theta_i)$ where each (E_i, θ_i) is μ_L-stable system of log Hodge bundles so that there is a Hodge metric h_i (unique up to a positive multiplication) for $(E_i|_{X-D}, \theta_i|_{X-D})$ which is adapted to (E_i, θ_i).*

Proof Let us first prove the proposition when (E, θ) is stable. By [40, Theorem 9.1 and Propositions 5.1–5.3], there is a harmonic metrics h for $(E|_{X-D}, \theta|_{X-D})$ which is adapted to (E, θ), and such a harmonic metric is unique up to a positive constant multiplication. We introduce automorphism $f_t : E \to E$ of E parametrized by $t \in U(1)$, defined by

$$f_t\left(\sum_{p+q=w} e^{p,q}\right) = \sum_{p+q=w} t^p e^{p,q}. \tag{2.4.1}$$

for every $e^{p,q} \in E^{p,q}$. Then $f_t : (E, \theta) \to (E, t\theta)$ is an isomorphism since $t\theta \circ f_t = f_t \circ \theta$. Hence by the uniqueness of harmonic metrics, there is a function $\lambda(t) : U(1) \to \mathbb{R}^+$ such that

$$f_t^* h = \lambda(t) \cdot h.$$

For every $e^{p,q} \in E^{p,q}$, one has

$$\begin{aligned}\lambda(t) \cdot h(e^{p,q}, e^{p,q}) &= f_t^* h(e^{p,q}, e^{p,q}) = h(f_t(e^{p,q}), f_t(e^{p,q})) \\ &= |t^p|^2 h(e^{p,q}, e^{p,q}) = h(e^{p,q}, e^{p,q})\end{aligned}$$

Hence $\lambda(t) \equiv 1$ for $t \in U(1)$, namely $f_t^* h = h$. On the other hand,

$$h(e^{p,q}, e^{r,s}) = f_t^* h(e^{p,q}, e^{r,s}) = h(f_t(e^{p,q}), f_t(e^{r,s})) = t^p t^{-r} h(e^{p,q}, e^{r,s})$$

for any $t \in U(1)$. Therefore, $h(e^{p,q}, e^{r,s}) = 0$ if $p \neq r$. Hence h is a direct sum of hermitian metrics for $E^{p,q}$, namely h is a Hodge metric. The proposition is proved if (E, θ) is stable.

Let us prove the general cases. By [40, Corollary 3.11 & Theorem 9.1 & Propositions 5.1-5.3], there is a *canonical and unique* decomposition $(E, \theta) = \oplus_{i\in I}(E_i, \theta_i) \otimes \mathbb{C}^{p_i}$ where I is a finite set and harmonic metrics h_i for $(E_i|_{X-D}, \theta_i|_{X-D})$ which is adapted to (E_i, θ_i) so that (E_i, θ_i) is a μ_L-stable log Higgs bundle. By the above arguments, it suffices to prove that each (E_i, θ_i) is system of log Hodge bundles. Since (E, θ) is a system of log Hodge bundles, $(E, t\theta)$ is isomorphic to (E, θ) for any $t \in U(1)$. We have the following decomposition $(E, t\theta) = \oplus_i (E_i, t\theta_i) \otimes \mathbb{C}^{p_i}$. Note that $(E_i, t\theta_i)$ is still μ_L-stable. By the uniqueness of the decomposition, $(E_i, t\theta_i) \simeq (E_{i_t}, \theta_{i_t})$ for some $i_t \in I$. Since I is a finite set, there exists t_1, t_2 so that t_1/t_2 is not a root of unity and $i_{t_1} = i_{t_2}$. In other words, $(E_i, t_1\theta_i) \simeq (E_i, t_2\theta_i)$. By [52, Lemma 4.1] or [53, Theorem 8], $(E_i, t_1\theta_i)$ is a system of log Hodge bundles, and so is (E_i, θ_i). Hence

(E, θ) is a direct sum of μ_L-stable system of log Hodge bundles (E_i, θ_i), and each $(E_i|_{X-D}, \theta_i|_{X-D})$ admits a Hodge metric h_i adapted to (E_i, θ_i). The proposition is proved. □

3 Principal system of log Hodge bundles

In this section, we will extend Simpson's *principal system of log Hodge bundles* in [51, § 8] to the log setting. We will provide all necessary proofs for the claims for completeness sake. Let us mention that most results in this section follows from [51, § 8 & § 9] with minor changes.

Let G_0 be a real algebraic group which is semi-simple with its Lie algebra denoted by $\mathfrak{g}_0$. Let G be the complexification of G_0 with its Lie algebra denoted by $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{g}_0 + \sqrt{-1}\mathfrak{g}_0$. G_0 is called a *Hodge group* if the following conditions hold.

- The Lie algebra $\mathfrak{g}$ of G admits a Hodge structure of weight 0, namely, one has a decomposition

$$\mathfrak{g} = \oplus \mathfrak{g}^{p,-p}$$

  so that $[\mathfrak{g}^{p,-p}, \mathfrak{g}^{q,-q}] \subset \mathfrak{g}^{p+q,-p-q}$.
- If $\overline{\bullet}$ denotes the complex conjugation with respect to $\mathfrak{g}_0$, then $\overline{\mathfrak{g}^{p,-p}} = \mathfrak{g}^{-p,p}$.
- The form

$$h_{\mathfrak{g}}(U, V) := (-1)^{p+1} Tr(ad_U ad_{\bar{V}}) \quad \text{for} \quad U, V \in \mathfrak{g}^{p,-p} \tag{3.0.1}$$

  is a positively definite hermitian metric for $\mathfrak{g}$.

let $K_0 \subset G_0$ be the Lie subgroup of G_0 so that its Lie algebra $\mathfrak{k}_0$ is $\mathfrak{g}_0 \cap \mathfrak{g}^{0,0}$. Let $K \subset G$ (resp. $\mathfrak{k}$) be the complexification of K_0 (resp. $\mathfrak{k}_0$), and thus the Lie algebra of K is $\mathfrak{k}$. Then the restriction of the Killing form of $\mathfrak{g}_0$ on $\mathfrak{k}_0$ is positively definite, and thus K_0 is a compact real Lie group.

In the rest of the paper, we shall use the above notations without recalling their meanings.

The following concrete example of the Hodge group will be used in this paper, especially in the proof of Theorem A.

Example 3.1 Consider the a direct sum of $\mathbb{C}$-vector spaces

$$V = \oplus_{i+j=w} V^{i,j}$$

Denote by $r_i := \operatorname{rank} V^{i,j}$, and $r := \operatorname{rank} V$. Fix a hermitian metric $h = \oplus_{i+j=w} h_i$ for V where h_i is a hermitian metric for $V^{i,j}$. We take a sesquilinear form $Q(u, v) := (\sqrt{-1})^{i-j} h(u, v)$ for $u, v \in V^{i,j}$. Define $G_0 := PU(V, Q) \simeq PU(p_0, q_0)$, where $p_0 := \sum_{i \text{ odd}} r_i$ and $q_0 := \sum_{i \text{ even}} r_i$. We shall show that G_0 is a *Hodge group*.

First we note that the complexification of G_0 is $G := PGL(V) \simeq PGL(r, \mathbb{C})$. Then the Lie algebra of G is $\mathfrak{g} = \mathfrak{sl}(V) \simeq \mathfrak{sl}(r, \mathbb{C})$, and the Lie algebra of G_0 is $\mathfrak{g}_0 = \mathfrak{su}(p_0, q_0)$. Let us define the *Hodge decomposition* as follows:

$$\mathfrak{g}^{p,-p}=\oplus_i \mathrm{Hom}(V^{i,j},V^{i+p,j-p})\cap\mathfrak{sl}(V).$$

Then $\mathfrak{g}=\oplus\mathfrak{g}^{p,-p}$. One can check that $\overline{\mathfrak{g}^{p,-p}}=\mathfrak{g}^{-p,p}$, where the conjugate is taken with respect to the real form $\mathfrak{g}_0$ of $\mathfrak{g}$.

Let K be the subgroup of G which fix each $V^{i,j}$. Then $K=P(\prod_{i+j=w}GL(V^{i,j}))$, and its Lie algebra is $\mathfrak{k}=\mathfrak{g}^{0,0}$. Define $K_0:=K\cap G_0=P(\prod_{i+j=w}U(V^{i,j},h_i))$, whose Lie algebra is $\mathfrak{k}_0=\mathfrak{g}^{0,0}\cap\mathfrak{g}_0$.

More precisely, if we fix a unitary frame $e_1,\ldots,e_{p_0}$ for $(\oplus_{iodd}V^{i,j},\oplus_{i\ odd}h_i)$ and a unitary frame $f_1,\ldots,f_{q_0}$ for $(\oplus_{i\ even}V^{i,j},\oplus_{iodd}h_i)$, elements in $\mathfrak{g}_0$ can be expressed as the ones in $M(r\times r,\mathbb{C})$ with the form

$$\begin{bmatrix} A & C\\ C^* & B\end{bmatrix}$$

where $A\in\mathfrak{u}(p_0)$ and $B\in\mathfrak{u}(q_0)$ so that $Tr(A)+Tr(B)=0$. Note that the Killing form

$$Tr(ad_u ad_v)=2rTr(uv),$$

if we consider u,v as elements in $\mathfrak{sl}(r,\mathbb{C})$. Moreover, for $u\in\mathfrak{g}^{p,-p}$, one can show that

$$\bar{u}=\begin{cases}-u^* & \text{if } p \text{ is even}\\ u^* & \text{if } p \text{ is odd.}\end{cases}$$

where u^* denotes the conjugate transpose of u. Hence the hermitian metric $h_{\mathfrak{g}}$ defined in (3.0.1) can be simply expressed as

$$h_{\mathfrak{g}}(u,v)=2rTr(uv^*)$$

once we consider u,v as elements in $\mathfrak{sl}(r,\mathbb{C})$. In other words, for the natural inclusion $\iota:\mathfrak{g}\hookrightarrow\mathfrak{gl}(V)$, one has $h_{\mathfrak{g}}=2r\cdot\iota^*h_{End(V)}$, where $h_{End(V)}$ is the hermitian metric on $\mathrm{End}(V)$ induced by h_V. This fact is an important ingredient in the proof of Theorem A.

Let us generalize Simpson's definition of *principal system of Hodge bundles* in [51, § 8] to the log setting as follows.

Definition 3.2 (*Principal system of log Hodge bundles*) A *principal system of log Hodge bundles* on a log pair (X,D) is a pair (P,τ), where P is a holomorphic K-fiber bundle endowed with a holomorphic map

$$\tau: T_X(-\log D)\to P\times_K\mathfrak{g}^{-1,1}$$

such that $[\tau(u),\tau(v)]=0$. A *metric* for $P|_{X-D}$ is a reduction $P_H\subset P|_{X-D}$ whose structure group is K_0. Let d_H be the Chern connection for P_H. Define $\bar{\tau}_H$ to be the complex conjugate of $\tau|_{X-D}$ with respect to the reduction P_H. Then

$$\overline{\tau}_H \in \mathscr{C}^\infty(X-D, (P_H \times_{K_0} \mathfrak{g}^{1,-1}) \otimes \Omega^{0,1}_{X-D}).$$

Set

$$D_H := d_H + \tau|_{X-D} + \overline{\tau}_H, \tag{3.0.2}$$

which is a connection on the smooth G_0-bundle $P_H \times_{K_0} G_0$. Such triple $(P|_{X-D}, \tau|_{X-D}, P_H)$ is called a *principal variation of Hodge structures* over $X-D$ of Hodge group G_0, if the induced connection D_H in (3.0.2) is *flat*, namely the curvature of D_H is zero.

Remark 3.3 Note that the metric reduction P_H for a principal system of Hodge bundles (P, τ) on a complex manifold X induces a hermitian metric h_H on $P \times_K \mathfrak{g} \simeq P_H \times_{K_0} \mathfrak{g}$ defined by

$$h_H\big((p,u),(p,v)\big) := h_{\mathfrak{g}}(u,v) \tag{3.0.3}$$

for any $p \in P_H$ and $u, v \in \mathfrak{g}$. Here $h_{\mathfrak{g}}$ is the hermitian metric defined in (3.0.1). Note that K_0 preserves the decomposition $\mathfrak{g} = \oplus_{p+q=w}\mathfrak{g}^{-p,p}$. It thus also preserves $h_{\mathfrak{g}}$. Indeed, for $u, v \in \mathfrak{g}^{-p,p}$ and $k \in K_0$, one has

$$(-1)^{p+1}h_{\mathfrak{g}}(Ad_k u, Ad_k v) = (-1)^{p+1}h_{\mathfrak{g}}(u,v).$$

By the equivalence relation $(p,u) \sim (pk^{-1}, Ad_k u)$, the metric h_H is thus well-defined.

Remark 3.4 (Period map of principal variation of Hodge structures) By Simpson [51, p. 900], for a principal variation of Hodge structures (P, τ, P_H) on a complex manifold X, one can also define its *period map* as follows. Denote by $\pi : \tilde{X} \to X$ the universal cover of X. Set $(\tilde{P} := \pi^*P, \tilde{\tau} := \pi^*\tau, \tilde{P}_H := \pi^*P_H)$, which is a principal variation of Hodge structures on the simply connected complex manifold $\tilde{X}$. The flat connection D_H thus induces a flat trivialization $\tilde{P}_H \times_{K_0} G_0 \simeq \tilde{X} \times G_0$. Denote by $\phi : \tilde{P}_H \to G_0$ the composition of the inclusion $\tilde{P}_H \subset \tilde{P}_H \times_{K_0} G_0 \simeq \tilde{X} \times G_0$ and the projection $\tilde{X} \times G_0 \to G_0$. It induces a map

$$\begin{aligned} f : \tilde{X} &\to G_0/K_0 =: \mathscr{D} \\ \tilde{x} &\mapsto \phi(e_x)\cdot K_0 \quad \forall e_x \in \tilde{P}_{H,\tilde{x}}. \end{aligned} \tag{3.0.4}$$

Alternatively, we view $G_0 \to \mathscr{D}$ as a principal K_0-fiber bundle over $\mathscr{D}$, and its pull-back on $\tilde{X}$ via f is nothing but the principal K_0-fiber bundle $\tilde{P}_H$ by our definition of f. Hence the complexified differential of f is

$$df^{\mathbb{C}} : T^{\mathbb{C}}_{\tilde{X}} \to f^*T^{\mathbb{C}}_{\mathscr{D}} \simeq f^*(G_0 \times_{K_0} \oplus_{p\neq 0}\mathfrak{g}^{p,-p}) = \tilde{P}_H \times_{K_0} \oplus_{p\neq 0}\mathfrak{g}^{p,-p}$$

One can prove that $df^{\mathbb{C}} = \tilde{\tau} + \overline{\tilde{\tau}}_H$, where $\overline{\tilde{\tau}}_H$ is the conjugate of $\tilde{\tau}$ with respect to $\tilde{P}_H$. Hence the restriction of $df^{\mathbb{C}}$ to the holomorphic tangent bundle $T_{\tilde{X}}$ is $\tilde{\tau}$, which is a holomorphic map since the holomorphic tangent bundle of $\mathscr{D}$ is $T_{\mathscr{D}} \simeq G_0 \times_{K_0} \oplus_{p<0} \mathfrak{g}^{p,-p}$. In conclusion, f is a holomorphic map, which is called the *period map* associated to the principal variation of Hodge structures (P, τ, P_H), whose differential is given by $df = \tilde{\tau}$.

The uniformization is related by Hodge group of Hermitian type.

Definition 3.5 [51, § 9] A Hodge group G_0 is called *Hermitian type* if the Hodge decomposition $\mathfrak{g}$ of the Lie algebra of G is

$$\mathfrak{g} = \mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{0,0} \oplus \mathfrak{g}^{1,-1}$$

and that G_0 has no compact factor. In this case, $K_0 \subset G_0$ is the maximal compact subgroup and $\mathscr{D} := G_0/K_0$ is a Hermitian symmetric space of non-compact type.

Let us generalize the definition of *uniformizing bundle* by Simpson [51, § 9] to the log setting.

Definition 3.6 (*Uniformizing bundle*) Let G_0 be a Hodge group of Hermitian type. A *uniformizing bundle* on a log pair (X, D) is a principal system of log Hodge bundles (P, τ) such that $\tau : T_X(-\log D) \xrightarrow{\simeq} P \times_K \mathfrak{g}^{-1,1}$ is an isomorphism. A *uniformizing variation of Hodge structures* is a uniformizing bundle on a complex manifold X together with a flat metric $P_H \subset P$.

Remark 3.7 (Uniformization via uniformizing bundles) It follows from Definition 3.6 that, for a uniformizing variation of Hodge structures (P, τ, P_H) over a complex manifold X, the period map $f : \tilde{X} \to \mathscr{D}$ defined in (3.0.4) is locally biholomorphic. This follows from the fact that $df = \tilde{\tau}$, which is isomorphic at any point of $\tilde{X}$ by the definition. Recall that in Remark 3.3 the metric reduction P_H together with the positively definite form $h_{\mathfrak{g}}$ for $\mathfrak{g}$ in (3.0.1) induce a metric h_H for $P \times_K \mathfrak{g}^{-1,1}$. For the period domain $\mathscr{D}$ which is a hermitian symmetric space, one can also define the hermitian metric $h_{\mathscr{D}}$ for $T_{\mathscr{D}} \simeq G_0 \times_{K_0} \mathfrak{g}^{-1,1}$ in a similar way. By Remark 3.4, $\tilde{P}_H = f^*G_0$ when we consider $G_0 \to \mathscr{D}$ as a principal K_0-fiber bundle over $\mathscr{D}$. One thus has

$$\pi^*\tau^*h_H = f^*h_{\mathscr{D}}. \tag{3.0.5}$$

In other words, $f : (\tilde{X}, h_{\tilde{X}} := \pi^*\tau^*h_H) \to (\mathscr{D}, h_{\mathscr{D}})$ is a *local isometry*. Hence for the action of $\pi_1(X)$ on $\tilde{X}$, the metric $h_{\tilde{X}}$ is invariant under this $\pi_1(X)$-action. If τ^*h_H is a complete metric, so is $\pi^*\tau^*h_H$, and by [13, Theorem IV.1.2], $f : \tilde{X} \to \mathscr{D}$ is a Riemannian covering map, which is thus a biholomorphism since $\tilde{X}$ and $\mathscr{D}$ are both simply connected. In other words, X is uniformized by the hermitian symmetric space $\mathscr{D}$ when the metric τ^*h_H on X is complete.

One can construct systems of log Hodge bundles from principal ones via Hodge representations.

Definition 3.8 [51, p. 900] Let $(V = \oplus_{p+q=w} V^{p,q}, h_V)$ be a polarized Hodge structure. A *Hodge representation* of G_0 is a complex representation $\rho : G \to GL(V)$ satisfying the following conditions.

- The action of $\mathfrak{g}$ is compatible with Hodge type, and such that K_0 preserves Hodge type. In other words,

$$d\rho(\mathfrak{g}^{r,-r})(V^{p,q}) \subset V^{p+r,q-r}$$

 and $\rho(K_0)(V^{p,q}) \subset V^{p,q}$.[1]
- The sesquilinear form Q defined by

$$Q(u, v) := (\sqrt{-1})^{p-q} h_V(u, v) \quad \text{for} \quad u, v \in V^{p,q} \tag{3.0.6}$$

 is G_0 invariant. Namely, one has $\rho(G_0) \subset U(V, Q)$.

Example 3.9 For the Hodge group G_0, $(\mathfrak{g} = \oplus_p \mathfrak{g}^{p,-p}, h_{\mathfrak{g}})$ is a polarized Hodge structure of weight 0, where $h_{\mathfrak{g}}$ is the polarization defined in (3.0.1) via the Killing form. One can easily check that the adjoint representation $Ad : G \to GL(\mathfrak{g})$ is a Hodge representation for this polarized Hodge structure.

A principal system of log Hodge bundles together with a Hodge representation induces a system of log Hodge bundles as follows.

Lemma 3.10 *If $\rho : G \to GL(V)$ is a Hodge representation of the Hodge group G_0 and (P, τ) is a principal system of log Hodge bundles on the log pair (X, D), then $(E := P \times_K V, \theta := d\rho(\tau))$ is a system of log Hodge bundles. A polarization h_V for V together with a metric P_H for $P|_{X-D}$ give a metric h_E on the system of Hodge bundles $(E, \theta)|_{X-D}$ over $X - D$. When $(P|_{X-D}, \tau|_{X-D}, P_H)$ is a principal variation of Hodge structures over $X-D$, $(E|_{X-D}, \theta|_{X-D}, h_E)$ gives rise to a complex variation of Hodge structures.*

Proof By Definition 3.8, one has $\rho(K)(V^{p,q}) \subset V^{p,q}$. Hence $E := P \times_K V$ admits a decomposition of holomorphic vector bundles $E = \oplus_{p+q=w} E^{p,q}$ with $E^{p,q} := P \times_K V^{p,q}$. Let us define $\theta := d\rho(\tau)$. Since $\tau : T_X(-\log D) \to P \times_K \mathfrak{g}^{-1,1}$ satisfies $[\tau(u), \tau(v)] = 0$, and $d\rho(g^{-1,1})(V^{p,q}) \subset V^{p-1,q+1}$, one thus has $\theta : E^{p,q} \to E^{p-1,q+1} \otimes \Omega^1_X(\log D)$, with $\theta \wedge \theta = 0$. Hence (E, θ) is a system of log Hodge bundles.

Let us now prove that $\rho|_{K_0} : K_0 \to GL(V)$ has image on $U(V, h_V)$. Since $\rho(K)(V^{p,q}) \subset V^{p,q}$, one thus has

$$\rho(K) \subset \prod_{p+q=w} GL(V^{p,q}).$$

Since the sesquilinear form Q in (3.0.6) is G_0 invariant, one thus has

$$\rho(G_0) = U(V, Q).$$

[1] As remarked by Simpson [51], this is not automatic if K_0 is not connected. However, in Example 3.1, K_0 is always connected, and thus such condition will be superfluous in that case.

Hence

$$\rho(K_0) \subset \rho(G_0 \cap K) \subset \prod_{p+q=w} U(V^{p,q}, h_{p,q}) \subset U(V, h_V). \tag{3.0.7}$$

Note that $E = P \times_K V \simeq P_H \times_{K_0} V$. We define the hermitian metric h_E for E by setting

$$h_E((p,u),(p,v)) := h_V(u,v) \tag{3.0.8}$$

for any $p \in P_H$ and for any $u, v \in V$. Since $\rho(K_0) \subset U(V, h_V)$, one can check as Remark 3.3 that h_E is well-defined.

If $(P|_{X-D}, \tau|_{X-D}, P_H)$ is a principal variation of Hodge structures on $X - D$, the connection $D_H := d_H + \tau + \overline{\tau}_H$ is flat. By construction, the connection $D_{h_E} := d_{h_E} + \theta + \overline{\theta}_{h_E}$ for $E|_{X-D}$ is also flat, where d_{h_E} is the Chern connection for the metrized vector bundle (E, h_E), and $\overline{\theta}_{h_E}$ is the conjugate of θ with respect to h_E. Indeed, it can be seen from that d_{h_E} is naturally induced by d_H, $\theta := d\rho(\tau)$, and $\overline{\theta}_{h_E} = d\rho(\overline{\tau}_H)$ by (3.0.8). By [51, p. 898], the triple $(E|_{X-D}, \theta|_{X-D}, h_E)$ gives rise to a complex variation of Hodge structures on $X - D$. □

Conversely, one can associate a system of log Hodge bundles with a principal one as follows. The following result shall be applied in the proof of Theorem A.

Proposition 3.11 *Let $(E,\theta) = (\oplus_{p+q=w} E^{p,q}, \theta)$ be a system of log Hodge bundles on a log pair (X, D). Then there is a principal system of log Hodge bundles (P, τ) with the structure group K associated to (E, θ), where K is the semi-simple Lie group in Example 3.1. Moreover, any hermitian metric (not necessarily harmonic) $h := \oplus_{p+q=w} h_p$ for $E|_{X-D}$ gives rise to a metric reduction P_H for $P|_{X-D}$ with the structure group K_0 defined in Example 3.1.*

Proof We shall adopt the same notions as those in Example 3.1. Denote by $r_p :=$ rank $E^{p,q}$, $r := \sum_{p+q=w} r_p$ and set $\ell_i := \sum_{p\geq i} r_i$. We consider the following frame bundle $\tilde{P}$. The fiber of $\tilde{P}$ over a point x is the set of all ordered bases $e_1, \ldots, e_r$ (or say frames) for E_x such that $e_{\ell_p - r_p + 1}, \ldots, e_{\ell_p}$ is a basis for $E_x^{p,q}$. The structure group of $\tilde{P}$ is thus $\prod_p GL(r_p, \mathbb{C})$, which is the subgroup of $GL(r, \mathbb{C})$. $\tilde{P}$ can be equipped with the holomorphic structure induced by E. Consider the homomorphism $f : GL(r, \mathbb{C}) \to PGL(r, \mathbb{C}) =: G$, and set $K = P\big(\prod_p GL(r_p, \mathbb{C})\big)$ to be the image of $\prod_p GL(r_p, \mathbb{C})$ under f. Set P to be the holomorphic K-fiber bundle obtained by extending the structure group of $\prod_p GL(r_p, \mathbb{C})$ using f.

Note that $P \times_K \mathfrak{g}^{-1,1} = \oplus_{i+j=w} \mathrm{Hom}(E^{i,j}, E^{i-1,j+1})$. Let us define $\tau := \theta$. The pair (P, τ) is a principal system of log Hodge bundles on the log pair (X, D).

Recall that the metric h for the Hodge bundle $(E, \theta)|_{X-D}$ is a direct sum $h = \oplus_{p+q=w} h_p$. We take a sesquilinear form Q of E defined by $Q(u, v) := (\sqrt{-1})^{p-q} h(u, v)$ for $u, v \in E^{p,q}$. We take $\tilde{P}_H$ to be a reduction of $\tilde{P}|_{X-D}$ consisting of unitary frames with respect to Q. In other words, The fiber of $\tilde{P}$ over a point x is the set of frames $e_1, \ldots, e_r$ for E_x such that $e_{\ell_p - r_p + 1}, \ldots, e_{\ell_p}$ is an orthonormal basis for $(E_x^{p,q}, h_p)$. Hence the structure group of $\tilde{P}_H$ is $\tilde{K}_0 := \prod_{p+q=w} U(r_p)$.

Define $K_0 := P\big(\prod_{p+q=w} U(r_p)\big)$, which is the image $f(\tilde{K}_0)$. Set P_H to be the smooth principal K_0-fiber bundle on $X - D$ obtained by extending the structure group of $\tilde{P}_H$ using $f : K \to K_0$. Then $P_H \subset P_{X-D}$ is also a metric reduction. The Hodge group G_0 will be $PU(p_0, q_0)$ where $p_0 := \sum_{p\ even} r_p$ and $q_0 := \sum_{p\ odd} r_p$, and $G := PGL(r, \mathbb{C})$ is the complexification of G_0. The proposition is proved. $\square$

4 Tannakian consideration

In this section, we shall construct *principal variation of Hodge structures* over quasi-projective manifolds. Its proof is based on Proposition 2.12 together with some Tannakian considerations in [38,40,52].

Theorem 4.1 *Let (X, D) be a projective log pair endowed with an ample polarization L. Let (P, τ) be a principal system of log Hodge bundles on (X, D), and let ρ be a Hodge representation $\rho : G \to GL(V)$ for some polarized Hodge structure $(V = \oplus_{i+j=w} V^{i,j}, h_V)$ so that $\rho|_{K_0} : K_0 \to GL(V)$ is faithful and $d\rho : \mathfrak{g}_0 \to \mathfrak{gl}(V)$ is injective. If the system of log Hodge bundles $(E := P \times_K V, \theta := d\rho(\tau))$ defined in Lemma 3.10 is μ_L-polystable with $\int_X ch_2(E) \cdot c_1(L)^{\dim X-2} = 0$, then there exists a metric reduction P_H for $P|_{X-D}$ so that the triple $(P|_{X-D}, \tau|_{X-D}, P_H)$ is a* principal variation of Hodge structures *on $X - D$. Moreover, such P_H together with the polarization h_V for V gives rise to a Hodge metric h for $(E, \theta)|_{X-D}$ (defined in Lemma 3.10) which is adapted to (E, θ).*

Proof We first prove that $(E, \theta)|_{X-D}$ admits a Hodge metric h over $(E, \theta)|_{X-D}$ which is adapted to (E, θ). Since K is a complex semi-simple Lie group, the Hodge representation $\rho' : K \to GL(\det V)$ induced by ρ has image contained in $SL(\det V) = 1$. Hence ρ' is trivial. Note that $\det E = P \times_K \det V$, which is thus a trivial line bundle on X. Hence $c_1(E) = 0$. Since we assume that (E, θ) is μ_L-polystable with $\int_X ch_2(E) \cdot c_1(L)^{\dim X-2} = 0$, it follows from Proposition 2.12 that $(E, \theta)|_{X-D}$ admits a Hodge metric h over $(E, \theta)|_{X-D}$ which is adapted to (E, θ).

Let us show that $\rho|_K : K \to GL(V)$ is faithful. By (3.0.7), one has $\rho(K_0) \subset U(V, h_V)$. Since K is the complexification of K_0 and $\rho|_{K_0} : K_0 \to GL(V)$ is assumed to be faithful, one concludes that $\rho|_K : K \to GL(V)$ is also faithful.

Let us now recall some Tannakian arguments. The representation ρ induces a representation $\rho_{a,b} : G \to GL(T^{a,b}V)$ for any $a, b \in \mathbb{N}$, where $T^{a,b}V := \mathrm{Hom}(V^{\otimes a}, V^{\otimes b})$. Since $\rho|_K : K \to GL(V)$ is faithful, we can consider K as a reductive algebraic subgroup of $GL(V)$. There is a one dimensional complex subspace $V_1 \in T^{a,b}V$ for some $(a, b) \in \mathbb{N}^2$ so that

$$K = \{g \in GL(V) \mid \rho_{a,b}(g)(V_1) = V_1\}. \tag{4.0.1}$$

Since K is reductive, there is a complementary subspace V_2 of $T^{a,b}V$ for V_1 which is invariant under K.

By Lemma 3.10, the Hodge representation $\rho_{a,b}$ and (P, τ) gives rise to a system of log Hodge bundles $(P \times_K T^{a,b}V, \theta^{a,b} := d\rho_{a,b}(\tau))$ over (X, D), which is nothing but

$T^{a,b}(E,\theta)$. Recall that $\rho_{a,b}(K)(V_1)=V_1$ and $\rho_{a,b}(K)(V_2)=V_2$. Consider the log Higgs bundles $(E_1,\theta_1):=(P\times_K V_1, d\rho_{a,b}(\tau))$ and $(E_2,\theta_2):=(P\times_K V_2, d\rho_{a,b}(\tau))$ over (X,D).

Note that $T^{a,b}(E,\theta)=(E_1,\theta_1)\oplus(E_2,\theta_2)$. By Theorem 2.8, $T^{a,b}(E,\theta)$ is μ_L-polystable with $\int_X c_1(T^{a,b}(E))\cdot c_1(L)^{\dim X-1}=0$ with respect to an arbitrary polarization L. Since $c_1(T^{a,b}(E))=c_1(E_1)+c_1(E_2)$, by the polystability of $T^{a,b}(E,\theta)$, we conclude that (E_1,θ_1) and (E_2,θ_2) are both μ_L-polystable. By Proposition 2.12, each $(E_i|_{X-D},\theta_i|_{X-D})$ admits a harmonic metric h_i which is adapted to (E_i,θ_i). Moreover, h coincides with $h_1\oplus h_2$ up to some obvious ambiguity.

In the rest of the proof, any object which appears is restricted over $X-D$. Let us first enlarge the structure group of P by defining $P_{GL(V)}:=P\times_K GL(V)$ via the faithful representation $\rho|_K: K\to GL(V)$. This is the holomorphic principal (frame) bundle associated to E. We can consider $P=P\times_K K\subset P_{GL(V)}$ as a reduction of $P_{GL(V)}$. The metric h for E gives rise to a reduction $P_{U(E,h)}$ of $P_{GL(V)}$ with the structure group $U(V,h_V)$. Indeed, note that

$$E=P_{GL(V)}\times_{GL(V)}V$$

and thus the metric h for E induces a family of hermitian metrics h_e for V parametrized by $e\in P_{GL(V)}$. It has the obvious relation $h_{e\cdot g}=g^*h_e$ for any $g\in GL(V)$. We define

$$P_{U(E,h)}:=\{e\in P_{GL(V)}\mid h_e=h_V\} \tag{4.0.2}$$

and it is obvious that if $e\in P_{U(E,h)}$, then $e\cdot g\in P_{U(E,h)}$ if and only if $g\in U(V,h_V)$. Hence the structure group of $P_{U(E,h)}$ is $U(V,h_V)$.

Let us define $P_H:=P\cap P_{U(E,h)}$ whose structure group is $U(V,h_V)\cap K\supset K_0$ by (3.0.7). Since K_0 is the maximal compact subgroup of K and $U(V,h_V)\cap K$ is also compact, one has moreover $U(V,h_V)\cap K=K_0$. Hence $P_H\subset P$ is a metric reduction with the structure group K_0.

Obviously, if we follow Lemma 3.10 to define a new metric h' for E by setting

$$h'((p,u),(p,v)):=h_V(u,v)$$

for any $p\in P_H$ and for any $u,v\in V$, then

$$h'=h \tag{4.0.3}$$

by (4.0.2). We shall prove that $(P|_{X-D},\tau|_{X-D},P_H)$ is a principal variation of Hodge structures on $X-D$ following the elegant arguments in [38, Proposition 3.7].

Let $A\in\mathscr{C}^\infty(P_{GL(V)},T^*_{P_{GL(V)}}\otimes\mathfrak{gl}(V))$ be the Chern connection 1-form for the principal bundle $P_{GL(V)}$ induced by the Chern connection d_h for (E,h). Fix a base point $p\in P\subset P_{GL(V)}$, and we denote by $\pi: P\to X$ the projection map. Recall that

$$T^{a,b}(E,h)=(E_1,h_1)\oplus(E_2,h_2),$$

and

$$E_i = P \times_K V_i.$$

Hence the holonomy $Hol(p, \gamma) \in GL(V)$ with respect to the connection A along any smooth loop γ based at $\pi(p)$ satisfies that

$$\rho_{a,b}\big(Hol(p, \gamma)\big)(V_i) \subset V_i$$

for $i = 1, 2$. By (4.0.1), one has $Hol(p, \gamma) \in K$. Hence the restriction of A to P is 1-form with values in $\mathfrak{k}$. In other words, A is induced by a connection on P.

On the other hand, by the definition of the Chern connection, A is also induced by a connection on $P_{U(E,h)}$; in other words, the restriction of A to $P_{U(E,h)}$ is 1-form with values in $\mathrm{Lie}(U(V, h_V))$, where $\mathrm{Lie}(U(V, h_V))$ denotes the Lie algebra of $U(V, h_V)$. Since $\mathfrak{k}_0 = \mathfrak{k} \cap \mathrm{Lie}(U(V, h_V))$, there is a connection $A_0 \in \mathscr{C}^\infty(P_H, T^*_{P_H} \otimes \mathfrak{k}_0)$ for the smooth principal K_0-fiber bundle $P_H := P_{U(E,h)} \cap P$ which induces the connection A. A_0 is moreover the Chern connection with respect to the reduction P_H of P by our construction. Let us define $F_H \in \mathscr{A}^{1,1}(P_H \times_{K_0} \mathfrak{g}_0)$ to be the curvature form of the connection $A_0 + \tau + \overline{\tau}_H$ over the smooth principal K_0-bundle $P_H \times_{K_0} G_0$, , where $\overline{\tau}_H$ is the adjoint of τ with respect to the metric reduction $P_H \subset P$. Recall that $\theta := d\rho(\tau)$. By (4.0.3), one has $\overline{\theta}_h = d\rho(\overline{\tau}_H)$. Hence

$$d\rho(F_H) = (d_h + \theta + \overline{\theta}_h)^2 = F_h(E) = 0 \tag{4.0.4}$$

where d_h is the Chern connection for (E, h). Since $d\rho : \mathfrak{g}_0 \to \mathfrak{gl}(V)$ is assumed to be injective, by (4.0.4) this implies that $F_H = 0$. In conclusion, $(P|_{X-D}, \tau|_{X-D}, P_H)$ is a principal variation of Hodge structures on $X - D$. □

5 Uniformization of quasi-projective manifolds by unit balls

This section is devoted to the proof of Theorem A. In Sect. 5.2 we shall prove a basic result for the extension of plurisubharmonic functions. This lemma will be used in the proof of Theorem A. We shall also give an application of this fact in Hodge theory: we can give a much simpler proof of the negativity of kernel of Higgs fields for tame harmonic bundles originally proven by Brunebarbe [9] (see also [60] for systems of log Hodge bundles). With all the tools developed above, we are able to prove Theorem A in Sect. 5.3.

5.1 Adaptedness to log order and acceptable metrics

We recall some notions in [41, § 2.2.2]. Let X be a $\mathscr{C}^\infty$-manifold, and E be a $\mathscr{C}^\infty$-vector bundle with a hermitian metric h. Let $\mathbf{v} = (v_1, \ldots, v_r)$ be a $\mathscr{C}^\infty$-frame of E. We obtain the $H(r)$-valued function $H(h, \mathbf{v})$,whose (i, j)-component is given by $h(v_i, v_j)$.

Let us consider the case $X = \Delta^n$, and $D = \sum_{i=1}^{\ell} D_i$ with $D_i = (z_i = 0)$. We have the coordinate $(z_1, \ldots, z_n)$. Let h, E and $\mathbf{v}$ be as above.

A frame $\mathbf{v}$ is called *adapted up to log order*, if the following inequalities hold over $X - D$

$$C^{-1}\left(-\sum_{i=1}^{\ell} \log |z_i|\right)^{-M} \le H(h, \mathbf{v}) \le C\left(-\sum_{i=1}^{\ell} \log |z_i|\right)^{M}$$

for some positive numbers M and C.

Definition 5.1 Let (X, D) be a log pair, and let E be a holomorphic vector bundle on X. A hermitian metric h for $E|_{X-D}$ is *adapted to log order* if for any point $x \in D$, there is an admissible coordinate $(U; z_1, \ldots, z_n)$, a holomorphic frame $\mathbf{v}$ for $E|_U$ which is adapted up to log order.

Definition 5.2 (*Acceptable metric*) Let (X, D) be a log pair and let (E, θ) be a log Higgs bundle over (X, D). We say that the metric h for $E|_{X-D}$ is *acceptable*, if for any $p \in D$ there is an admissible coordinate $(U; z_1, \ldots, z_n)$ around p, so that the norm $|F_h|_{h,\omega_P} \le C$ for some $C > 0$ over $U - D$. Such triple (E, θ, h) is called an *acceptable bundle* on (X, D).

One can easily check that acceptable metrics and adaptedness to log order defined above are invariant under bimeromorphic transformations.

Lemma 5.3 *Let (X, D) be a log pair, and let $\mu : \tilde{X} \to X$ be a bimeromorphic morphism so that $\mu^{-1}(D) = \tilde{D}$. For a log Higgs bundle (E, θ) over (X, D), one can define a log Higgs bundle $(\tilde{E}, \tilde{\theta})$ on $(\tilde{X}, \tilde{D})$ by setting $\tilde{E} = \mu^* E$ and $\tilde{\theta}$ to be the composition*

$$\mu^* E \xrightarrow{\mu^*\theta} \mu^*(E \otimes \Omega_X^1(\log D)) \to \mu^* E \otimes \Omega_{\tilde{X}}^1(\log \tilde{D}).$$

If the metric h for $(E, \theta)|_{X-D}$ is acceptable or adapt to log order, so is the metric $\mu^* h$ for $(\tilde{E}, \tilde{\theta})|_{\tilde{X}-\tilde{D}}$. □

5.2 Extension of psh functions and negativity of kernel of Higgs fields

In this subsection we shall prove a result on the extension of plurisubharmonic (psh for short) functions, which will be used in the proof of Theorem A and Proposition 6.6. As a byproduct, we give a very simple proof of the negativity of kernels of Higgs fields of tame harmonic bundles by Brunebarbe [9, Theorem 1.3], which generalizes the earlier work by Zuo [60] for system of log Hodge bundles.

Lemma 5.4 *Let $X = \Delta^n$, and $D = \sum_{i=1}^{\ell} D_i$ with $D_i = (z_i = 0)$. Let φ be a psh function on X^*. We assume that for any $\delta > 0$, there is a positive constant C_δ so that*

$$\varphi(z) \le \delta \sum_{j=1}^{\ell}(-\log |z_j|^2)) + C_\delta$$

on X^*. Then φ extends uniquely *to a psh function on X.*

Proof Define $\varphi_\varepsilon := \varphi + \varepsilon\sum_{j=1}^{\ell}(\log|z_j|^2)$ for any $\varepsilon > 0$. Then for each $\varepsilon > 0$, φ_ε is locally bounded from above, which thus extends to a psh $\tilde{\varphi}_\varepsilon$ on the whole X by the well-known fact in pluripotential theory. By the maximum principle, for any $0 < r < 1$, there is a point $\xi_\varepsilon \in S(0,r) \times \cdots S(0,r)$ so that

$$\sup_{z\in\Delta(0,r)\times\cdots\times\Delta(0,r)} \varphi_\varepsilon(z) \leq \varphi_\varepsilon(\xi_\varepsilon) \leq \varphi(\xi_\varepsilon)$$

where $S(0,r) := \{z \in \Delta \mid |z| = r\}$. Note that the compact set $S(0,r) \times \cdots S(0,r)$ is contained in $X - D$. Since φ is psh on $X - D$, there exists $z_0 \in S(0,r) \times \cdots S(0,r)$ so that

$$\sup_{z\in S(0,r)\times\cdots S(0,r)} \varphi(z) \leq \varphi(z_0) < +\infty.$$

Hence φ_ε is *uniformly* locally bounded from above.

We define the *upper envelope* $\tilde{\varphi} := \sup_{\varepsilon>0}\tilde{\varphi}_\varepsilon$, and define the *upper semicontinuous regularization* of $\tilde{\varphi}$ by $\tilde{\varphi}^\star(x) := \lim_{\delta\to 0^+}\sup_{\mathbb{B}(x,\delta)}\tilde{\varphi}(z)$, where $\mathbb{B}(x,\delta)$ is the unit ball of radius δ centered at x. Then by the well-known result in pluripotential theory [19, Chapter 1, Theorem 5.7], $\tilde{\varphi}^\star$ is a psh function on X. By our construction, $\tilde{\varphi}^\star(z) = \varphi(z)$ on $X - D$. This proves our result. □

A direct consequence of the above lemma is the following extension theorem of positive currents.

Lemma 5.5 *Let (X, D) be a log pair and let L be a line bundle on X. Assume that h is a smooth hermitian metric for $L|_{X-D}$, which is adapted to log order. Assume further that the curvature form $\sqrt{-1}R_h(L|_{X-D}) \geq 0$. Then h extends to a singular hermitian metric $\tilde{h}$ for L with zero Lelong numbers so that the curvature current $\sqrt{-1}R_{\tilde{h}}(L)$ is closed and positive. In particular, L is a nef line bundle.* □

Let us show how to apply Lemma 5.4 to reprove the negativity of kernels of Higgs fields of tame harmonic bundles.

Theorem 5.6 (Brunebarbe) *Let X be a compact Kähler manifold and let D be a simple normal crossing divisor on X. Let (E, θ, h) be a tame harmonic bundle on $X - D$, and let $({}^\diamond E, \theta)$ be the prolongation defined in* [39, § 4.1]. *Let $\mathcal{F}$ be any coherent torsion free subsheaf of ${}^\diamond E$ which lies in the kernel of the Higgs field $\theta : {}^\diamond E \to {}^\diamond E \otimes \Omega^1_X(\log D)$, namely $\theta(\mathcal{F}) = 0$. Then*

(i) *the singular hermitian metric $h|_{\mathcal{F}}$ for $\mathcal{F}$, is* semi-negatively curved *in the sense of* [49, Definition 2.4.1].
(ii) *The dual $\mathcal{F}^*$ of $\mathcal{F}$ is* weakly positive over $X^\circ - D$ *in the sense of Viehweg, where $X^\circ \subset X$ is the Zariski open set so that $\mathcal{F}|_{X^\circ} \to {}^\diamond E|_{X^\circ}$ is a subbundle.*
(iii) *If the harmonic metric h is adapted to log order and $\mathcal{F}$ is a subbundle of ${}^\diamond E$ so that $\theta(\mathcal{F}) = 0$, then the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{F}^*)}(1)$ admits a singular hermitian metric g with zero Lelong numbers so that the curvature current $\sqrt{-1}R_g(\mathcal{O}_{\mathbb{P}(\mathcal{F}^*)}(1)) \geq 0$; in particular, $\mathcal{F}^*$ is a nef vector bundle.*

Proof By [49, Definition 2.4.1], it suffices to prove that for any open set U and any $s \in \mathcal{F}(U)$, $\log|s|_h^2$ extends to a psh function on U. Pick any point $x \in D$. By the definition of ${}^\diamond E$, for any $\delta > 0$, there are an admissible coordinate $(U; z_1, \ldots, z_n)$ centered at x, and a positive constant C_δ so that

$$\log|s|_h^2 \leq \delta \sum_{j=1}^{\ell}(-\log|z_j|^2)) + C_\delta$$

on $U - D$. Recall that $R_h(E) + [\theta, \overline{\theta}_h] = F_h(E) = 0$. Since $\theta(s) = 0$, we have

$$\begin{aligned}\sqrt{-1}\partial\overline{\partial}\log|s|_h^2 &\geq -\frac{\sqrt{-1}\{\theta s, \theta s\}}{|s|_h^2} - \frac{\sqrt{-1}\{\overline{\theta}_h s, \overline{\theta}_h s\}}{|s|_h^2}\\ &= -\frac{\sqrt{-1}\{\overline{\theta}_h s, \overline{\theta}_h s\}}{|s|_h^2} \geq 0.\end{aligned}$$

over $X - D$. Hence $\log|s|_h^2$ is a psh function on $X - D$. By Lemma 5.4, we conclude that $\log|s|_h^2$ extends to a psh function on U. This proves that $(\mathcal{F}, h)$ is negatively curved in the sense of Păun-Takayama.

The metric h induces a negatively curved singular hermitian metric h_1 (in the sense of [49, Definition 2.2.1]) on the subbundle $\mathcal{F}|_{X^\circ}$. By Lemma 5.5, h_1 induces a singular metric g for the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{F}^*|_{X^\circ})}(1)$ so that $\sqrt{-1}R_g(\mathcal{O}_{\mathbb{P}(\mathcal{F}^*|_{X^\circ})}(1)) \geq 0$. Note that $X - X^\circ$ is a codimension at least two subvariety. The second statement then follows from Hörmander's L^2-techniques in [49, Proof of Theorem 2.5.2].

Let us prove the last statement. Since $\mathcal{F}$ is a subbundle of ${}^\diamond E$, one has $X^\circ = X$. Since h is assumed to be adapted to log order, the singular hermitian metric g for $\mathcal{O}_{\mathbb{P}(\mathcal{F}^*)}(1)$ thus has zero Lelong numbers everywhere. This implies the nefness of the vector bundle $\mathcal{F}^*$. □

5.3 Characterization of non-compact ball quotient

Let us state and prove our first main theorem in this paper.

Theorem 5.7 *Let X be an n-dimensional complex projective manifold and let D be a simple normal crossing divisor on X. Let L be an ample polarization on X. For the log Hodge bundle $(\Omega_X^1(\log D) \oplus \mathcal{O}_X, \theta)$ on (X, D) with θ defined in (1.1.1), we assume that it is μ_L-polystable. Then one has the following inequality*

$$\left(2c_2(\Omega_X^1(\log D)) - \frac{n}{n+1}c_1(\Omega_X^1(\log D))^2\right) \cdot c_1(L)^{n-2} \geq 0. \tag{5.3.1}$$

When the above equality holds,

(i) *if D is smooth, then $X - D \simeq \mathbb{B}^n/\Gamma$ for some torsion free lattice $\Gamma \subset PU(n,1)$ acting on $\mathbb{B}^n$. Moreover, X is the (unique) toroidal compactification of $\mathbb{B}^n/\Gamma$, and each connected component of D is the* smooth *quotient of an Abelian variety A by a finite group acting freely on A.*

(ii) *If D is not smooth, then the universal cover $\widetilde{X-D}$ of $X-D$ is* not *biholomorphic to $\mathbb{B}^n$, though there exists a holomorphic map $\widetilde{X-D} \to \mathbb{B}^n$ which is locally biholomorphic.*

In both cases, $K_X + D$ is big, nef and ample over $X - D$.

Proof Denote the log Hodge bundle $(E, \theta) = (E^{1,0} \oplus E^{0,1}, \theta)$ by

$$E^{1,0} := \Omega^1_X(\log D), \quad E^{0,1} := \mathcal{O}_X.$$

By [40, Theorem 6.5] we have the following Bogomolov-Gieseker inequality for (E, θ)

$$\begin{aligned} &\big(2c_2(\Omega^1_X(\log X)) - \frac{n}{n+1} c_1(\Omega^1_X(\log D))^2\big) \cdot c_1(L)^{n-2} \\ &= \big(2c_2(E) - \frac{\text{rank } E - 1}{\text{rank } E} c_1(E)^2\big) \cdot c_1(L)^{n-2} \geq 0 \end{aligned} \tag{5.3.2}$$

This shows the desired inequality (5.3.1).

The rest of the proof will be divided into three steps. In Step 1, we shall construct a uniformizing variation of Hodge structures on $X - D$ so that the corresponding period map defined in (3.0.4) induces a holomorphic map (so-called *period map* in Remark 3.7) from the universal cover of $X - D$ to $\mathbb{B}^n$ which is locally biholomorphic. By (3.0.5), this period map is moreover an *isometry* if we equip $X - D$ with hermitian metric induced by the Hodge metric. In Step two we will prove that, when D is smooth, the hermitian metric on $X - D$ induced by the Hodge metric is *complete*. Together with arguments in Remark 3.7, this proves that the above period map is indeed a biholomorphism. In Step three we shall prove Theorem 5.7.(ii) and the positivity of $K_X + D$.

Step 1. We apply Proposition 3.11 to the above system of log Hodge bundles $(E^{1,0} \oplus E^{0,1}, \theta)$. Then there is a principal system of log Hodge bundles (P, τ) on (X, D) with the structure group $K = P(GL(V^{1,0}) \times GL(V^{0,1}))$ with rank $V^{1,0} = \text{rank } E^{1,0} = n$, and rank $V^{0,1} = \text{rank } E^{0,1} = 1$. Here we use the notations in Example 3.1. Then by Proposition 3.11 the Hodge group relative to (P, τ) is $G_0 = PU(n, 1)$, and $K_0 = K \cap G_0 = P(U(n) \times U(1)) = U(n)$. For the complexified group $G = PGL(V)$ of G_0, its adjoint representation $Ad : G \to GL(\mathfrak{g}) = GL(\mathfrak{sl}(V))$ is faithful. By Example 3.9, this is a Hodge representation. By Example 3.10, such Hodge representation Ad induces a system of log Hodge bundles $(P \times_{Ad} \mathfrak{g}, d(Ad)(\tau))$ over (X, D). It follows our construction of (P, τ) that

$$(P \times_{Ad} \mathfrak{g}, d(Ad)(\tau)) = (\text{End}(E)^{\perp}, \theta_{End(E)^{\perp}}),$$

where $\text{End}(E)^{\perp}$ is the trace-free subbundle of $\text{End}(E)$, and $\theta_{End(E)^{\perp}}$ is the induced Higgs field from (E, θ).

On the other hand, an easy computation shows that $c_1(\text{End}(E)) = 0$, and

$$ch_2(\text{End}(E)) = -2\text{rank } E \cdot c_2(E) + (\text{rank } E - 1)c_1(E)^2$$

$$= nc_1^2(K_X + D) - 2(n+1)c_2(\Omega_X^1(\log D)) = 0$$

since the equality in (5.3.2) holds by our assumption. Since we assume that (E, θ) is μ_L-polystable, by Theorem 2.9, $(\mathrm{End}(E), \theta_{End(E)})$ is also μ_L-polystable. We now apply Proposition 2.12 to find a Hodge metric h for the system of log Hodge bundle $(\mathrm{End}(E)|_{X-D}, \theta_{End(E)}|_{X-D})$ which is adapted to $(\mathrm{End}(E), \theta_{End(E)})$. Since $(\mathrm{End}(E), \theta_{End(E)}) = (\mathrm{End}(E)^{\perp}, \theta_{End(E)^{\perp}}) \oplus (\mathcal{O}_X, 0)$, we conclude that $h = h_1 \oplus h_2$, where h_1 is the harmonic metric for $(\mathrm{End}(E)^{\perp}|_{X-D}, \theta_{End^{\perp}(E)}|_{X-D})$ which is adapted to the log Higgs bundle $(\mathrm{End}^{\perp}(E), \theta_{End^{\perp}(E)})$, and h_2 is the canonical metric for the trivial Higgs bundle $(\mathcal{O}_X, 0)$.

We now apply Theorem 4.1 to conclude that h_1 induces a reduction P_H for $P|_{X-D}$ with the structure group $K_0 = P(U(n) \times U(1)) \simeq U(n)$, which is compatible with h_1 such that $(P|_{X-D}, \tau|_{X-D}, P_H)$ is a principal variation of Hodge structures on $X - D$. Note that

$$T_X(-\log D) \xrightarrow{\tau} P \times_K \mathfrak{g}^{-1,1} = \mathrm{Hom}(E^{1,0}, E^{0,1}) \simeq \mathrm{Hom}(\Omega_X^1(\log D), \mathcal{O}_X)$$

is an isomorphism. Hence $(P|_{X-D}, \tau|_{X-D}, P_H)$ is moreover a *uniformizing variation of Hodge structures* over $X - D$ in the sense of Definition 3.6. By Remark 3.7, it gives rise to a holomorphic map, the so-called period map,

$$\widetilde{X - D} \to G_0/K_0 = PU(n,1)/U(n) \simeq \mathbb{B}^n \tag{5.3.3}$$

defined in (3.0.4), which is locally *biholomorphic*. Here $\widetilde{X - D}$ is the *universal cover* of $X - D$.

Note that the reduction P_H together with the hermitian metric $h_{\mathfrak{g}}$ in (3.0.1) gives rise to a natural metric h_H over $P \times_K \mathfrak{g}|_{X-D}$ defined in (3.0.3). By Remark 3.7 again, if the pull back $\tau^* h_H$ is a *complete metric* on $X - D$, then $X - D$ is uniformized by $G_0/K_0 = PU(n,1)/U(n)$ which is the complex unit ball of dimension n, denoted by $\mathbb{B}^n$. It follows from (4.0.3) that $h_1 = h_H$. It now suffices to show that $\tau^* h$ is complete if we want to prove that $X - D$ is uniformized by $\mathbb{B}^n$, where we recall

$$\tau : T_X(-\log D) \xrightarrow{\simeq} \mathrm{Hom}(E^{1,0}, E^{0,1}) \subset \mathrm{End}(E).$$

In next step, we will apply similar ideas by Simpson [52, Corollary 4.2] to prove this. Note that until now we made no assumption on the smoothness of D.

Step 2. Throughout Step 2, we will assume that D is smooth. Consider now the system of log Hodge bundles $(\mathcal{E}, \eta) := (\mathrm{End}(E), \theta_{End(E)})$. We first mention that the above Hodge metric h for $(\mathcal{E}, \eta)|_{X-D}$ is adapted to log order in the sense of Definition 5.1. Indeed, it follows from [39, Corollary 4.9] that the eigenvalues of monodromies of the flat connection $D := d_h + \eta + \overline{\eta}_h$ around the divisor D are 1. By the "weak" norm estimate in [39, Lemma 4.15], we conclude that h is adapted to log order[2].

[2] Indeed, a strong norm estimate has already been obtained by Cattani–Kaplan–Schmid in [14]. Here we only need to know that h is adapted to log order, which is a bit easier to obtain using Andreotti–Vesentini type results by Simpson [52] and Mochizuki [39, Lemma 4.15].

We first give an estimate for τ^*h. For any point $x \in D$, consider an admissible coordinates $(U; z_1, \ldots, z_n)$ centered at x as Definition 2.3 so that $D \cap U = (z_1 = 0)$. To distinguish the sections of log Higgs bundles and log forms, we write $e_1 := d\log z_1$ and $e_i = dz_i$ for $i = 2, \ldots, n$. Denote by $e_0 = 1$ the constant section of $\mathcal{O}_X$.

Let us introduce a new metric $\tilde{h}$ on $(E, \theta)|_{U^*}$ as follows.

$$|e_1|^2_{\tilde{h}} := (-\log|z_1|^2); \quad \langle e_i, e_j\rangle_{\tilde{h}} := 0 \quad \text{for} \quad i \neq j;$$
$$|e_i|^2_{\tilde{h}} := 1 \quad \text{for} \quad i = 2, \ldots, n; \quad |e_0|^2_{\tilde{h}} := (-\log|z_1|^2)^{-1}.$$

Write $h_{ii} := |e_i|^2_{\tilde{h}}$, and $F_{\tilde{h}}(E) := F_{\tilde{h}}(E)_{kj} \otimes e_j^* \otimes e_k$. Then for $i, j = 2, \ldots, n$, one has

$$\begin{aligned}
F_{\tilde{h}}(E)_{11} &= F_{\tilde{h}}(E)_{10} = F_{\tilde{h}}(E)_{01} = F_{\tilde{h}}(E)_{0i} = F_{\tilde{h}}(E)_{j0} = 0\\
F_{\tilde{h}}(E)_{ij} &= (-\log|z_1|^2)^{-1} d\bar{z}_i \wedge dz_j\\
F_{\tilde{h}}(E)_{1i} &= \frac{1}{(-\log|z_1|^2)^2 \bar{z}_1} d\bar{z}_1 \wedge dz_i\\
F_{\tilde{h}}(E)_{i1} &= \frac{1}{(-\log|z_1|^2) z_1} d\bar{z}_i \wedge dz_1\\
F_{\tilde{h}}(E)_{00} &= \sum_{i=2}^{n} (-\log|z_1|^2)^{-1} dz_i \wedge d\bar{z}_i.
\end{aligned}$$

In conclusion, there is a constant $C_1 > 0$ so that one has

$$|F_{\tilde{h}}(E)|^2_{h,\omega_e} = \sum_{0 \le j,k \le n} |F_{\tilde{h}}(E)_{kj} \otimes e_j^* \otimes e_k|^2_{h,\omega_e} \le \frac{C_1}{(-\log|z_1|^2)^3 |z_1|^2} \tag{5.3.4}$$

over $U^*(\frac{1}{2})$ (notation defined in Definition 2.3), where $\omega_e = \sqrt{-1}\sum_{i=1}^n dz_i \wedge d\bar{z}_i$ is the Euclidean metric on U^*.

We abusively denote by $\tilde{h}$ the induced metric on $(\mathcal{E}, \eta)|_{U^*} := (\mathrm{End}(E), \theta_{End(E)})|_{U^*}$, which is adapted to log order on $(U, D \cap U)$ in the sense of Definition 5.1 by our construction. Then

$$\begin{aligned}
F_{\tilde{h}}(\mathcal{E}) &= F_{\tilde{h}}(E) \otimes \mathbb{1}_{E^*} + \mathbb{1}_E \otimes F_{\tilde{h}^*}(E^*)\\
&= F_{\tilde{h}}(E) \otimes \mathbb{1}_{E^*} - \mathbb{1}_E \otimes F_{\tilde{h}}(E)^{\dagger}
\end{aligned}$$

where $F_{\tilde{h}}(E)^{\dagger}$ is the transpose of $F_{\tilde{h}}(E)$. Hence

$$F_{\tilde{h}}(\mathcal{E})(e_i \otimes e_j^*) = \sum_{k,\ell} (\delta_{j\ell} F_{\tilde{h}}(E)_{ik} - \delta_{ik} F_{\tilde{h}}(E)_{\ell j})(e_k \otimes e_\ell^*)$$

for $0 \le i, j, k, \ell \le n$. It then follows from (5.3.4) that

$$|F_{\tilde{h}}(\mathcal{E})|^2_{h,\omega_e} \le \frac{C_2}{(-\log|z_1|^2)^3|z_1|^2} \tag{5.3.5}$$

over $U^*(\frac{1}{2})$ for some constant $C_2 > 0$. Consider the identity map s for $\mathcal{E}$, which can be seen as a holomorphic section of $\mathrm{End}(\mathcal{E},\mathcal{E})$. We denote by $(\mathcal{F},\Phi) := (\mathrm{End}(\mathcal{E},\mathcal{E}), \eta_{End(\mathcal{E})})$ the induced Higgs bundle by $(\mathcal{E},\eta)$. One can check that

$$\Phi(s) = 0. \tag{5.3.6}$$

We equip $\mathcal{F}|_{U^*}$ with the metric $h_{\mathcal{F}} := \tilde{h} \otimes h^*$, where h is the harmonic metric for $(\mathcal{E},\eta)|_{X-D}$ constructed in Step one. Note that

$$\begin{aligned} F_{h_{\mathcal{F}}}(\mathcal{F}) &= F_{\tilde{h}}(\mathcal{E}) \otimes \mathbb{1}_{\mathcal{E}^*} + \mathbb{1}_{\mathcal{E}} \otimes F_{h^*}(\mathcal{E}^*) \\ &= F_{\tilde{h}}(\mathcal{E}) \otimes \mathbb{1}_{\mathcal{E}^*} \end{aligned}$$

By (5.3.5), there is a constant $C_0 > 0$ so that one has

$$|F_{h_{\mathcal{F}}}(\mathcal{F})|_{h_{\mathcal{F}},\omega_e} \le \frac{C_0}{(-\log|z_1|^2)^{\frac{3}{2}}|z_1|} \tag{5.3.7}$$

over $U^*(\frac{1}{2})$. Then

$$\begin{aligned} \sqrt{-1}\partial\overline{\partial}\log|s|^2_{h_{\mathcal{F}}} &\ge -\frac{\sqrt{-1}\{R_{h_{\mathcal{F}}}s,s\}}{|s|^2_{h_{\mathcal{F}}}} \\ &= -\frac{\sqrt{-1}\{\Phi s,\Phi s\}}{|s|^2_{h_{\mathcal{F}}}} - \frac{\sqrt{-1}\{\overline{\Phi}_{h_{\mathcal{F}}}s,\overline{\Phi}_{h_{\mathcal{F}}}s\}}{|s|^2_{h_{\mathcal{F}}}} - \frac{\sqrt{-1}\{F_{h_{\mathcal{F}}}(\mathcal{F})s,s\}}{|s|^2_{h_{\mathcal{F}}}} \\ &\ge -\frac{\sqrt{-1}\{F_{h_{\mathcal{F}}}(\mathcal{F})s,s\}}{|s|^2_{h_{\mathcal{F}}}}. \end{aligned}$$

Here the third inequality follows from (5.3.6). For any $\xi = (\xi_2,\ldots,\xi_n)$ with $0 \le \xi_2,\ldots,\xi_n \le \frac{1}{2}$, we define a smooth function f_ξ over Δ^* parametrized by ξ by

$$f_\xi(z_1) := \log|s|^2_{h_{\mathcal{F}}}(z_1,\xi_2,\ldots,\xi_n).$$

Then the above inequality together with (5.3.7) implies that

$$\Delta f_\xi \ge -|F_{h_{\mathcal{F}}}(\mathcal{F})|_{h_{\mathcal{F}},\omega_e} \ge -\frac{C_0}{(-\log|z_1|^2)^{\frac{3}{2}}|z_1|} =: \varphi$$

where C_0 is some uniform constant which does not depend on ξ. Note that

$$\|\varphi\|_{L^2} := \int_{0<|z_1|<\frac{1}{2}} |\varphi(z_1)|^2 dz_1 d\bar{z}_1 < C_4 \tag{5.3.8}$$

for some constant $C_4 > 0$. For any fixed $0 \leq \xi_2, \ldots, \xi_n \leq \frac{1}{2}$, consider the Dirichlet problem

$$\begin{cases} \phi = f_\xi \quad \text{on} \quad \{z_1 \mid |z_1| = \frac{1}{2}\} \\ \Delta\phi = \varphi \quad \text{on} \quad \{z_1 \mid 0 < |z_1| < \frac{1}{2}\} \end{cases} \tag{5.3.9}$$

By (5.3.8) and the elliptic estimate, one has

$$\sup_{0<|z_1|<\frac{1}{2}} |\phi(z_1)| \leq C_5(\|\varphi\|_{L^2} + \sup_{|z_1|=\frac{1}{2}} f_\xi). \tag{5.3.10}$$

over $\{z_1 \mid 0 < |z_1| < \frac{1}{2}\}$ for some uniform positive constant C_5 which does not depending on ξ. Hence $\Delta(f_\xi - \phi) \geq 0$ over $\{z_1 \mid 0 < |z_1| < \frac{1}{2}\}$. Since both h and $\tilde{h}$ are adapted to log order, so is $h_{\mathcal{F}}$. Hence there is a constant $C_6 > 0$ so that

$$\log |s|^2_{h_{\mathcal{F}}} \leq C_6 \log \left(-\sum_{i=1}^{\ell} \log |z_i| \right)$$

over $U^*(\frac{1}{2})$. By Lemma 5.4, we conclude that $f_\xi - \phi$ extends to a subharmonic function on $\{z_1 \mid |z_1| < \frac{1}{2}\}$. Note that $f_\xi(z_1) - \phi(z_1) = 0$ when $|z_1| = \frac{1}{2}$. Hence by maximum principle,

$$f_\xi(z_1) \leq \phi(z_1)$$

for any $0 < |z_1| < \frac{1}{2}$. Let

$$C_7 := \sup_{|z_1|=\frac{1}{2}, 0\leq\xi_2,\ldots,\xi_n\leq\frac{1}{2}} f_\xi(z_1)$$

which is finite. By (5.3.8) and (5.3.10), we have

$$\sup_{0<|z_1|<\frac{1}{2}, 0\leq z_2,\ldots,z_n\leq\frac{1}{2}} \log |s|^2_{h_{\mathcal{F}}}(z_1, \ldots, z_n) \leq C_5(C_4 + C_7).$$

This implies that $h \geq C_8 \cdot \tilde{h}$ over $U^*(\frac{1}{2})$ for some constant $C_8 > 0$. By (5.3.5), one has

$$|F_{\tilde{h}^*}(\mathcal{E}^*)|^2_{h^*,\omega_e} \leq \frac{C_0}{(-\log |z_1|^2)^3 |z_1|^2}.$$

Hence if we use the metric $h \otimes \tilde{h}^*$ for $\mathcal{F}$ and do the same proof, we can prove that $h \leq C_9 \cdot \tilde{h}$ over $U^*(\frac{1}{2})$ for some constant $C_9 > 0$. Therefore, $\tilde{h}$ and h are *mutually bounded* on $U^*(\frac{1}{2})$. By

$$\tau\left(z_1\frac{\partial}{\partial z_1}\right) = e_1^* \otimes e_0 \tag{5.3.11}$$

$$\tau\left(\frac{\partial}{\partial z_j}\right) = e_j^* \otimes e_0 \quad \text{for} \quad j = 2, \ldots, n, \tag{5.3.12}$$

we obtain the norm estimate for the metric

$$\tau^* h \sim \tau^* \tilde{h} = \frac{\sqrt{-1}dz_1 \wedge d\bar{z}_1}{|z_1|^2(\log|z_1|^2)^2} + \sum_{k=2}^{n} \frac{\sqrt{-1}dz_k \wedge d\bar{z}_k}{-\log|z_1|^2} \tag{5.3.13}$$

Though $\tau^* h$ is strictly less than the Poincaré metric near D, one can easily prove that it is still a *complete metric*. Therefore, the hermitian metric $\tau^* h_H = \tau^* h$ on $X - D$ is also complete. Based on Remark 3.7, we conclude that $X - D$ is uniformized by the complex unit ball of dimension n, namely, there is a torsion free lattice $\Gamma \subset PU(n, 1)$ so that $X - D \simeq \mathbb{B}^n/\Gamma$. By (5.3.11) and (5.3.12), the canonical Kähler–Einstein metric $\omega := \tau^* h$ for $T_X(-\log D)|_U$ is adapted to log order. It follows from Theorem A.7 that X is the unique toroidal compactification for the non-compact ball quotient $\mathbb{B}^n/\Gamma$. We accomplish the proof of Theorem 5.7.(i).

Step 3. Assume now D is not smooth. By (5.3.3), the period map $\widetilde{X - D} \to \mathbb{B}^n$ is locally biholomorphic. Assume by contradiction that it is an isomorphism. Since h is adapted to log order, the canonical Kähler–Einstein metric $\omega := \tau^* h$ for $T_X(-\log D)|_U$ is also adapted to log order. It follows from Theorem A.7 that D cannot be singular. The contradiction is obtained, and thus the period map is not a uniformizing mapping. We proved Theorem 5.7.(ii).

Let us show that $K_X + D$ is big, nef and ample over $X - D$. Note that the metric $\det \omega^{-1}$ for $(K_X + D)|_U$ is adapted to log order, and that

$$\frac{\sqrt{-1}}{2\pi} R_{\det \omega^{-1}}((K_X + D)|_U) = (n + 1)\omega.$$

By Lemma 5.5, the hermitian metric $\det \omega^{-1}$ extends to a singular hermitian metric h_{K_X+D} for $K_X + D$ with zero Lelong numbers. Hence $K_X + D$ is nef. Since $\sqrt{-1}R_{h_{K_X+D}}(K_X + D) > 0$ on $X - D$, $K_X + D$ is thus big and ample over $X - D$. We finish the proof of the theorem. □

Remark 5.8 Note that the asymptotic behavior of the metric (5.3.13) is exactly the same as that of the Kähler–Einstein metric for the ball quotient near the boundary of its toroidal compactification (see [44, eq. (8) on p. 338]). This is indeed the hint for our construction of $\tilde{h}$.

Remark 5.9 We expect that Theorem 5.7.(ii) cannot happen. This is the case when $\dim X = 2$. Indeed, when the Miyaoka–Yau type equality in (1.1.2) holds, together with the conclusion that $K_X + D$ is big, nef and ample over $X - D$ in Theorem 5.7, it follows from [34] that $X - D$ is uniformized by $\mathbb{B}^2$, which is a contradiction to Theorem 5.7.(ii). This is not surprising: consider the smooth toroidal compactification X of a two dimensional ball quotient $\mathbb{B}^2/\Gamma$ with $D := X - \mathbb{B}^2/\Gamma$, (1.1.3) holds by Theorem B. Let $x \in D$ and let $\pi : Y = \mathrm{Bl}_x X \to X$. Then Y is a projective surface compactifying $\mathbb{B}^2/\Gamma$ with the boundary $D_Y := \pi^* D$ a simple normal crossing (not smooth) divisor. However, one has

$$3c_2(\Omega_Y^1(\log D_Y)) - c_1(\Omega_Y^1(\log D_Y))^2 = 1,$$

which violates the condition of uniformization in Theorem 5.7.

6 Higgs bundles associated to non-compact ball quotients

In this section, we will prove Theorem B. Sections 6.1 and 6.2 are technical preliminaries. In Sect. 6.3 we prove that a log Higgs bundle (E, θ) on a compact Kähler log pair is slope polystable with respect to some polarization by big and nef cohomology (1, 1)-class, if (E, θ) admits a Hermitian–Yang–Mills metric with "mild singularity" near the boundary divisor. In Sect. 6.4 we use the Bergman metric for quotients of complex unit balls by torsion free lattices to construct such Hermitian–Yang–Mills metric. This proves Theorem B.

6.1 Notions of positivity for curvature tensors

We recall some notions of positivity for Higgs bundles in [20, §1.3].

Let (E, θ) be a Higgs bundle endowed with a smooth metric h. For any $x \in X$, let $e_1, \ldots, e_r$ be a frame of E at x, and let $e^1, \ldots, e^r$ be its dual in E^*. Let $z_1, \ldots, z_n$ be a local coordinate centered at x. We write

$$F_h(E) = R_h(E) + [\theta, \overline{\theta}_h] = R_{j\bar{k}\alpha}^{\beta} dz_j \wedge d\bar{z}_k \otimes e^\alpha \otimes e_\beta$$

Set $R_{j\bar{k}\alpha\bar{\beta}} := h_{\gamma\bar{\beta}} R_{j\bar{k}\alpha}^{\gamma}$, where $h_{\gamma\bar{\beta}} = h(e_\gamma, e_\beta)$. $F_h(E)$ is called *Nakano semi-positive* at x if

$$\sum_{j,k,\alpha,\beta} R_{j\bar{k}\alpha\bar{\beta}} u^{j\alpha} \overline{u^{k\beta}} \geq 0$$

for any $u = \sum_{j,\alpha} u^{j\alpha} \frac{\partial}{\partial z_j} \otimes e_\alpha \in (T_X^{1,0} \otimes E)_x$. (E, θ, h) is called Nakano semipositive if $F_h(E)$ is Nakano semi-positive at every $x \in X$. When $\theta = 0$, this reduces to the same positivity concepts in [19, Chapter VII, §6] for vector bundles.

We write

$$F_h(E) \geq_{\mathrm{Nak}} \lambda(\omega \otimes \mathbb{1}_E) \quad \text{for } \lambda \in \mathbb{R}$$

if

$$\sum_{j,k,\alpha,\beta}(R_{j\bar{k}\alpha\bar{\beta}}-\lambda\omega_{j\bar{k}}h_{\alpha\bar{\beta}})(x)u^{j\alpha}\overline{u^{k\beta}}\geq 0$$

for any $x\in X$ and any $u=\sum_{j,\alpha}u^{j\alpha}\frac{\partial}{\partial z_j}\otimes e_\alpha\in(T_X^{1,0}\otimes E)_x$.

Let us recall the following lemma in [20, Lemma 1.8].

Lemma 6.1 *Let (E,θ,h) be a Higgs bundle on a Kähler manifold (X,ω). If there is a positive constant C so that $|F_h(x)|_{h,\omega}\leq C$ for any $x\in X$, then*

$$C\omega\otimes\mathbb{1}_E\geq_{Nak}F_h\geq_{Nak}-C\omega\otimes\mathbb{1}_E.$$

The following easy fact in [20, Lemma 1.9] will be useful in this paper.

Lemma 6.2 *Let (E_1,θ_2,h_1) and (E_2,θ_2,h_2) are two metrized Higgs bundles over a Kähler manifold (X,ω) such that $|F_{h_1}(x)|_{h_1,\omega}\leq C_1$ and $|F_{h_2}(x)|_{h_2,\omega}\leq C_2$ for all $x\in X$. Then for the hermitian vector bundle $(E_1\otimes E_2,h_1h_2)$, one has*

$$|F_{h_1\otimes h_2}(x)|_{h_1\otimes h_2,\omega}\leq\sqrt{2r_2C_1^2+2r_1C_2^2}$$

for all $x\in X$. Here $r_i:=\operatorname{rank}E_i$.

6.2 Some pluripotential theories

In this subsection we recall some results of deep pluripotential theories in [4,26]. The results in this subsection will be used in the proof of Proposition 6.6. Let us first recall the definitions of big or nef cohomology $(1,1)$-classes in [18, §6].

Definition 6.3 Let (X,ω) be a compact Kähler manifold. Let $\alpha\in H^{1,1}(X,\mathbb{R})$ be a cohomology $(1,1)$-class of X. The class α is *nef* (numerically eventual free) if for any $\varepsilon>0$, there is a smooth closed $(1,1)$-form $\eta_\varepsilon\in\alpha$ so that $\eta_\varepsilon\geq-\varepsilon\omega$. The class α is *big* if there is a closed positive $(1,1)$-current $T\in\alpha$ so that $T\geq\delta\omega$ for some $\delta>0$. Such a current T will be called a *Kähler current*.

Let X be a complex manifold of dimension n and let $U\subset X$ be a Zariski open set of X. Pick a smooth hermitian form ω on X. For any smooth differential form η of degree p on U so that

$$\int_U|\eta|_\omega\wedge\omega^n<+\infty,$$

one can *trivially* extend η to *a current T_η on X of degree $n-p$* by setting

$$\langle T_\eta,u\rangle:=\int_U\eta\wedge u \tag{6.2.1}$$

where u is the any *test form* of degree p which has compact support. In general, T_η might not be closed even if η is closed.

Let (X, ω) be a compact Kähler manifold of dimension n. Let $\alpha_1, \ldots, \alpha_n$ be big cohomology classes. Let $T_i \in \alpha_i$ be positive closed (1, 1)-currents whose local potential is locally bounded outside a closed analytic subvariety of X (a particular case of *small unbounded locus* of [4, Definition 1.2]). In this celebrated work by Boucksom-Eyssidieux-Guedj-Zariahi [4], they defined non-pluripolar product for these currents

$$\langle T_1 \wedge \cdots \wedge T_p \rangle$$

which is a closed positive (p, p)-current, and does not charge on any closed proper analytic subsets. Therefore, if we assume further that T_i is smooth over $X - A$ where A is a closed analytic subvariety of X, then $\langle T_1 \wedge \cdots \wedge T_p \rangle$ is nothing but the trivial extension of the (p, p)-form $(T_1 \wedge \cdots \wedge T_p)|_{X-A}$ to X.

Following [4, Definition 1.21], for a big class α, a positive (1, 1)-current $T \in \alpha$ has *full Monge-Ampère mass* if

$$\int_X \langle T_i^n \rangle = \mathrm{Vol}(\alpha).$$

The set of such positive currents in α with full Monge-Ampère mass is denoted by $\mathcal{E}(\alpha)$. We will not recall the definition of the *volume of big classes* by Boucksom in [8]. We just mention that when the class α is big and nef, one has

$$\mathrm{Vol}(\alpha) = \alpha^n.$$

The following lemma will be used in Sect. 6.3.

Lemma 6.4 *Let (X, ω) be a compact Kähler manifold and let D be a simple normal crossing divisor on X. Let S be a closed positive* (1, 1)*-current on X so that $S|_{X-D}$ is a smooth* (1, 1)*-form over $X - D$ which is strictly positive at one point and has at most* Poincaré growth *near D. Then the cohomology class $\alpha := \{S\}$ is big and nef, and $S \in \mathcal{E}(\alpha)$.*

Proof Let T be the Kähler current on X constructed in Remark 2.5. Since $T|_{X-D}$ has at most Poincaré growth near D, there exists a constant $C_1 > 0$ so that

$$C_1 T - S \geq 0.$$

Pick any point $x \in D$. Then there exists some admissible coordinates $(U; z_1, \ldots, z_n)$ centered at x so that the local potential φ of S satisfies that

$$\varphi \geq -C_1 \log \left(- \prod_{i=1}^{\ell} \log |z_1|^2 \right) - C_2$$

for some constant $C_2 > 0$. Hence S has zero Lelong numbers everywhere and thus α is nef. Since S is strictly positive at one point on $X - D$, it is big by [8]. It follows from [26, Proposition 2.3] that $S \in \mathcal{E}(\alpha)$. The lemma is proved. □

Let us recall an important theorem in [4].

Theorem 6.5 *[4, Corollary 2.15] Let (X, ω) be a compact Kähler manifold of dimension n. Let $\alpha_1, \ldots, \alpha_n$ be big and nef classes on X. For $T_i \in \mathcal{E}(\alpha_i)$ which are all smooth outside a closed proper analytic subset A, one has*

$$\int_{X-A} T_1 \wedge \cdots \wedge T_n = \int_X \langle T_1 \wedge \cdots \wedge T_n \rangle = \alpha_1 \cdots \alpha_n.$$

6.3 Hermitian–Yang–Mills metric and stability

Let (X, ω) be a compact Kähler manifold and let D be a simple normal crossing divisor on X. For applications of birational geometry, one usually considers more general polarization by big and nef line bundles. In this subsection, we will prove that a log Higgs bundle (E, θ) on (X, D) is μ_α-polystable if $(E, \theta)|_{X-D}$ admits a Hermitian–Yang–Mills metric whose growth at infinity is "mild", where α is certain big and nef cohomology class. When dim $X = 1$ or $D = \varnothing$ and the polarization is Kähler, this has been proved by Simpson [51,52]. As we have seen in Theorem 2.8, when X is projective and both the first and second Chern classes of E vanish and the polarization is an ample line bundle, this result has been proved by Mochizuki.

We start with the following technical result, which is strongly inspired by the deep result of Guenancia [27, Proposition 3.8].

Proposition 6.6 *Let (X, ω_0) be a compact Kähler manifold and let D be a simple normal crossing divisor on X. Let (E, θ) be a log Higgs bundle on (X, D). Let α be a big and nef cohomology $(1, 1)$-class containing a positive closed $(1, 1)$-current $\omega \in \alpha$ so that $\omega|_{X-D}$ is a smooth Kähler form and has at most Poincaré growth near D. Assume that there is a hermitian metric h for $(E, \theta)|_{X-D}$ which is adapted to log order (in the sense of Definition 5.1) and is acceptable (in the sense of Definition 5.2). Then for any saturated Higgs subsheaf $G \subset E$, one has*

$$c_1(G) \cdot \alpha^{n-1} = \int_{X-D-Z} Tr(\sqrt{-1} R_{h_G}(G)) \wedge \omega^{n-1} \tag{6.3.1}$$

where Z is the analytic subvariety of codimension at least two so that $G|_{X-Z} \subset E|_{X-Z}$ is a subbundle, and h_G is the metric on G induced by h.

Proof By Remark 2.5, one can construct a *Kähler current*

$$T := \omega_0 - \sqrt{-1}\partial\bar{\partial} \log \left(- \prod_{i=1}^{\ell} \log |\varepsilon \cdot \sigma_i|^2_{h_i} \right), \tag{6.3.2}$$

over X, whose restriction on $X - D$ is a complete Kähler form ω_P, which has the same Poincaré growth near D. Here σ_i is the section $H^0(X, \mathcal{O}_X(D_i))$ defining D_i, and h_i is some smooth metric for the line bundle $\mathcal{O}_X(D_i)$. Since we assume that h is acceptable, (after rescaling T by multiplying a constant) one thus has

$$|F_h(E)|_{h,\omega_P} \leq 1.$$

By Lemma 6.1, one has

$$-\mathbb{1} \otimes \omega_P \leq_{Nak} F_h(E) \leq_{Nak} \mathbb{1} \otimes \omega_P$$

over $X - D$.

We first consider the case that G is an invertible saturated subsheaf of E which is invariant under θ. Then the metric h of E induces a *singular hermitian metric* h_G for G defined on the whole X, which is smooth on on $X^\circ := X - D - Z$. The curvature current $\sqrt{-1}R_{h_G}(G)$ is a closed (1, 1)-current on $X - D$, which is a smooth (1, 1)-form on X°. Define by $\pi : E|_{X^\circ} \to G|_{X^\circ}$ the orthogonal projection with respect to h and $\pi^\perp : E|_{X^\circ} \to G^\perp|_{X^\circ}$ the projection to its orthogonal complement. By the Chern–Weil formula (see for example [51, Lemma 2.3]), over X°, we have

$$R_{h_G}(G) = F_{h_G}(G) = F_h(E)|_G + \overline{\beta}_h \wedge \beta - \varphi \wedge \overline{\varphi}_h \tag{6.3.3}$$

where $F_h(E)|_G$ is the orthogonal projection of $F_h(E)$ on $\mathrm{Hom}(G, G)|_{X^\circ} = \mathcal{O}_{X^\circ}$, and $\beta \in \mathscr{A}^{1,0}(X^\circ, \mathrm{Hom}(G, G^\perp))$ is the second fundamental form, and $\varphi \in \mathscr{A}^{1,0}(X^\circ, \mathrm{Hom}(G^\perp, G))$ is equal to $\theta|_{G^\perp}$. Hence $\sqrt{-1}R_{h_G}(G) \leq \sqrt{-1}F_h(E)|_G$.

For any local frame e of $G|_{X^\circ}$, note that

$$|e|_h^2 \cdot \sqrt{-1}F_h(E)|_G = \langle \sqrt{-1}F_h(E)(e), e\rangle_h \leq \langle \mathbb{1} \otimes \omega_P e, e\rangle_h = |e|_h^2 \cdot \omega_P$$

Hence $\sqrt{-1}F_h(E)|_G - \omega_P$ is a semi-negative (1, 1)-form on X°, and thus over X° one has

$$-\sqrt{-1}R_{h_G}(G) + T \geq \omega_P - \sqrt{-1}F_h(E)|_G \geq 0$$

Since we assume that (E, h) is adapted to log order, $(G^{-1}|_{X-Z}, h_G^{-1}|_{X-Z})$ is thus adapted to log order for the log pair $(X - Z, D - Z)$. By Lemma 5.5 and (6.3.2), $-\sqrt{-1}R_{h_G}(G) + T$ extends to a closed positive (1, 1)-current on $X - Z$.

Since Z is of codimension at least two, a standard fact in pluripotential theory (see [48, Theorem 3.3.42]) shows that $-\sqrt{-1}R_{h_G}(G) + T$ extends to a positive closed (1, 1)-current on the whole X.

Denote by $s \in H^0(X, E \otimes G^{-1})$ the section defining the inclusion $G \to E$. We fix a smooth hermitian metric h_0 for G and we define a function $H := |s|^2_{h \cdot h_0^{-1}} = h_G \cdot h_0^{-1}$ on $X - D$. Then

$$\sqrt{-1}\partial\overline{\partial} \log H = \sqrt{-1}R_{h_0}(G) - \sqrt{-1}R_{h_G}(G). \tag{6.3.4}$$

Hence there is a constant $C_0 > 0$ so that

$$\sqrt{-1}\partial\overline{\partial}\log H + C_0 T \geq T. \tag{6.3.5}$$

By Lemma 6.4, $\omega \in \mathcal{E}(\alpha)$. Since $\sqrt{-1}R_{h_0}(G)$ is a smooth (1, 1)-form on X, it follows from Theorem 6.5 that

$$\int_{X^\circ} \sqrt{-1}R_{h_0}(G) \wedge \omega^{n-1} = c_1(G) \cdot \alpha^{n-1}.$$

To prove (6.3.1), by (6.3.4) and the above equality it suffices to prove that

$$\int_{X^\circ} \sqrt{-1}\partial\overline{\partial}\log H \wedge \omega^{n-1} = 0. \tag{6.3.6}$$

We will pursue the ideas in [27, Proposition 3.8] to prove this equality.

Let us take a log resolution $\mu : \tilde{X} \to X$ of the ideal sheaf $\mathscr{I}$ defined by $s \in H^0(X, E \otimes G^{-1})$, with $\mathcal{O}_{\tilde{X}}(-A) = \mu^*\mathscr{I}$ and $\tilde{D} := \mu^{-1}(D)$ a simple normal crossing divisor. Let us denote by $(\tilde{E}, \tilde{\theta})$ the induced log Higgs bundle on $(\tilde{X}, \tilde{D})$ by pulling back (E, θ) via μ. Then the metric $\tilde{h} := \mu^* h$ for $(\tilde{E}, \tilde{\theta})|_{\tilde{X}-\tilde{D}}$ is also adapted to log order and acceptable by Lemma 5.3.

Note that $\mathrm{Supp}(\mathcal{O}_X/\mathscr{I}) = Z$. Write $\tilde{G} := \mu^* G$. There is a nowhere vanishing section

$$\tilde{s} \in H^0(\tilde{X}, \tilde{E} \otimes \tilde{G}^{-1} \otimes \mathcal{O}_{\tilde{X}}(-A))$$

so that $\mu^* s = \tilde{s} \cdot \sigma_A$, where σ_A is the canonical section in $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(A))$ which defines the effective exceptional divisor A.

Fix a Kähler form $\tilde{\omega}$ on $\tilde{X}$, as Remark 2.5 we construct another Kähler current

$$\tilde{T} := \tilde{\omega} - \sqrt{-1}\partial\overline{\partial}\log\left(-\prod_{i=1}^{m}\log|\varepsilon \cdot \tilde{\sigma}_i|^2_{\tilde{h}_i}\right), \tag{6.3.7}$$

over $\tilde{X}$, whose restriction on $\tilde{X} - \tilde{D}$ is a complete Kähler form, which has the same Poincaré growth near $\tilde{D}$. Here $\tilde{\sigma}_i$ is the section $H^0(X, \mathcal{O}_X(\tilde{D}_i))$ defining $\tilde{D}_i$, and $\tilde{h}_i$ is some smooth metric for the line bundle $\mathcal{O}_{\tilde{X}}(\tilde{D}_i)$.

Let us fix a smooth hermitian metric h_A for $\mathcal{O}_{\tilde{X}}(A)$. Write $\tilde{H} := |\tilde{s}|^2_{\tilde{h}\cdot\mu^* h_0^{-1}\cdot h_A^{-1}}$. Since $\tilde{h}$ is adapted to log order and $\tilde{s}$ is nowhere vanishing, there is a constant $C_1, C_2 > 0$ so that

$$\log \tilde{H} \geq C_1 \varphi_P - C_2, \tag{6.3.8}$$

where we denote by

$$\varphi_P := -\log\left(-\prod_{i=1}^{\ell}\log|\varepsilon\cdot\tilde{\sigma}_i|^2_{\tilde{h}_i}\right).$$

Since $\tilde{h} := \mu^* h$ for $(\tilde{E},\tilde{\theta})|_{\tilde{X}-\tilde{D}}$ is acceptable, by same arguments as those for (6.3.5), one can show that

$$\sqrt{-1}\partial\overline{\partial}\log\tilde{H} + C_3\tilde{T} \geq \tilde{T}$$

over $\tilde{X}-\tilde{D}$ for some constant $C_3 > 0$. Note that the local potential of $\sqrt{-1}\partial\overline{\partial}\log\tilde{H} + C_3\tilde{T}$ is bounded from below by $(C_1+C_3)\varphi_P$ according to (6.3.8). By [26, Proposition 2.3], one has

$$\sqrt{-1}\partial\overline{\partial}\log\tilde{H} + C_3\tilde{T} \in \mathcal{E}(\{C_3\tilde{T}\}).$$

One can check that $\mu^*\omega \leq C_4\tilde{T}$ for some constant $C_4 > 0$. By Lemma 6.4 again, $\mu^*\omega \in \mathcal{E}(\mu^*\alpha)$. Hence by Theorem 6.5 one has

$$\int_{\mu^{-1}(X^\circ)}(\sqrt{-1}\partial\overline{\partial}\log\tilde{H} + C_3\tilde{T})\wedge\mu^*\omega^{n-1} = \{C_3\tilde{T}\}\cdot\mu^*\alpha^{n-1}.$$

Recall that $\tilde{T} \in \mathcal{E}(\{\tilde{T}\})$ by Lemma 6.4. Hence

$$\int_{\mu^{-1}(X^\circ)} C_3\tilde{T}\wedge\mu^*\omega^{n-1} = \{C_3\tilde{T}\}\cdot\mu^*\alpha^{n-1}.$$

One thus has

$$\int_{\mu^{-1}(X^\circ)}\sqrt{-1}\partial\overline{\partial}\log\tilde{H}\wedge\mu^*\omega^{n-1} = 0. \tag{6.3.9}$$

Note that over $\tilde{X}-\tilde{D}$, one has

$$\sqrt{-1}\partial\overline{\partial}\log\tilde{H} + [A] - \sqrt{-1}R_{h_A}(A) = \mu^*\sqrt{-1}\partial\overline{\partial}\log H$$

where $[A]$ is the current of integration of A. Hence over $\mu^{-1}(X^\circ) \simeq X^\circ$, one has

$$\sqrt{-1}\partial\overline{\partial}\log\tilde{H} - \sqrt{-1}R_{h_A}(A) = \mu^*\sqrt{-1}\partial\overline{\partial}\log H. \tag{6.3.10}$$

By Theorem 6.5 again,

$$\int_{\mu^{-1}(X^\circ)}\sqrt{-1}R_{h_A}(A)\wedge\mu^*\omega^{n-1} = c_1(A)\cdot\mu^*\alpha^{n-1} = 0, \tag{6.3.11}$$

where the last equality follows from the fact that A is μ-exceptional. (6.3.9), (6.3.10) together with (6.3.11) shows the desired equality (6.3.6). We finish the proof of (6.3.1) when rank $G = 1$.

Assume that rank $G = r$. We replace (E, θ, h) by the wedge product $(\tilde{E}, \tilde{\theta}, \tilde{h}) := \Lambda^r(E, \theta, h)$. By Lemma 6.2, the induced metric $\tilde{h}$ is also acceptable and one can easily check that it is also adapted to log order. Note that det G is also invariant under $\tilde{\theta}$, and that det $G \to \Lambda^r E$. We then reduce the general cases to rank 1 cases. The proposition is thus proved. □

Let us state and prove the main result in this section.

Theorem 6.7 *Let X be a compact Kähler manifold and let D be a simple normal crossing divisor on X. Let α be a big and nef cohomology $(1, 1)$-class containing a positive closed $(1, 1)$-current $\omega \in \alpha$ so that $\omega|_{X-D}$ is a smooth Kähler form and has at most Poincaré growth near D. Let (E, θ) be a log Higgs bundle on (X, D). Assume that there is a hermitian metric h on $(E, \theta)|_{X-D}$ such that*

- *it is adapted to log order (in the sense of Definition 5.1);*
- *it is acceptable (in the sense of Definition 5.2);*
- *it is* Hermitian–Yang–Mills*:*

$$\Lambda_\omega F_h(E)^\perp = 0.$$

Then (E, θ) is μ_α-polystable.

Proof We shall use the same notations as those in Proposition 6.6.

Let G be any saturated Higgs-subsheaf $G \subset E$, and denote by Z the analytic subvariety of codimension at least two so that $G|_{X-Z} \subset E|_{X-Z}$ is a subbundle. By the Chern–Weil formula again, over $X^\circ := X - Z - D$ we have

$$\Lambda_\omega F_{h_G}(G) = \frac{\Lambda_\omega Tr(F_h(E))}{\text{rank } E} \otimes \mathbb{1}_G + \Lambda_\omega(\overline{\beta}_h \wedge \beta - \varphi \wedge \overline{\varphi}_h).$$

where $\beta \in \mathscr{A}^{1,0}(X^\circ, \text{Hom}(G, G^\perp))$ is the second fundamental form of G in E with respect to the metric h, and $\varphi \in \mathscr{A}^{1,0}(X^\circ, \text{Hom}(G^\perp, G))$ is equal to $\theta|_{G^\perp}$.

Hence

$$\begin{aligned}
&\int_{X^\circ} Tr(\sqrt{-1}F_{h_G}(G)) \wedge \omega^{n-1} \\
&\quad = \int_{X^\circ} \frac{\text{rank } G}{\text{rank } E} Tr(\sqrt{-1}F_h(E)) \wedge \omega^{n-1} - (|\beta|_h^2 + |\varphi|_h^2)\frac{\omega^n}{n}.
\end{aligned}$$

By Proposition 6.6 together with the above inequality, one concludes the slope inequality

$$\mu_\alpha(G) \leq \mu_\alpha(E)$$

and the equality holds if and only if $\beta \equiv 0$ and $\varphi \equiv 0$. We shall prove that if the above slope equality holds, G is a *sub-Higgs bundle* of E, and we have the decomposition

$$(E, \theta) = (G, \theta|_G) \oplus (F, \theta_F)$$

where (F, θ_F) is another sub-Higgs bundle of E.

Set rank $E = r$ and rank $G = m$. We first prove that G is a subbundle of E. It is equivalent to show that $\det G \to \Lambda^r E$ is a subbundle, and we thus reduce the problem to the case that rank $G = 1$. Assume that $\mu_\alpha(G) = \mu_\alpha(E)$ and thus $\beta \equiv 0$ and $\varphi \equiv 0$. By (6.3.3), over X° one has

$$\sqrt{-1}R_{h_G}(G) = \sqrt{-1}F_h(E)|_G \geq -T|_{X^\circ}, \tag{6.3.12}$$

where T is the Kähler current defined in (6.3.2). By Lemma 5.5, $\sqrt{-1}R_{h_G}(G) + T$ extends to a closed positive (1, 1)-current on $X - Z$, and thus to the whole X.

Assume now $x_0 \in X$ is a point where $(E/G)_{x_0}$ is not locally free. Take a local holomorphic frame e of G on some open neighborhood $(U; z_1, \ldots, z_n)$ of x, and a holomorphic frame $e_1, \ldots, e_r$ of E. Then $e = \sum_{i=1}^r f_i(x)e_i$, where $f_i \in \mathcal{O}(U_i)$ so that $f_1(x_0) = \cdots = f_r(x_0) = 0$. By the asssumption that h is adapted to log order, one concludes that

$$\log |e|_h^2 \leq C_1 \log(|z_1|^2 + \cdots + |z_n|^2) + C_2 \log\left(-\log\left(\prod_{i=1}^{\ell} |z|_i^2\right)\right) \tag{6.3.13}$$

for some positive constants C_1 and C_2. On the other hand, by (6.3.12) on U we have

$$\sqrt{-1}\partial\overline{\partial} \log |e|_h^2 = -\sqrt{-1}R_{h_G}(G) \leq T.$$

By the construction of T, we conclude that

$$\log |e|_h^2 \geq -C_3 \log\left(-\log\left(\prod_{i=1}^{\ell} |z|_i^2\right)\right) - C_4,$$

for some positive constants C_3 and C_4. This contradicts with (6.3.13). Hence we conclude that when the slope equality holds, G is a subbundle of E.

We now find the desired decomposition of (E, θ). By the above argument, when the slope equality holds, $(G, \theta|_G)$ is a Higgs subbundle of (E, θ) (not assumed to be rank 1 now), and $\beta \equiv 0$ and $\varphi \equiv 0$. This means that the orthogonal projection $\pi : E|_{X-D} \to G|_{X-D}$ is holomorphic, that $G^\perp$ is a holomorphic subbundle of $E|_{X-D}$, and that

$$(E, \theta)|_{X-D} = (G, \theta|_G)|_{X-D} \oplus (G^\perp, \theta|_{G^\perp}). \tag{6.3.14}$$

We shall prove that π extends to a morphism $\tilde{\pi} : E \to G$ so that $\pi \circ \iota = \mathbb{1}$. For any point $x_0 \in D$, we pick an admissible coordinate $(U; z_1, \ldots, z_n)$ centered at x_0 and a holomorphic fame $(e_1,\ldots,e_r)$ for $E|_U$ adapted to log order so that $(e_1,\ldots,e_m)$ is a holomorphic fame for $G|_U$. Write $\pi(e_j|_{X-D}) = \sum_{i=1}^r f_i(x)e_i$, where $f_i(x) \in \mathcal{O}(U-D)$. For $j=1,\ldots,m$, one has $\pi(e_j|_{X-D}) = e_j$ and it extends naturally. For $j>m$ and some $1<r<1$, over $U^*(r)$ one has

$$C\left(-\log\left(\prod_{i=1}^{\ell}|z|_i^2\right)\right)^M \geq |e_j|_h^2 \geq |\pi(e_j)|_h^2 \geq C^{-1}\left(-\log\left(\prod_{i=1}^{\ell}|z|_i^2\right)\right)^{-M}\sum_{i=1}^{r}|f_i|^2$$

for some $C, M>0$, where the second inequality is due to the fact that π is the orthogonal projection with respect to h, and the last inequality follows from the fact that h is adapted to log order. Hence each $|f_i|$ must be locally bounded from above on U, and it thus extends to a holomorphic function on U. We conclude that π extends to a morphism $\tilde{\pi}: E \to G$, whose rank is constant and $\tilde{\pi}\circ\iota = \mathbb{1}$, where $\iota: G\to E$ denotes the inclusion. Let us define by $F := \ker\tilde{\pi}$, which is a subbundle of E so that $E = G\oplus F$. Note that $F|_{X-D} = G^{\perp}$. By (6.3.14) together with the continuity propery we conclude that F is a sub-Higgs bundle of (E,θ), and that $(E,\theta) = (G,\theta|_G)\oplus(F,\theta|_F)$. Since $h|_G$ (resp. $h|_F$) is a Hermitian–Yang–Mills metric for $(G,\theta|_G)$ (resp. $(F,\theta|_F)$) satisfying the three conditions in the theorem, we can argue in the same way as above to decompose $(G,\theta|_G)$ and $(F,\theta|_F)$ further to show that (E,θ) is a direct sum of μ_α-stable log Higgs bundles with the same slope. Hence (E,θ) is μ_α-polystable. We prove the theorem. □

6.4 Application to toroidal compactification of ball quotient

Let $\Gamma \in PU(n,1)$ be a torsion free lattice, and let $\mathbb{B}^n/\Gamma$ be the associated ball quotient. By the work of Baily–Borel, Siu–Yau and Mok [44], $\mathbb{B}^n/\Gamma$ has a unique structure of a quasi-projective complex algebraic variety (see for example [10, Theorem 3.1.12]). When the parabolic subgroups of Γ are unipotent, by the work of Ash et al. [2] and Mok [44, Theorem 1], $\mathbb{B}^n/\Gamma$ admits a *unique* smooth toroidal compactification, which we denote by X. Let us denote by $D := X - \mathbb{B}^n/\Gamma$ the boundary divisor, which is a disjoint union of abelian varieties. Let g_B be the Bergman metric for $\mathbb{B}^n$, which is complete, invariant under $PU(n,1)$ and has constant holomorphic sectional curvature -1. Hence it descends to a metric ω on $X-D$. If we consider ω as a metric for $T_X(-\log D)|_{X-D}$, by [54, Proposition 2.1] it is *good* in the sense of Mumford [45, Section 1]. Therefore, for any $k\geq 1$, it follows from [45, Theorem 1.4] that the trivial extension of the Chern form $c_k(T_{X-D},\omega)$ onto X defines a (k,k)-current $[c_k(T_{X-D},\omega)]$ on X, which represents the cohomology class $c_k(T_X(-\log D)) \in H^{k,k}(X)$. Let us first prove (1.1.3), which is indeed an easy computation.

For any $x_0 \in X-D$, we take a normal coordinate system $(z_1,\ldots,z_n)$ centered at x_0 so that

$$\omega = \sqrt{-1}\sum_{1\leq\ell,m\leq n}\delta_{\ell m}dz_\ell\wedge d\bar{z}_m - \sum_{j,k,\ell,m}c_{jk\ell m}z_j\bar{z}_k + O(|z|^3)$$

where $c_{jk\ell m}$ is the coefficients of the Chern curvature tensor

$$R_\omega(T_X) = \sum_{j,k,\ell,m} c_{jk\ell m} dz_j \wedge d\bar{z}_k \otimes \left(\frac{\partial}{\partial z_\ell}\right)^* \otimes \frac{\partial}{\partial z_m}.$$

By [43, p. 177], one has

$$c_{jk\ell m}(x_0) = -(\delta_{jk}\delta_{\ell m} + \delta_{jm}\delta_{k\ell}). \tag{6.4.1}$$

One can check that

$$nc_1(T_{X-D}, \omega)^2 - 2(n+1)c_2(T_{X-D}, \omega) \equiv 0.$$

We thus conclude that the Chern classes $c_k(\Omega_X^1(\log D))$ satisfies

$$nc_1(\Omega_X^1(\log D))^2 - 2(n+1)c_2(\Omega_X^1(\log D)) = 0.$$

Hence (1.1.3) in Theorem B holds.

For the log Hodge bundle $(E, \theta) = (E^{1,0} \oplus E^{0,1}, \theta)$, given by

$$E^{1,0} := \Omega_X^1(\log D), \quad E^{0,1} := \mathcal{O}_X$$

with the Higgs field θ defined in (1.1.1), we shall prove that it is μ_α-polystable for the big and nef polarization α in Theorem 6.7. We equipped $(E^{1,0} \oplus E^{0,1})|_{X-D}$ with the metric

$$h := \omega^{-1} \oplus h_c \tag{6.4.2}$$

where h_c is the canonical metric on $\mathcal{O}_{X-D}$ so that $|1|_{h_c} = 1$. Recall that the curvature $F_h(E)$ of the connection $D_h := d_h + \theta + \overline{\theta}_h$ is

$$F_h(E) = R_h(E) + [\theta, \overline{\theta}_h],$$

where $R_h(E)$ is the Chern curvature of (E, h). An easy exercise shows that

$$\sqrt{-1} F_h(E) = \omega \otimes \mathbb{1}.$$

In particular, h is a Hermitian–Yang–Mills metric for $(E, \theta)|_{X-D}$. We shall show that it satisfies the three conditions in Theorem 6.7. Indeed, we only have to check the first two conditions since $\sqrt{-1}F_h(E)^\perp \equiv 0$.

We first note that ω has at most Poincaré growth near D in the sense of Definition 2.4. Indeed, this follows easily from the Ahlfors–Schwarz lemma (see for example [47, Lemma 2.1]) since the holomorphic sectional curvature of ω is -1. Hence for any admissible coordinate system $(U; z_1, \ldots, z_n)$ as in Definition 2.3, one has $|F_h(E)|_{h,\omega_P} \leq C$, where ω_P is the Poincaré metric on U^*.

By the following result, we see that h is adapted to log order.

Lemma 6.8 ([44, eq. (8) on p. 338]) *Let (X, D) be as above. Then for any $x \in D$, there is an admissible coordinate $(U; z_1, \ldots, z_n)$ at x so that the frame $z_1\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_{n-1}}, \frac{\partial}{\partial z_n}$ is adapted to log order (in the sense of Sect. 5.1) with respect to the above metric ω.*

Therefore, the metric h for $(E, \theta)|_{X-D}$ satisfies the three conditions in Theorem 6.7. In conclusion, (E, θ) is μ_α-polystable for the big and nef class α in Theorem 6.7

To finish the proof of Theorem B, we have to show that $c_1(K_X + D)$ can be made as a polarization in Theorem 6.7, which follows from the following result.

Lemma 6.9 [44, Proposition 1] *The Kähler form $\frac{(n+1)}{2\pi}\omega$ on $X - D$ defined above extends to a closed positive $(1,1)$-current $\varpi \in c_1(K_X + D)$ with zero Lelong numbers. In particular, $K_X + D$ is big and nef.*

6.5 Proof of Corollary C

We shall show how to apply Theorems A and B to derive Corollary C.

Proof of Corollary C We first assume that parabolic subgroups of Γ are unipotent. By [44, Theorem 1], there is a toroidal compactification $\overline{X}$ for the ball quotient $X := \mathbb{B}^n/\Gamma$, so that $D := \overline{X} - X$ is a smooth divisor. Moreover, $\overline{X}$ is projective. Fix any ample polarization L on X. By Theorem B, the log Higgs bundle $(E, \theta) := (\Omega^1_{\overline{X}}(\log D) \oplus \mathcal{O}_{\overline{X}}, \theta)$ on $(\overline{X}, D)$ defined in (1.1.1) is μ_L-polystable with

$$2c_2(\Omega^1_{\overline{X}}(\log D)) - \frac{n}{n+1}c_1(\Omega^1_{\overline{X}}(\log D))^2 = 0. \tag{6.5.1}$$

Let us denote by $\overline{X}^\sigma$ and D^σ the conjugate varieties of $\overline{X}$ and D under σ. Hence $\overline{X}^\sigma$ is a smooth projective variety and D^σ is a smooth divisor on $\overline{X}^\sigma$. For any coherent sheaf $\mathcal{E}$ on $\overline{X}^\sigma$, we denote by $\mathcal{E}^\sigma$ its conjugate under σ, which is also a coherent sheaf on $\overline{X}^\sigma$. Note that the conjugate action induces a canonical isomorphism between cohomology groups

$$\Phi_k : H^{2k}(\overline{X}, \mathbb{C}) \xrightarrow{\simeq} H^{2k}(\overline{X}^\sigma, \mathbb{C}),$$

and that Chern classes of vector bundles over $\overline{X}$ are preserved under σ in the sense that $\Phi_k(c_k(F)) = c_k(F^\sigma)$ for any holomorphic vector bundle F over $\overline{X}$. Since $\left(\Omega^1_{\overline{X}}(\log D)\right)^\sigma = \Omega^1_{\overline{X}^\sigma}(\log D^\sigma)$, (6.5.1) also holds for the log cotangent bundle $\Omega^1_{\overline{X}}(\log D)$. Moreover, the conjugate of (E, θ) under σ is the log Higgs bundle $(E^\sigma, \theta^\sigma) := (\Omega^1_{\overline{X}^\sigma}(\log D^\sigma) \oplus \mathcal{O}_{X^\sigma}, \theta^\sigma)$ on $(\overline{X}^\sigma, D^\sigma)$ defined in (1.1.1). Hence for any Higgs subsheaf $\mathcal{F}$ of (E, θ), $\mathcal{F}^\sigma$ is also a Higgs subsheaf of $(E^\sigma, \theta^\sigma)$ with the slope inequality $\mu_L(\mathcal{F}) = \mu_{L^\sigma}(\mathcal{F}^\sigma)$. Hence $(E^\sigma, \theta^\sigma)$ is μ_{L^σ}-polystable. By Theorem A, $\overline{X}^\sigma - D^\sigma$ is also a ball quotient, with $\overline{X}^\sigma$ its toroidal compactification.

For a general torsion free lattice $\Gamma \subset PU(n, 1)$, there is a finite index subgroup $\Gamma' \subset \Gamma$ so that parabolic subgroups of Γ' are unipotent (see for example [10, §3.3]). Denote by $X := \mathbb{B}^n/\Gamma$ and $Y := \mathbb{B}^n/\Gamma'$. Since the base change of an étale morphism is étale, we conclude that $Y^\sigma \to X^\sigma$ is also a finite étale surjective morphism. By the above result, Y^σ is the ball quotient, and thus so is X^σ. Corollary C is proved. □

Remark 6.10 In the above proof we show that a toroidal compactification of a ball quotient is also a toroidal compactification of another ball quotient. As pointed out by the referee, this fact follows from Corollary C directly. His/her elegant argument is as follows: since an Abelian variety is simply an algebraic variety with a group law defined by regular functions, a Galois conjugate of the Abelian variety as a component of the compactifying divisor is by itself an Abelian variety, which sits in the compactifying divisor of the ball quotient obtained as the conjugate of the original one according to Corollary C.

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Data availability All data generated or analysed during this study are included in this article.

Appendix A: Metric rigidity for toroidal compactification of non-compact ball quotients

The main motivation of this appendix is to provide one building block for Theorem 5.7.(ii). Our main result, Theorem A.7, says that there is no other smooth compactification for non-compact ball quotient than the toroidal one, so that the Bergman metric grows “mildly” near the boundary. Besides its own interests, this result is applied to show that the smoothness of D in Theorem A is necessary if one would like to characterize non-compact ball quotients.

A.1: Toroidal compactifications of quotients by non-neat lattices

In this section, we recall a well known way of constructing the toroidal compactifications of ball quotients in the case where the lattice has torsion at infinity. The reader will find more details about the natural orbifold structure on these compactifications in [21]. For our purposes, the basic result given in Proposition A.1 will be sufficient.

Recall that we say that a lattice $\Gamma \subset PU(n, 1)$ is *neat* (cf. [7]) if for any $g \in \Gamma$, the subgroup of $\mathbb{C}^*$ generated by the eigenvalues of g is torsion free. This implies that Γ is torsion free and that all parabolic elements of Γ are unipotent, so that the toroidal compactifications of $\mathbb{B}^n/\Gamma$ provided by [2,44] are *smooth* (there is no "torsion at infinity"). Note that by [7, Proposition 17.4] in the arithmetic case, and [6], or [50, Theorem 6.11] in general, any lattice in $PU(n, 1)$ admits a finite index neat normal sublattice.

Proposition A.1 *Let $\Gamma \subset PU(n,1)$ be a torsion free lattice, and let $\Gamma' \subset \Gamma$ be a finite index normal* neat *sublattice. Let $U = \mathbb{B}^n/\Gamma$, $U' = \mathbb{B}^n/\Gamma'$, and denote by X' the smooth toroidal compactification of $U' = \mathbb{B}^n/\Gamma'$ as constructed in* [2,44].

Then the natural action of the finite group $G = \Gamma/\Gamma'$ on U' extends to X', and the quotient $X = X'/G$ is a normal projective space, with boundary $X - U$ made of quotient of abelian varieties by finite groups. Moreover, when Γ is arithmetic, X coincides with the toroidal compactification of U constructed in [2].

Before explaining how to prove Proposition A.1, let us recall the construction of X' as it is defined in [44] (see also [11] for a similar discussion).

Each component D of $X' - U'$ is associated to a certain Γ'-orbit of points of $\partial\mathbb{B}^n$, whose points are called the Γ'-*rational boundary components* of ∂B^n (cf. [2, Chapter 3] or [44, §1.3]). Let $b \in \partial\mathbb{B}^n$ be such a point, and let $N_b \subset PU(n,1)$ be its stabilizer. This is a maximal parabolic real subgroup of $PU(n,1)$; let us denote by W_b its unipotent radical. This group is an extension $1 \to U_b \to W_b \xrightarrow{\pi} A_b \to 1$, where $A_b \cong \mathbb{C}^{n-1}$, and $U_b \cong \mathbb{R}$ is the center of W_b. Let $L_b = {N_b}/{W_b}$. This reductive group can be embedded as a Levi subgroup in N_b, so that $N_b = W_b \cdot L_b$. Moreover, we have a further decomposition $L_b = U(n-1) \times \mathbb{R}$. (all this description can be obtained e.g. by specializing the discussion of [3, Section 1.3] or [2, Section 4.2] to the case of the ball).

This Lie theoretic description of N_b can be understood more easily by expressing the action of the previous groups on the horoballs tangent to b. Let $(S_b^{(N)})_{N\geq 0}$ be the family of these horoballs. Each $S_b^{(N)} \subset \mathbb{B}^n$ can be described as an open subset in a Siegel domain of the third kind, as follows:

$$S_b^{(N)} \simeq \{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} \mid \operatorname{Im} z_n > ||z'||^2 + N\}. \tag{A.1.1}$$

We have $S_b^{(0)} \cong \mathbb{B}^n$, and when $b = (0, ..., 0, 1)$, the change of coordinates between the two descriptions of the ball is given by the *Cayley transform*

$$\begin{aligned}(w_1, \dots, w_{n-1}, w_n) \in \mathbb{B}^n &\mapsto (z', z_n)\\ &= \left(\frac{w_1}{1-w_n}, \dots, \frac{w_{n-1}}{1-w_n}, i\frac{1+w_n}{1-w_n}\right) \in S^{(0)}_{(0,\dots,0,1)}.\end{aligned}$$

Now, if $g \in W_b$, we can write $g = (s, a)$ accordingly to the decomposition $W_b \overset{sets}{=} U_b \times A_b$ ($U_b \cong \mathbb{R}$, $A_b \cong \mathbb{C}^{n-1}$), and we have, for any $(w', w_n) \in S_b^{(N)}$:

$$g \cdot (z', z_n) = (z' + a, z_n + i||a||^2 + 2i\overline{a} \cdot z' + s). \tag{A.1.2}$$

We check easily that $S_b^{(N)}$ is preserved by W_b. Also, for any $g \in L_b \simeq U(n-1) \times \mathbb{R}$, we can write $g = (r, t)$, and we then have

$$g \cdot (z', z_n) = (e^t(r \cdot z'), e^{2t} z_n). \tag{A.1.3}$$

We are now ready to describe the quotients of $S_b^{(N)}$ by the action of $\Gamma' \cap N_b$. Note first that since Γ' is neat, we have $\Gamma' \cap N_b \subset W_b$. Then, by the discussion above, we obtain a decomposition as *sets* $N_b \overset{sets}{=} (\mathbb{C}^{n-1} \times \mathbb{R}) \times (U(n-1) \times \mathbb{R})$, in which the elements of $\Gamma' \cap N_b$ can be written as $(a, t, \mathrm{Id}, 0)$. It also follows from [44] that $\Gamma' \cap U_b = \mathbb{Z}\tau$ for some $\tau \in U_b \simeq \mathbb{R}$. This last fact permits to form the quotient $G_b^{(N)} = S_b^{(N)}/U_b \cap \Gamma'$; using (A.1.1), we can also express the latter quotient as an open subset of $\mathbb{C}^{n-1} \times \mathbb{C}^*$:

$$G_b^{(N)} = \{(w', w_n) \in \mathbb{C}^{n-1} \times \mathbb{C}^* \ \mid \ |w_n| e^{\frac{2\pi}{\tau}||w'||^2} < e^{-\frac{2\pi}{\tau}N}\},$$

and the quotient is then realized by the map $(z', z_n) \in S_b^{(N)} \to (z', e^{\frac{2i\pi}{\tau}z_n}) \in G_b^{(N)}$.

The group $\Lambda_b := \pi(\Gamma' \cap W_b) \subset \mathbb{C}^{n-1}$ is an abelian lattice, acting on $G_b^{(N)} \subset \mathbb{C}^{n-1} \times \mathbb{C}^*$ as

$$a \cdot (z', z_n) = (z' + a, e^{-\frac{2\pi}{\tau}||a||^2 - \frac{4\pi}{\tau}\overline{a}\cdot z'} z_n),$$

Clearly, the closure $\overline{G_b^{(N)}}$ in $\mathbb{C}^n$ is an open neighborhood of $\mathbb{C}^{n-1} \times \{0\}$. We can form the quotient

$$\Omega_b^{(N)} = \overline{G_b^{(N)}}/\Lambda_b$$

which is then isomorphic to a tubular neighborhood of the abelian variety $\mathbb{C}^{n-1}/\Lambda_b$ in some negative line bundle. Finally, the toroidal compactification X' can be obtained by glueing the open varieties $\Omega_b^{(N)}$ to U' (as b runs among a system of representatives of the rational boundary components, and N is large enough).

Our claims about X can be derived from the following lemma.

Lemma A.2 *Let $b \in \partial\mathbb{B}^n$ be a Γ'-rational boundary component, and let $g \in \Gamma$. Then the point $b' = g \cdot b$ is also Γ'-rational, and there exists $N, N' > 0$, for which g induces an isomorphism $S_b^{(N)} \xrightarrow{g} S_{b'}^{(N')}$, yielding in turn a unique compatible biholomorphism $\Omega_b^{(N)} \to \Omega_{b'}^{(N')}$.*

Proof As Γ' is torsion free, a point $z \in \partial\mathbb{B}^n$ is Γ'-rational if and only if $W_b \cap \Gamma' \neq \{e\}$ (see [44, §1.3]). Since g normalizes Γ', we have $g(W_b \cap \Gamma')g^{-1} \subset W_{b'} \cap \Gamma'$ so b' is Γ'-rational if b is.

As for our second claim, since the set of horoballs is preserved by the action of $PU(n, 1)$, we may find N, N' such that g induces a isomorphism $S_b^{(N)} \to S_{b'}^{(N)}$. Let (x', x_n) (resp. (y', y_n)) be standard coordinates on $S_b^{(N)}$ (resp. $S_b^{(N')}$) as in (A.1.1), chosen so that $(y', y_n) = (x', x_n) \circ u$ for some $u \in U(n)$ satisfying $u \cdot b' = b$. Then $ug \in N_b$, and by (A.1.2) and (A.1.3), we have $(y', y_n) \circ g = f(x', x_n)$ for some *affine* map f.

Since g normalizes Γ', we have $g(\Gamma' \cap U_b)g^{-1} = \Gamma' \cap U_{b'}$, so the map $S_b^{(N)} \xrightarrow{g} S_{b'}^{(N')}$ passes to the quotient to give a map $\widetilde{g} : G_b^{(N)} \to G_{b'}^{(N')}$. Using an explicit expression for the affine map f, we find an (*a priori* multivaluate) expression for $\widetilde{g}$ as

$$(z', z_n) \in G_b^{(N)} \overset{\widetilde{g}}{\mapsto} (A \cdot z' + u \log z_n + z_0',\; C\, z_n^a\, e^{b \cdot z'}) \in G_{b'}^{(N')}$$

for some $A \in M_{n-1}(\mathbb{C})$, some vectors $u, b, z_0' \in \mathbb{C}^{n-1}$ and $C, a \in \mathbb{C}$. The formula above induces a well-defined, invertible map $G_b^{(N)} \to G_{b'}^{(N')}$, so we have $u = 0$, $a = \pm 1$. This implies that $\widetilde{g}$ extends holomorphically to $\widetilde{g} : \overline{G_b^{(N)}} \to \overline{G_{b'}^{(N')}}$. Finally, as g normalizes Γ', $\widetilde{g}$ passes to the quotient by Λ_b and $\Lambda_{b'}$, giving a biholomorphism $\Omega_b^{(N)} \to \Omega_{b'}^{(N')}$. □

Going back to the proof of Proposition A.1, we see that Lemma A.2 permits to define a unique action of the quotient $G = \Gamma/\Gamma'$ on X', compatible with its natural action on U'. We then let $X := X'/G$. The following lemma ends the proof of Proposition A.1, and clarifies the link with the construction of [2].

Lemma A.3 *The variety X defined above does not depend on the choice of Γ'. When Γ is arithmetic, X coincides with the toroidal compactification of U constructed in* [2].

Proof Let $\Gamma', \Gamma'' \subset \Gamma$ be two neat lattices of finite index, and let us show that the varieties constructed from Γ' and Γ'' are the same. Since $\Gamma \cap \Gamma'$ also has finite index in Γ, we may assume $\Gamma'' \subset \Gamma'$. By Lemma A.2, the action of two lattices $\Gamma'' \subset \Gamma'$ are compatible with each other over each set $\Omega_b^{(N)}$, which suffices to prove the first point.

Let us prove the second point. The construction of the toroidal compactification of [2] depends on a certain choice of Γ-*admissible polyhedra* for each rational boundary component (see [2, Definition 5.1]). In the case of the ball, since $\dim_{\mathbb{R}} U_b = 1$ for any $b \in \partial \mathbb{B}^n$, there is only one such possible choice (cf. [*loc. cit.*, Theorem 4.1.(2)]). The claim now follows from the functoriality of compatible toroidal compactifications (see [28, Lemma 2.6]), since "choices" of polyhedra admissible for two lattices $\Gamma' \subset \Gamma$ are thus automatically compatible with each other. □

Note that even though this construction of X is well adapted to our purposes, it should not be used to define X as an *orbifold*, as it has the drawback of producing artificial ramification orders along the boundary components of X. As explained in [21], a better way of proceeding is to construct directly open neighborhoods of the components of $X - U$ as *stacks*, before glueing them to U.

A.2. Main results

Let us first begin with the following lemma.

Lemma A.4 *Let Y be the toroidal compactification of the ball quotient $U := \mathbb{B}^n/\Gamma$ by a torsion free lattice $\Gamma \subset PU(n, 1)$ whose parabolic isometries are all unipotent. Let X be another projective compactification of U, and assume that X has at most klt singularities.*

Then the identity map of U extends to a birational morphism $f : X \to Y$.

Proof The identity map of U defines a birational map $f : X \dashrightarrow Y$. Assume by contradiction that f is not regular. One can take a resolution of indeterminacies $\mu : \tilde{X} \to X$ for f so that $\mu|_{\mu^{-1}(U)} : \mu^{-1}(U) \xrightarrow{\sim} U$ is an isomorphism:

$$\begin{array}{ccccc} & & \tilde{X} & & \\ & \swarrow^{\mu} & & \searrow^{\tilde{f}} & \\ X & & \overset{f}{\dashrightarrow} & & Y \end{array}$$

By the rigidity result (see [17, Chapter 3, Lemma 1.15]), there is a fiber $\mu^{-1}(z)$ with $z \in D$ which is not contracted by $\tilde{f}$. Clearly, we have $\tilde{f}(\mu^{-1}(z)) \subset Y - U$.

Since X has klt singularities, the work of Hacon–McKernan [29] implies that every fiber of μ is rationally chain connected. Thus, $\widetilde{f}(\mu^{-1}(z))$ is a point since abelian varieties do not contain rational curves. This gives a contradiction. □

Remark A.5 If we make the more restrictive hypothesis that X has at most quotient singularities, we can replace the use of [29] by the work of Kollar [36], which implies that each fiber of μ is simply connected. As $Y - U$ is a disjoint union of abelian varieties, this also implies that the image of $\tilde{f} : \mu^{-1}(z) \to Y - U$ must be a point.

Let us introduce a natural class of pairs under which our rigidity theorem will hold.

Definition A.6 Let (X, D) be a pair consisting of normal algebraic variety and a reduced divisor. We say that (X, D) has *algebraic quotient singularities* if it admits a finite affine cover $(X_i)_{i \in I}$, such that each $(X_i, D \cap X_i)$ is the quotient of a smooth SNC pair (U_i, D_i) by a finite group G_i leaving D_i invariant.

Note that for any lattice $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$, if X is the toroidal compactification of $U = \mathbb{B}^n/\Gamma$ described in Section 1, then $(X, X - U)$ has algebraic quotient singularities.

We can now state our main result as follows.

Theorem A.7 *Let $U := \mathbb{B}^n/\Gamma$ be an n-dimensional ball quotient by a torsion free lattice $\Gamma \subset PU(n, 1)$. Let X be a klt compactification of U, and let $D := X - U$.*

Let $D^{(1)} \subset D$ be the divisorial part of D. If the Kähler–Einstein metric ω on the bundle $T_X(-\log D^{(1)})|_U$ is adapted to log order near the generic point of any component of $D^{(1)}$, then (X, D) identifies with the toroidal compactification of U.

Remark A.8 (1) Under the more restrictive assumption that (X, D) has algebraic quotient singularities, the use of Lemma A.4 in our proof below can be made without appealing to the difficult result of [29] (see Remark A.5).

(2) As an easy consequence of Theorem A.7, we can remark that there is no klt compactification X of U such that $X - U$ has codimension ≥ 2.

Corollary A.9 *With the same assumptions as in Theorem A.7, if X is smooth and D has simple normal crossings, then D is in fact smooth, and each component is a* smooth *quotient of an abelian variety A by some finite group acting freely on A.*

Let us prove Theorem A.7. Let $\Gamma' \subset \Gamma$ be a subgroup of finite index so that all parabolic elements of Γ' are unipotent. Writing $U' := \mathbb{B}^n/\Gamma'$, this gives a finite étale surjective morphism $U' \to U$.

Let X' be the normalization of X in the function field of U': this is a normal projective variety X' compactifying U', with a compatible finite surjective morphism $\mu : X' \to X$ (see e.g. [1, Chapter 12, §9]). Since klt singularities are preserved under finite surjective morphisms, the variety X' has at most klt singularities (see [31, Corollary 5.20]).

Remark A.10 If (X, D) has algebraic quotient singularities, one sees easily that this is also the case for X'. To see this, form the fiber product $Z' = Z \times_X X'$, where $Z \to X$ is an affine covering as in Definition A.6. By [37, Theorem 2.23], the variety Z', endowed with it natural boundary divisor, has algebraic quotient singularities. Finally, Lemma A.14 shows that $Z' \to X'$ is a quotient map, which gives the result.

Let Y' be the toroidal compactification of U', so that the boundary $A := Y' - U'$ is a smooth divisor.

Lemma A.11 *The identity map on U' extends as an isomorphism $f : X' \to Y'$. In particular, there is a finite surjective morphism $g : Y' \to X$, which identifies with the étale and surjective map $U' \to U$ over $X - D$.*

Proof Since X' is klt, Lemma A.4 shows that the identity map of U' extends to a birational morphism $f : X' \to Y'$. Assume by contradiction that f is not an isomorphism. As Y' is smooth, it follows from [31, Corollary 2.63] that the exceptional set $\mathrm{Ex}(f)$ is of pure codimension one. Thus, the birational morphism f must contract an irreducible divisorial component E of the boundary $D' := X' - U'$.

Denote by D^{sing} the singular locus of D, and let $\omega' := \mu^*\omega$, be the canonical Kähler Einstein metric on U'. Lemma A.12 below shows that ω' is adapted to log-order for $T_{X'^\circ}(-\log E^\circ)$, where $X'^\circ := \mu^{-1}(X - D^{\mathrm{sing}})$, and $E^\circ := X'^\circ \cap E$. We are going to derive a contradiction with the fact the E is contracted. Let A_1 be the component of A containing $f(E)$. We can take admissible coordinates $(\mathcal{W}; z_1, \ldots, z_n)$ and $(\mathcal{U}; w_1, \ldots, w_n)$ centered at some well-chosen $x' \in E \cap X'^\circ$ and $y := f(x') \in A_1$ respectively so that $f(\mathcal{W}) \subset \mathcal{U}$, and $f|_E : E \to f(E)$ is smooth at x'. Denote by $(f_1(z), \ldots, f_n(z))$ the expression of f within these coordinates. Then if the admissible coordinates are chosen properly, one has

$$(f_1(z), \ldots, f_n(z)) = (z_1^{m_1} g_1(z), \ldots, z_1^{m_k} g_k(z), g_{k+1}, \ldots, g_n)$$

where $g_1(z), \ldots, g_k(z)$ are holomorphic functions defined on $\mathcal{W}$ so that $g_i(z) \neq 0$ and $m_i \geq 1$ for $i = 1, \ldots, k$. Since E is exceptional, one has $k \geq 2$. By the norm estimate in [44, eq. (8) on p. 338], the Kähler–Einstein metric ω for $T_Y(-\log A)|_U$ is adapted to log order. More precisely, one has

$$|dw_2|^2_{\omega^{-1}} \sim (-\log |w_1|^2).$$

Since $f^*d\log w_2 = m_2 d\log z_1 + d\log g_2(w)$, one thus has the following norm estimate

$$|d\log z_1|^2_{\omega'^{-1}} \geq \frac{1}{m_2^2}\mu^*|d\log w_2|^2_{\omega^{-1}} - \frac{1}{m_2^2}\mu^*|\frac{dg_2}{g_2}|^2_{\omega^{-1}} \geq \frac{C(-\log|z_1|^2)}{|z_1|^{2m_2}}$$

for some constants $C > 0$. Since $d\log z_1$ is a local nowhere vanishing section for $\Omega^1_{X'}(\log D')$, we conclude that the metric ω'^{-1} for $\Omega^1_{X'^\circ}(-\log D'^\circ)$ is *not* adapted to log order, and so is ω' for $T_{X'^\circ}(-\log D'^\circ)$. This gives a contradiction, and ends the proof of the lemma. □

Lemma A.12 *With the notations of the proof of Lemma A.11, the metric ω' is adapted to log-order for $T_{X'^\circ}(-\log E^\circ)$.*

Proof Write $\mathcal{W} := \mu^{-1}(\mathcal{V})$. Since $\mu|_{\mathcal{W}-D'} : \mathcal{W} - D' \to \mathcal{V} - D$ is a finite unramified cover, the image of $(\mu|_{\mathcal{W}-D'})_*\big(\pi_1(\mathcal{W} - D')\big)$ is a subgroup of $\pi_1(\mathcal{W} - D) \simeq \mathbb{Z}$ of index m. Letting $\nu(z_1, \cdots, z_n) = (z_1^m, z_2, \ldots, z_n)$, one has thus the following commutative diagram

$$\begin{array}{ccc} \Delta^* \times \Delta^{n-1} & \xrightarrow{h^\circ} & \mathcal{W} \\ \downarrow{\scriptstyle \nu|_{\Delta^*\times\Delta^{n-1}}} & & \downarrow{\scriptstyle \mu|_{\mathcal{W}}} \\ \Delta^n & \xrightarrow{\simeq} & \mathcal{V} \end{array}$$

so that $h^\circ_{\Delta^*\times\Delta^{n-1}} : \Delta^* \times \Delta^{n-1} \to \mathcal{W} \cap U'$ is an isomorphism. By the Riemann removable singularities theorem, h extends to a holomorphic map $h : \Delta^n \to \mathcal{W}$, which is easily seen to be surjective with finite fibers. Hence h is moreover biholomorphic. $(\mathcal{W}; z_1, \ldots, z_n; h)$ is therefore an admissible coordinate centered at x' with $(z_1 = 0) = \mathcal{W} \cap D'$ so we can now identify μ with ν. Hence,

$$\mu^* d\log x_1 = md\log z_1, \mu^* dx_2 = dz_2, \ldots, \mu^* dx_n = dz_n,$$

and the frame $(d\log z_1, dz_2, \ldots, dz_n)$ for $\Omega^1_{X'}(\log D')|_{\mathcal{W}}$ is adapted to log order. This shows that the metric ω' is adapted to log order for $T_{X'^\circ}(-\log D'^\circ)$. □

We can now conclude the case discussed in Corollary A.9, where (X, D) is assumed to be a smooth log-pair. Since the boundary of $Y' - U'$ is smooth, this implies that D must also be smooth. Moreover, for each connected component A_i of A, there is a connected component D_j of D so that $g|_{A_i} : A_i \to D_j$ is a finite surjective morphism, which is also étale by the local description of μ given in the proof of Lemma A.12. Hence in this case, D_i is a smooth quotient of an abelian variety by the free action of some finite group G_i. This suffices to establish Corollary A.9.

The proof of Theorem A.7 will be complete with the following lemma.

Lemma A.13 *The variety X identifies with the quotient of Y' by the natural action of $G = \Gamma/\Gamma'$. In particular, $X \cong Y$.*

Proof This result comes right away from Lemma A.14 below, taking $M = Y'$, $N = X$, and $G = \mathcal{G}$. Remark that we have $R(Y')^G = R(U')^G = R(U) = R(X)$ since $U = U'/G$. For the second statement, remark that by Proposition A.1, the toroidal compactification Y of U also identifies with the quotient Y'/G. Thus, there is an isomorphism $Y \cong X$ compatible with the identity on U. Theorem A.7 is proved. □

Lemma A.14 *Let $f : M \to N$ be a finite surjective morphism between two normal reduced schemes. Assume that M is acted upon by a finite groupoid $\mathcal{G}$, and that f is $\mathcal{G}$-invariant. Suppose in addition that $R(M)^{\mathcal{G}} = R(N)$, where $R(M)$, $R(N)$ are the rings of rational functions on M, N. Then N is the quotient of M by $\mathcal{G}$.*

Proof It suffices to show that $f_*(\mathcal{O}_M)^{\mathcal{G}} = \mathcal{O}_N$. This is a local statement on the base, so we may assume that $N = \operatorname{Spec} A$, $M = \operatorname{Spec} B$, with A is integrally closed. We have $B^{\mathcal{G}} \subset R(B)^{\mathcal{G}} = R(A)$ by assumption. Since $A \subset B$ is finite, and A is integrally closed, this implies $B^{\mathcal{G}} \subset A$, as required. □

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