LINEAR SHAFAREVICH CONJECTURE IN POSITIVE CHARACTERISTIC, HYPERBOLICITY AND APPLICATIONS

by

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Abstract. — Given a complex quasi-projective normal variety X and a linear representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ with K any field of positive characteristic, we mainly establish the following results:

(a) the construction of the Shafarevich morphism $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$ associated with ϱ.

(b) In cases where X is projective, ϱ is faithful and the Γ-dimension of X is at most two (e.g. $\dim X = 2$), we prove that the Shafarevich conjecture holds for X: the universal covering of X is holomorphically convex.

(c) In cases where ϱ is big, we prove that the Green-Griffiths-Lang conjecture holds for X: X is of log general type if and only it is pseudo Picard or Brody hyperbolic.

(d) When ϱ is big and the Zariski closure of $\varrho(\pi_1(X))$ is a semisimple algebraic group, we prove that X is pseudo Picard hyperbolic, and strongly of log general type.

(e) If X is special or h-special, then $\varrho(\pi_1(X))$ is virtually abelian.

We also prove Claudon-Höring-Kollár's conjecture for complex projective manifolds with linear fundamental groups of any characteristic.

Contents

0. Main results

0.1. Existence of the Shafarevich morphism. — The Shafarevich conjecture stipulates that the universal covering of a complex projective variety is holomorphically convex. If this conjecture holds true, it implies the existence of the *Shafarevich morphism*. Over the past three decades, this conjecture has been extensively studied when considering cases where fundamental groups are subgroups of *complex* general linear groups, referred to as the *linear Shafarevich conjecture*. Drawing upon the robust techniques of non-abelian Hodge theories established by Simpson [Sim88, Sim92] and Gromov-Schoen [GS92], linear Shafarevich conjecture has been studied in [Kat97, KR98, Eys04, EKPR12, CCE15, DYK23], to quote only a few. It is natural to ask whether the conjecture holds when the fundamental groups of algebraic varieties are subgroups of general

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linear groups in positive characteristic. In this paper we address this question along with the exploration of hyperbolicity and algebro-geometric properties of these algebraic variety. The first result of this paper is the construction of the Shafarevich morphism.

***Theorem A* (=Theorem 2.9). —** *Let X be a quasi-projective normal variety and $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a linear representation, where K is a field of positive characteristic. Then there exists a dominant (algebraic) morphism $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$ over a quasi-projective normal variety $\mathrm{Sh}_\varrho(X)$ with connected general fibers such that for any connected Zariski closed subset $Z \subset X$, the following properties are equivalent:*

(a) *$\mathrm{sh}_\varrho(Z)$ is a point;*
(b) *$\varrho(\mathrm{Im}[\pi_1(Z) \to \pi_1(X)])$ is finite;*
(c) *for each irreducible component Z_o of Z, $\varrho^{ss}(\mathrm{Im}[\pi_1(Z_o^{\mathrm{norm}}) \to \pi_1(X)])$ is finite, where $\varrho^{ss} : \pi_1(X) \to \mathrm{GL}_N(\bar{K})$ is the semisimplification of ϱ and Z_o^{norm} denotes the normalization of Z_o.*

The above morphism $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$ will be called the *Shafarevich morphism* associated with ϱ. We remark that in our previous work [DYK23], Theorem A was proved when $\mathrm{char}\, K = 0$ and ϱ is semisimple, with a weaker statement that $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$ is algebraic in the function field level.

We also prove the following theorem on the Shafarevich conjecture.

***Theorem B* (=Theorem 7.15). —** *Let X be a projective normal variety and let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a faithful representation where K is a field of positive characteristic. If the Γ-dimension (see Definition 7.14) of X is at most two (e.g. when $\dim X \leq 2$), then the universal covering $\widetilde{X}$ of X is holomorphically convex.*

0.2. On the Green-Griffiths-Lang conjecture. — Building on the methods utilized in establishing Theorem A, together with the techniques developed in [CDY22], we prove the following theorem on the generalized Green-Griffiths-Lang conjecture. A stronger and more refined result will be stated in Theorem D.

***Theorem C* (=Theorem 3.1 $\subsetneq$ Theorem D). —** *Let X be a complex quasi-projective normal variety. Let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a big representation where K is a field of positive characteristic. Then the following properties are equivalent:*

(i) *X is of log general type;*
(ii) *X is strongly of log general type;*
(iii) *X is pseudo Picard hyperbolic, that is, there exists a proper Zariski closed subset $\Xi \subsetneq X$ such that any holomorphic map $f : \mathbb{D}^* \to X$ from the punctured disk with essential singularity at the origin has image $f(\mathbb{D}^*) \subset \Xi$.*
(iv) *X is pseudo Brody hyperbolic, that is, there exists a proper Zariski closed subset $\Xi \subsetneq X$ such that any non-constant holomorphic map $f : \mathbb{C} \to X$ has image $f(\mathbb{C}) \subset \Xi$.*

We say a quasi-projective variety X is *strongly of log general type* if there exists a proper Zariski closed subset $\Xi \subsetneq X$ such that any positive dimensional closed subvariety $V \subset X$ is of log general type provided that $V \not\subset \Xi$. Recall that a representation $\varrho : \pi_1(X) \to G(K)$ is said to be *big*, (or *generically large* in [Kol95]), if for any closed irreducible subvariety $Z \subset X$ containing a *very general* point of X, $\varrho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)])$ is infinite. It is worth noting that a stronger notion of largeness exists, where ϱ is called *large* if $\varrho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)])$ is infinite for any closed subvariety Z of X. We remark that in [CDY22] we prove Theorem C when $\mathrm{char}\, K = 0$ and ϱ is semisimple. It is worthwhile to mention that in the case $\mathrm{char}\, K > 0$, ϱ is not required to be semisimple.

We would like to refine Theorem C to compare the *non-hyperbolicity locus* of the hyperbolicity notions in Theorem C. We first introduce a notion of special loci $\mathrm{Sp}(\varrho)$ for any big representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ which measures the "non-large locus" of ϱ.

Definition 0.1. — Let X be a smooth quasi-projective variety. Let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a representation where K is a field. We define

$$\mathrm{Sp}(\varrho) := \overline{\bigcup_{\iota : Z \hookrightarrow X} Z}^{\mathrm{Zar}},$$

where $\iota : Z \hookrightarrow X$ ranges over all positive dimensional closed subvarieties of X such that $\iota^*\varrho(\pi_1(Z))$ is finite.

Additionally, as in [CDY22, Definition 0.1], we can introduce four *special subsets* $\mathrm{Sp}_{\mathrm{sab}}$, $\mathrm{Sp}_{\mathrm{alg}}(X)$, $\mathrm{Sp}_{\mathrm{h}}(X)$ and $\mathrm{Sp}_{\mathrm{p}}(X)$ of X that measure the *non-hyperbolicity locus* of the hyperbolicity notions in Theorem C from different perspectives.

Definition 0.2. — Let X be a quasi-projective normal variety. We define

(i) $\mathrm{Sp}_{\mathrm{sab}}(X) := \overline{\bigcup_f f(A_0)}^{\mathrm{Zar}}$, where f ranges over all non-constant rational maps $f : A \dashrightarrow X$ from all semi-abelian varieties A to X such that f is regular on a Zariski open subset $A_0 \subset A$ whose complement $A \backslash A_0$ has codimension at least two;
(ii) $\mathrm{Sp}_{\mathrm{h}}(X) := \overline{\bigcup_f f(\mathbb{C})}^{\mathrm{Zar}}$, where f ranges over all non-constant holomorphic maps from $\mathbb{C}$ to X;
(iii) $\mathrm{Sp}_{\mathrm{alg}}(X) := \overline{\bigcup_V V}^{\mathrm{Zar}}$, where V ranges over all positive-dimensional closed subvarieties of X which are not of log general type;
(iv) $\mathrm{Sp}_{\mathrm{p}}(X) := \overline{\bigcup_f f(\mathbb{D}^*)}^{\mathrm{Zar}}$, where f ranges over all holomorphic maps from the punctured disk $\mathbb{D}^*$ to X with essential singularity at the origin, i.e., f has no holomorphic extension $\bar{f} : \mathbb{D} \to \overline{X}$ to a projective compactification $\overline{X}$.

Subsequently, we establish a theorem concerning these special subsets, thereby refining Theorem C.

***Theorem D* (=Lemma 3.4 and Theorem 3.6).** — *Let X be a quasi-projective normal variety. Let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a big representation where K is a field of positive characteristic. Then $\mathrm{Sp}(\varrho)$ is a proper Zariski closed subset of X, and we have*

$$\mathrm{Sp}_{\mathrm{sab}}(X) \setminus \mathrm{Sp}(\varrho) = \mathrm{Sp}_{\mathrm{alg}}(X) \setminus \mathrm{Sp}(\varrho) = \mathrm{Sp}_{\mathrm{p}}(X) \setminus \mathrm{Sp}(\varrho) = \mathrm{Sp}_{\mathrm{h}}(X) \setminus \mathrm{Sp}(\varrho).$$

We have $\mathrm{Sp}_\bullet(X) \subsetneq X$ if and only if X is of log general type, where $\mathrm{Sp}_\bullet$ denotes any of $\mathrm{Sp}_{\mathrm{sab}}$, $\mathrm{Sp}_{\mathrm{alg}}$, Sp_{h} or Sp_{p}.

0.3. How fundamental groups determine hyperbolicity. — It is natural to explore how the fundamental groups of algebraic varieties determine their hyperbolicity properties. In our previous work [CDY22], we provided a characterization based on representations of fundamental groups into complex general linear groups. In this paper, we establish analogous results concerning representations in positive characteristic fields.

***Theorem E* (=Theorem 3.7).** — *Let X be a quasi-projective normal variety. Let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a big representation where K is an algebraically closed field of positive characteristic. If the Zariski closure $\varrho(\pi_1(X))$ is a semisimple algebraic group over K, then $\mathrm{Sp}_\bullet(X) \subsetneq X$, where $\mathrm{Sp}_\bullet$ denotes any of $\mathrm{Sp}_{\mathrm{sab}}$, $\mathrm{Sp}_{\mathrm{alg}}$, Sp_{h} or Sp_{p}.*

It is worthwhile mentioning that when $\mathrm{char}\, K = 0$, Theorem E was proved in [CDY22, Theorem A]. We also remark that the condition in Theorem 3.7 is sharp (cf. Remark 3.9).

0.4. Some applications. — Theorem E has various applications. We begin by addressing the conjecture proposed by Claudon, Höring, and Kollár concerning algebraic varieties with compactifiable universal coverings (cf. Conjecture 4.1).

***Theorem F* (=Theorem 4.7).** — *Let X be a smooth projective variety with an infinite fundamental group $\pi_1(X)$, such that its universal covering $\widetilde{X}$ is a Zariski open subset of some compact Kähler manifold. If there exists a faithful representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$, where K is any field of any characteristic, then the Albanese map of X is (up to finite étale cover) locally isotrivial with simply connected fiber F. In particular we have $\widetilde{X} \simeq F \times \mathbb{C}^{q(X)}$ with $q(X)$ the irregularity of X.*

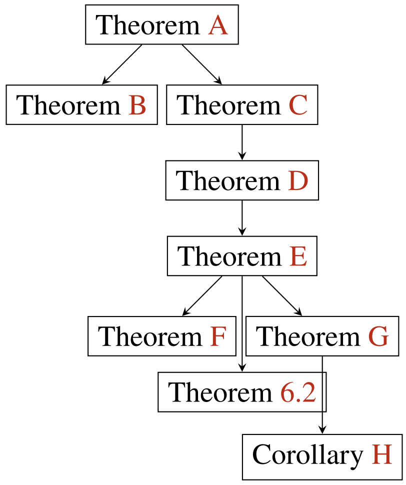

FIGURE 1. Relationships between Main Theorems

Campana's abelianity conjecture [Cam04] predicts that a smooth projective variety X that is *special* has a virtually abelian fundamental group. Another application of Theorem E is the proof of this conjecture in the context of representations in positive characteristic.

***Theorem G* (=Theorem 5.11). —** *Let X be a smooth quasi-projective variety, and let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be any representation where K is a field of positive characteristic. If X is special or h-special (cf. Definitions 5.1 and 5.2), then $\varrho(\pi_1(X))$ is virtually abelian.*

Note that in cases when X is projective and $\operatorname{char} K = 0$, Theorem G was proved by Campana [Cam04] (for X special) and the second author [Yam10] (for X Brody special). It is important to mention that Theorem G does not hold when $\operatorname{char} K = 0$ as in [CDY22, Example 11.26] we constructed a special and Brody special smooth quasi-projective variety with nilpotent fundamental group but not virtually abelian. In cases where $\operatorname{char} K = 0$, in [CDY22] we prove that $\varrho(\pi_1(X))$ is virtually nilpotent (cf. Theorem 5.7).

As a consequence of Theorem G, we provide a characterization of semiabelian variety.

***Corollary H* (=Proposition 5.12). —** *Let X be a smooth quasi-projective variety, and let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a big representation where K is a field of positive characteristic.*

(i) *If X is special or h-special, then after replacing X by some finite étale cover, its Albanese map $\alpha : X \to A$ is birational and $\alpha_* : \pi_1(X) \to \pi_1(A)$ is an isomorphism.*
(ii) *If the logarithmic Kodaira dimension $\bar{\kappa}(X) = 0$, then after replacing X by some finite étale cover, its Albanese map $\alpha : X \to A$ is birational and proper in codimension one, i.e. there exists a Zariski closed subset $Z \subset A$ of codimension at least two such that α is proper over $A \backslash Z$.*

It is worth mentioning that in [CDY22] we proved Corollary H in cases where $\operatorname{char} K = 0$ and ϱ is big and reductive.

Lastly, we apply Theorem E to obtain a structure theorem for quasi-projective varieties X for which there exists a big representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ where K is a field of positive characteristic. See Theorem 6.2.

0.5. Structure of the paper. — The paper presents several results from different perspectives. For the readers' convenience, we list in Figure 1 the relationships between main theorems.

We remark that Theorems A to E are entirely novel results, even in cases where X is a projective variety. Their proofs differ from those used in studying complex reductive representations of fundamental groups. In a forthcoming work, we will extend Theorem B to arbitrary projective normal varieties.

Convention and notation. — In this paper, we use the following conventions and notations:

- Quasi-projective varieties and their closed subvarieties are assumed to be positive-dimensional and irreducible unless specifically mentioned otherwise. Zariski closed subsets, however, may be reducible.
- Fundamental groups are always referred to as topological fundamental groups.
- If X is a complex space, its normalization is denoted by X^{norm}.
- $\mathbb{D}$ denotes the unit disk in $\mathbb{C}$, and $\mathbb{D}^*$ denotes the punctured unit disk.
- For an algebraic group G, we denote by $\mathcal{D}G$ its derived group.
- For any prime number p, we denote by $\mathrm{GL}(N, \mathbb{F}_p)$ the general linear group over $\mathbb{F}_p$. If K is a field with $\operatorname{char} K = p$, we denote by $\mathrm{GL}_N(K)$ its K-points.
- For a finitely generated group Γ, any field K and any representation $\varrho : \Gamma \to \mathrm{GL}_N(K)$, we denote by $\varrho^{ss} : \Gamma \to \mathrm{GL}_N(\bar{K})$ the semisimplification of ϱ, where $\bar{K}$ denotes the algebraic closure of K. ϱ is *reductive* if the Zariski closure of $\varrho(\pi_1(X))$ is a reductive group. We note that ϱ^{ss} is a semisimple representation, thus reductive (cf. [Mil17, Corollary 19.18]).

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1. Technical preliminary

1.1. Katzarkov-Eyssidieux reduction. —

***Definition 1.1* (Katzarkov-Eyssidieux reduction). —** Let X be a complex smooth quasi-projective variety, and let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a linear representation where K is a non-archimedean local field. A morphism $s_\varrho : X \to S_\varrho$ to a complex normal quasi-projective variety S_ϱ is called Katzarkov-Eyssidieux reduction map if

- s_ϱ is dominant and has connected general fibers, and
- for every connected Zariski closed subset T of X, the image $s_\varrho(T)$ is a point if and only if the image $\varrho(\mathrm{Im}[\pi_1(T) \to \pi_1(X)])$ is a bounded subgroup of $\mathrm{GL}_N(K)$.

When X is projective, we may easily see that $s_\varrho : X \to S_\varrho$ is unique up to isomorphism, if it exists. In our previous work [CDY22] jointly with Cadorel, we establish the existence of Katzarkov-Eyssidieux reduction map for reductive representations. This generalized previous work by Katzarkov [Kat97] and Eyssidieux [Eys04] from projective varieties to the quasi-projective cases. Here we state a stronger result, which is implicitly contained in our paper [DYK23].

***Theorem 1.2*. —** *Let X be a complex smooth quasi-projective variety, and let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a* linear *representation where K is a non-archimedean local field. Then there exists a quasi-projective normal variety S_ϱ and a dominant morphism $s_\varrho : X \to S_\varrho$ with connected general fibers, such that for any connected Zariski closed subset T of X, the following properties are equivalent:*

(a) *the image $\varrho(\mathrm{Im}[\pi_1(T) \to \pi_1(X)])$ is a bounded subgroup of $G(K)$.*
(b) *For every irreducible component T_o of T, the image $\varrho(\mathrm{Im}[\pi_1(T_o^{\mathrm{norm}}) \to \pi_1(X)])$ is a bounded subgroup of $G(K)$.*
(c) *The image $s_\varrho(T)$ is a point.* □

Proof. — Let $\varrho^{ss} : \pi_1(X) \to \mathrm{GL}_N(\bar{K})$ be the semisimplification of ϱ. We assume that L/K is a finite extension such that $\varrho^{ss} : \pi_1(X) \to \mathrm{GL}_N(L)$. It is proven in [CDY22, Theorem H] that the Katzarkov-Eyssidieux reduction map $s_{\varrho^{ss}} : X \to S_{\varrho^{ss}}$ for ϱ^{ss} exists and satisfies the properties in the theorem.

On the other hand, we have the following result.

Claim 1.3 **([DYK23, Lemma 3.7]). —** *Let K be a non-archimedean local field and Γ be a finitely generated group. If $\{\varrho_i : \Gamma \to \mathrm{GL}_N(\bar{K})\}_{i=1,2}$ are two linear representations such that there semisimplifications are conjugate, then ϱ_1 is bounded if and only if ϱ_2 is bounded.* □

Therefore, if we define s_ϱ to be $s_{\varrho^{ss}} : X \to S_{\varrho^{ss}}$, it satisfies the properties required in the theorem. Indeed, let $T \subset X$ be a connected Zariski closed subset. Set $\Gamma = \pi_1(T)$. Then Γ is finitely generated. Let $\iota : \Gamma \to \pi_1(X)$ be a natural morphism. Note that the semisimplifications of two composite representations $\varrho \circ \iota : \Gamma \to \mathrm{GL}_N(\bar{K})$ and $\varrho^{ss} \circ \iota : \Gamma \to \mathrm{GL}_N(\bar{K})$ are conjugate. Hence by Claim 1.3, $\varrho \circ \iota$ is bounded iff $\varrho^{ss} \circ \iota$ is bounded. Hence the image $s_{\varrho^{ss}}(T)$ is a point iff $\varrho(\mathrm{Im}[\pi_1(T) \to \pi_1(X)])$ is a bounded subgroup of $G(K)$. Similarly $s_{\varrho^{ss}}(T)$ is a point iff $\varrho(\mathrm{Im}[\pi_1(T_o^{\mathrm{norm}}) \to \pi_1(X)])$ is a bounded subgroup of $G(K)$ for every irreducible component T_o of T. Thus $s_{\varrho^{ss}} : X \to S_{\varrho^{ss}}$ satisfies the properties required in the theorem. □

According to this theorem, the two properties (a) and (b) are equivalent for every connected Zariski closed subset T of X. Hence for every Katzarkov-Eyssidieux reduction map $s_\varrho : X \to S_\varrho$, the three statements (a), (b) and (c) are equivalent.

***Remark 1.4.* —** The proof of Theorem 1.2 shows that for every linear representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$, where K is a non-archimedean local field, the Katzarkov-Eyssidieux reduction map $s_{\varrho^{ss}} : X \to S_{\varrho^{ss}}$ for the semisimplification ϱ^{ss} is the Katzarkov-Eyssidieux reduction map for ϱ.

The following lemmas proved in [DYK23] will be used throughout this paper.

Lemma 1.5 **([DYK23, Lemma 1.28]). —** *Let V be a quasi-projective normal variety and let $(f_\lambda : V \to S_\lambda)_{\lambda\in\Lambda}$ be a family of morphisms into quasi-projective varieties S_λ. Then there exist a quasi-projective normal variety S_∞ and a morphism $f_\infty : V \to S_\infty$ such that*

- *f_∞ is dominant and has connected general fibers,*
- *for every subvariety $Z \subset V$, $f_\infty(Z)$ is a point if and only if $f_\lambda(Z)$ is a point for every $\lambda \in \Lambda$, and*
- *there exist $\lambda_1, \ldots, \lambda_n \in \Lambda$ such that $f_\infty : V \to S_\infty$ is the quasi-Stein factorization of $(f_1, \ldots, f_n) : V \to S_{\lambda_1} \times \cdots S_{\lambda_n}$.* □

Such $f_\infty : V \to S_\infty$ is called the *simultaneous Stein factorization* of $(f_\lambda : V \to S_\lambda)_{\lambda\in\Lambda}$.

***Lemma 1.6.* —** *Let p be a prime number. Let Λ be a non-empty set of reductive representations $\tau : \pi_1(X) \to \mathrm{GL}_{N_\tau}(K_\tau)$, where K_τ are local fields of* char $K_\tau = p$. *Then there exist*

- *a local field K of* char $K = p$,
- *a positive integer $N > 0$,*
- *a reductive representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$*

such that the simultaneous Stein factorization of Katzarkov-Eyssidieux reduction maps $(s_\tau : X \to S_\tau)_{\tau\in\Lambda}$ coincides with Katzarkov-Eyssidieux reduction map of ϱ. Moreover we have $\cap_{\tau\in\Lambda}\ker(\tau^{ss}) \subset \ker(\varrho)$.

Proof. — Let $\sigma : X \to \Sigma$ be the simultaneous Stein factorization of $(s_\tau : X \to S_\tau)_{\tau\in\Lambda}$. Then Σ is normal, and $\sigma : X \to \Sigma$ is dominant and has connected general fibers. By Lemma 1.5, there exist $\tau_1, \ldots, \tau_n \in \Lambda$ such that $\sigma : X \to \Sigma$ is the quasi-Stein factorization of $(s_{\tau_1}, \ldots, s_{\tau_n}) : X \to S_{\tau_1} \times \cdots S_{\tau_n}$. We may take a local field K of char $K = p$ such that $K_{\tau_i} \subset K$ and $\tau_i^{ss} : \pi_1(X) \to \mathrm{GL}_N(K)$ for all $i = 1, \ldots, n$. Set $N = N_{\tau_1} + \cdots + N_{\tau_n}$. We define $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ by

$$(\tau_1^{ss}, \ldots, \tau_n^{ss}) : \pi_1(X) \to \mathrm{GL}_{N_{\tau_1}}(K) \times \cdots \times \mathrm{GL}_{N_{\tau_n}}(K) \subset \mathrm{GL_N(K_0)}.$$

Then ϱ is semisimple. For every connected Zariski closed subset T of X, $\sigma(T)$ is a point iff $s_{\tau_i}(T)$ is a point for every $i = 1, \ldots, n$. This happens iff $\tau_i^{ss}(\mathrm{Im}[\pi_1(T) \to \pi_1(X)])$ is a bounded subgroup of $\mathrm{GL}_{N_{\tau_i}}(K)$ for every $i = 1, \ldots, n$ (cf. Remark 1.4). The later is equivalent to the boundedness of $\varrho(\mathrm{Im}[\pi_1(T) \to \pi_1(X)])$. Hence the map $\sigma : X \to \Sigma$ is a Katzarkov-Eyssidieux reduction map for ϱ. Note that $\ker(\varrho) = \ker(\tau_1^{ss}) \cap \cdots \cap \ker(\tau_n^{ss})$. Hence $\cap_{\tau\in\Lambda}\ker(\tau^{ss}) \subset \ker(\varrho)$. □

1.2. Some facts on algebraic group. — Note that an algebraic group G over a field of characteristic zero is reductive if and only if G is *linearly reductive*, i.e, for every finite dimensional representation of G is semisimple (cf. [Mil17, Corollary 22.43]). However, this fact fails for algebraic group defined over positive characteristic field. We recall the following example in [Mil17, Example 12.55].

Example 1.7. **—** Let $k = \overline{\mathbb{F}_2}$ and let V be the standard 2-dimensional representation of $\mathrm{SL}_2(k)$. Then $\mathrm{Sym}^2 V$ is not semisimple as a representation. Indeed, let $e_1 := (1,0)$ and $e_2 := (0,1)$. Within the basis $\{e_1e_2, e_1^2, e_2^2\}$ of $\mathrm{Sym}^2 k^2$, we can express $\mathrm{Sym}^2 V$ in the matrix form as

$$\mathrm{SL}_2(k) \to \mathrm{GL}_3(k)$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ ab & a^2 & b^2 \\ cd & c^2 & d^2 \end{pmatrix}$$

It is easy to see that it is not a semisimple representation since the SL_2-invariant subspace spanned by $\{e_1^2, e_2^2\}$ have no SL_2-equivariant complement because ab and cd are not linear polynomials in a^2, b^2, c^2, d^2.

It is worthwhile mentioning that an algebraic group G over a field of characteristic $p \neq 0$ is linearly reductive if and only if its identity component G° is a torus and p does not divide the index $(G : G^\circ)$ (cf. [Mil17, Remark 12.56]). Therefore, in this paper, we must distinguish between *semisimple* and *reductive* representations as we are working over linear algebraic groups over fields of positive characteristic.

2. Shafarevich morphism in positive characteristic

In this section we will prove Theorem A.

2.1. A lemma on finite group. —

Lemma 2.1. **—** *Let K be an algebraically closed field of positive characteristic and let Γ be a finitely generated group. Let $\varrho : \Gamma \to \mathrm{GL}_N(K)$ be a representation such that its semisimplification has finite image. Then $\varrho(\Gamma)$ is finite.*

Proof. — Since the semisimplification ϱ^{ss} of ϱ has finite image, we can replace Γ by a finite index subgroup such that $\varrho^{ss}(\Gamma)$ is trivial. Therefore, some conjugation σ of ϱ has image in the subgroup $\mathrm{U}_N(K)$ consisting of all upper-triangular matrices in $\mathrm{GL}_N(K)$ with 1's on the main diagonal.

Note $\mathrm{U}_N(K)$ admits a central normal series whose successive quotients are isomorphic to $\mathbb{G}_{a,K}$. We remark that a finitely generated subgroup of $\mathbb{G}_{a,K}$ is a finite group, for K is positive characteristic. By [ST00, Proposition 4.17], any finite index subgroup of a finitely generated group is also finitely generated. Consequently, $\sigma(\Gamma)$ admits a central normal series whose successive quotients are finitely generated subgroups of $\mathbb{G}_{a,K}$, which are all finite groups. It follows that $\sigma(\Gamma)$ is finite. The lemma is proved. □

2.2. Consideration of character varieties. — We first briefly explain the character varieties for finitely generated groups in positive characteristic and refer the readers to [LM85, Ses77] for more details. Let Γ be a finitely generated group. The variety of N-dimensional linear representations of Γ in characteristic zero is represented by an affine $\mathbb{Z}$-scheme $R(\Gamma, N)$ of finite type. Namely, given a commutative ring A, the set of A-points of $R(\Gamma, N)$ is:

$$R(\Gamma, N)(A) = \mathrm{Hom}(\Gamma, \mathrm{GL}_N(A)).$$

Let p be a prime number. Consider $R(\Gamma, N)_{\mathbb{F}_p} := R(\Gamma, N) \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{F}_p$ and note that the general linear group over $\mathbb{F}_p$, denoted by $\mathrm{GL}(N, \mathbb{F}_p)$, acts on $R(\Gamma, N)_{\mathbb{F}_p}$ by conjugation. Using Seshadri's extension of geometric invariant theory quotients for schemes of arbitrary field [Ses77, Theorem 3], we can take the GIT quotient of $R(\Gamma, N)_{\mathbb{F}_p}$ by $\mathrm{GL}(N, \mathbb{F}_p)$, denoted by $M_{\mathrm{B}}(\Gamma, N)_{\mathbb{F}_p}$. Then $M_{\mathrm{B}}(\Gamma, N)_{\mathbb{F}_p}$ is also an affine $\mathbb{F}_p$-scheme of finite type. For any algebraically closed field K of characteristic p, the K-points $M_{\mathrm{B}}(\Gamma, N)_{\mathbb{F}_p}(K)$ is identified with the conjugacy classes of semisimple representations $\Gamma \to \mathrm{GL}_N(K)$. Namely we have the following.

Proposition 2.2. — *Let K be an algebraically closed field of characteristic p. Then:*

(i) *Given a linear representation $\varrho : \Gamma \to \mathrm{GL}_N(K)$, we have $[\varrho^{ss}] = [\varrho]$.*
(ii) *For each point $x \in M_{\mathrm{B}}(\Gamma, N)_{\mathbb{F}_p}(K)$, there exists a semisimple representation $\varrho : \Gamma \to \mathrm{GL}_N(K)$ such that $[\varrho] = x$.*
(iii) *Let $\varrho : \Gamma \to \mathrm{GL}_N(K)$ and $\varrho' : \Gamma \to \mathrm{GL}_N(K)$ be two semisimple representations such that $[\varrho] = [\varrho']$. Then ϱ and ϱ' are conjugate.*

Proof. — These are well-known when the field is zero characteristic (cf. [LM85, Thm 1.28]). The proofs also work for positive characteristic case as well, as we briefly present below.

Let $\varrho \in R(\Gamma, N)_{\mathbb{F}_p}(K)$ be a linear representation. We denote by $O(\varrho)$ the orbit of ϱ by the conjugacy action, which is a constructible subset of $R(\Gamma, N) \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\, K$. Let $\overline{O(\varrho)}$ be the closure of $O(\varrho)$. Then we have ϱ^{ss} is a closed point of $\overline{O(\varrho)}$, whose proof is the same as [LM85, Lem 1.26]. For the GIT quotient $\pi : R(\Gamma, N)_{\mathbb{F}_p} \to M_{\mathrm{B}}(\Gamma, N)_{\mathbb{F}_p}$, we know that $\overline{O(\varrho)} \subset \pi^{-1}([\varrho])$. Therefore, we have $[\varrho] = [\varrho^{ss}]$.

Next, let $x \in M_{\mathrm{B}}(\Gamma, N)_{\mathbb{F}_p}(K)$. Since the GIT quotient is surjective, so is $\pi_K : R(\Gamma, N)_{\mathbb{F}_p}(K) \to M_{\mathrm{B}}(\Gamma, N)_{\mathbb{F}_p}(K)$. Hence we may take $\varrho_0 \in R(\Gamma, N)_{\mathbb{F}_p}(K)$ such that $[\varrho_0] = x$. Set $\varrho = (\varrho_0)^{ss}$. Then ϱ is semisimple and $[\varrho] = [\varrho_0] = x$.

Finally, for any matrix $A \in \mathrm{GL}_N(K)$, we denote by $\chi(A) = T^N + \sigma_1(A)T^{N-1} + \cdots + \sigma_N(A)$ its characteristic polynomial. Given $\gamma \in \Gamma$, we define a map $f_\gamma : R(\Gamma, N)_{\mathbb{F}_p}(K) \to \mathbb{A}^N(K)$ by $f_\gamma(\tau) = (\sigma_1(\tau(\gamma)), \ldots, \sigma_N(\tau(\gamma)))$, where $\tau \in R(\Gamma, N)_{\mathbb{F}_p}(K)$. Note that f_γ is induced by a morphism of $\mathbb{F}_p$-schemes, which we still denote by $f_\gamma : R(\Gamma, N)_{\mathbb{F}_p} \to \mathbb{A}^N_{\mathbb{F}_p}$. Note that f_γ is $\mathrm{GL}(N, \mathbb{F}_p)$ invariant. Hence f_γ factors the GIT quotient $\pi : R(\Gamma, N)_{\mathbb{F}_p} \to M_{\mathrm{B}}(\Gamma, N)_{\mathbb{F}_p}$. Now let $\varrho, \varrho' \in R(\Gamma, N)_{\mathbb{F}_p}(K)$ be two semisimple representations such that $[\varrho] = [\varrho']$. Then we have $f_\gamma(\varrho) = f_\gamma(\varrho')$ for all $\gamma \in \Gamma$. In other words, $\varrho(\gamma)$ and $\varrho'(\gamma)$ have the same characteristic polynomials for all $\gamma \in \Gamma$. Hence by the Brauer–Nesbitt theorem, the two semisimple representations ϱ and ϱ' are conjugate. □

Lemma 2.3. — *Let $M \subset M_{\mathrm{B}}(X, N)_{\mathbb{F}_p}$ be a constructible set. Assume that, for every reductive representation $\tau : \Gamma \to \mathrm{GL}_N(K)$ with K a local field of characteristic p such that $[\tau] \in M(K)$, the image $\tau(\Gamma) \subset \mathrm{GL}_N(K)$ is bounded. Then M is zero dimensional.*

Proof. — Let $\pi : R(\Gamma, N)_{\mathbb{F}_p} \to M_{\mathrm{B}}(\Gamma, N)_{\mathbb{F}_p}$ be the GIT quotient, which is a surjective $\mathbb{F}_p$-morphism. Let $T \subset \pi^{-1}(M)$ be any irreducible affine curve defined over $\overline{\mathbb{F}_p}$. We shall show that $\pi(T)$ is a point. This proves that M is zero dimensional.

To prove that $\pi(T)$ is a point, we take $\bar{C}$ as the compactification of the normalization C of T, and let $\{P_1, \ldots, P_\ell\} = \bar{C} \setminus C$. There exists $q = p^n$ for some $n \in \mathbb{Z}_{>0}$ such that $\bar{C}$ is defined over $\mathbb{F}_q$ and $P_i \in \bar{C}(\mathbb{F}_q)$ for each i. By the universal property of the representation scheme $R(\Gamma, N)$, C gives rise to a representation $\varrho_C : \Gamma \to \mathrm{GL}_N(\mathbb{F}_q[C])$, where $\mathbb{F}_q[C]$ is the coordinate ring of C. Consider the discrete valuation $v_i : \mathbb{F}_q(C) \to \mathbb{Z}$ defined by P_i, where $\mathbb{F}_q(C)$ is the function field of C. Let $\widehat{\mathbb{F}_q(C)}_{v_i}$ be the completion of $F_q(C)$ with respect to v_i. Then we have the isomorphism $\big(\widehat{\mathbb{F}_q(C)}_{v_i}, v_i\big) \simeq \big(\mathbb{F}_q((t)), v\big)$, where $\big(\mathbb{F}_q((t)), v\big)$ is the formal Laurent field of $\mathbb{F}_q$ with the valuation v defined by $v(\sum_{i=m}^{+\infty} a_i t^i) = \min\{i \mid a_i \neq 0\}$. Let $\varrho_i : \Gamma \to \mathrm{GL}_N(\mathbb{F}_q((t)))$ be the extension of ϱ_C with respect to $\widehat{\mathbb{F}_q(C)}_{v_i}$. Then we have

$$[\varrho_i] \in M(\mathbb{F}_q((t))).$$

Hence, by our assumption, $\varrho_i^{ss}(\Gamma)$ is bounded for each i. Hence, by Claim 1.3, $\varrho_i(\Gamma)$ is bounded for each i. Thus after we replace ϱ_i by some conjugation, we have $\varrho_i(\Gamma) \subset \mathrm{GL}_N(\mathbb{F}_q[[t]])$, where the $\mathbb{F}_q[[t]]$ is the ring of integers of $\mathbb{F}_q((t))$, i.e.

$$\mathbb{F}_q[[t]] := \{\sum_{i=0}^{+\infty} a_i t^i \mid a_i \in \mathbb{F}_q\}.$$

For any matrix $A \in \mathrm{GL}_N(K)$, we denote by $\chi(A) = T^N + \sigma_1(A)T^{N-1} + \cdots + \sigma_N(A)$ its characteristic polynomial. Since we have assumed that $\varrho_i(\Gamma) \subset \mathrm{GL}_N(\mathbb{F}_q[[t]])$ for each i, it follows that $\sigma_j(\varrho_i(\gamma)) \in \mathbb{F}_q[[t]]$ for each i. Therefore, by the definition of ϱ_i, $v_i(\sigma_j(\varrho_C(\gamma))) \geq 0$ for each i. It follows that $\sigma_j(\varrho_C(\gamma))$ extends to a regular function on $\overline{C}$, which is thus constant. This implies that for any two representations $\eta_1 : \Gamma \to \mathrm{GL}_N(K_1)$ and $\eta_2 : \Gamma \to \mathrm{GL}_N(K_2)$ such that $\operatorname{char} K_1 = \operatorname{char} K_2 = p$ and $\eta_i \in C(K_i)$, we have $\chi(\eta_1(\gamma)) = \chi(\eta_2(\gamma))$ for each $\gamma \in \Gamma$. In other words, η_1 and η_2 has the same characteristic polynomial. It follows that $[\eta_1] = [\eta_2]$ by the Brauer–Nesbitt theorem. Hence $\pi(T)$ is a point. Thus M is zero dimensional. □

Corollary 2.4. — *Let $M \subset M_{\mathrm{B}}(X,N)_{\mathbb{F}_p}$ be the same as in Lemma 2.3 satisfying the same assumption. Let $\varrho : \Gamma \to \mathrm{GL}_N(L)$ be a linear representation with L a field of characteristic p and $[\varrho] \in M(L)$. Then the image $\varrho(\Gamma) \subset \mathrm{GL}_N(L)$ is finite.*

Proof. — By Lemma 2.3, M is zero dimensional. Thus we can find a point $\eta : \Gamma \to \mathrm{GL}_N(\overline{\mathbb{F}_p})$ such that $[\eta] \in M(\overline{\mathbb{F}_p})$. Since $\eta(\Gamma)$ is finite, the semisimplification η^{ss} of η has also finite image. As the semisimplification of ϱ is isomorphic to η^{ss}, by virtue of Lemma 2.1, we conclude that $\varrho(\Gamma)$ is finite. □

Lemma 2.5. — *Let $\varphi : \Gamma' \to \Gamma$ be a group morphism from another finitely generated group Γ'. Let $M \subset M_{\mathrm{B}}(\Gamma,N)_{\mathbb{F}_p}$ be a constructible set. Then the following two statements are equivalent:*

(a) *For every reductive representation $\tau : \Gamma \to \mathrm{GL}_N(K)$ with K a local field of characteristic p such that $[\tau] \in M(K)$, the image $\tau \circ \varphi(\Gamma') \subset \mathrm{GL}_N(K)$ is bounded.*
(b) *For every linear representation $\varrho : \Gamma \to \mathrm{GL}_N(L)$ with L a field of characteristic p and $[\varrho] \in M(L)$, the image $\varrho \circ \varphi(\Gamma') \subset \mathrm{GL}_N(L)$ is finite.*

Proof. — The implication (b) $\Longrightarrow$ (a) is trivial. In the following we prove the implication (a) $\Longrightarrow$ (b). We have a induced map $\iota : M_{\mathrm{B}}(\Gamma,N)_{\mathbb{F}_p} \to M_{\mathrm{B}}(\Gamma',N)_{\mathbb{F}_p}$. Set $M' = \iota(M)$. Then $M' \subset M_{\mathrm{B}}(\Gamma',N)_{\mathbb{F}_p}$ is a constructible set. We shall show that M' satisfies the assumption of Lemma 2.3. Indeed let $\sigma : \Gamma \to \mathrm{GL}_N(K')$ be a reductive representation such that $[\sigma] \in M'(K')$, where K' is a local field of characteristic p. Then we may take a reductive representation $\tau : \Gamma \to \mathrm{GL}_N(\overline{K'})$ such that $[\tau] \in M(\overline{K'})$ and $\iota([\tau]) = [\sigma]$. By the assumption (a), $\tau \circ \varphi(\Gamma')$ is bounded. By $\iota([\tau]) = [\tau \circ \varphi]$, we have $[\tau \circ \varphi] = [\sigma]$. Thus by Claim 1.3, $\sigma(\Gamma')$ is bounded. Thus M' satisfies the assumption of Lemma 2.3.

Now let $\varrho : \Gamma \to \mathrm{GL}_N(L)$ be a linear representation such that $[\varrho] \in M(L)$ with L a field of characteristic p. Then we have $[\varrho \circ \varphi] \in M'(L)$. By Corollary 2.4, $\varrho \circ \varphi(\Gamma')$ is finite. □

2.3. Factorization through non-rigidity. — Let X be a quasi-projective smooth variety. We write $M_{\mathrm{B}}(X,N)_{\mathbb{F}_p} = M_{\mathrm{B}}(\pi_1(X),N)_{\mathbb{F}_p}$. Let $M \subset M_{\mathrm{B}}(X,N)_{\mathbb{F}_p}$ be a Zariski closed subset.

Definition 2.6. — The *reduction map* $s_M : X \to S_M$ is obtained through the simultaneous Stein factorization of the reductions $\{s_\tau : X \to S_\tau\}_{[\tau]\in M(K)}$. Here $\tau : \pi_1(X) \to \mathrm{GL}_N(K)$ ranges over all reductive representations with K a local field of characteristic p such that $[\tau] \in M(K)$ and $s_\tau : X \to S_\tau$ is the reduction map defined in Theorem 1.2.

The reduction map $s_M : X \to S_M$ enjoys the following crucial property.

Theorem 2.7. — *Let M be a Zariski closed subset of $M_{\mathrm{B}}(X,N)_{\mathbb{F}_p}$. The reduction map $s_M : X \to S_M$ is the Shafarevich morphism for M. That is, for any connected Zariski closed subset Z of X, the following properties are equivalent:*

(a) *$s_M(Z)$ is a point;*
(b) *for any linear representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ with K a field of characteristic p and $[\varrho] \in M(K)$, we have $\varrho(\mathrm{Im}[\pi_1(Z) \to \pi_1(X)])$ is finite;*
(c) *for any semisimple representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ with K a field of characteristic p such that $[\varrho] \in M(K)$, we have $\varrho(\mathrm{Im}[\pi_1(Z_o^{\mathrm{norm}}) \to \pi_1(X)])$ is finite, where Z_o is any irreducible component of Z.*

Proof. — (a) $\implies$ (b): Let $\tau : \pi_1(X) \to \mathrm{GL}_N(K)$ be a reductive representation with K a local field of characteristic p such that $[\tau] \in M(K)$. Then by the assumption (a), $s_\tau(Z)$ is a point. Hence $\tau(\mathrm{Im}[\pi_1(Z) \to \pi_1(X)])$ is bounded. Hence by Lemma 2.5, $\varrho(\mathrm{Im}[\pi_1(Z) \to \pi_1(X)])$ is finite for every linear representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ with K a field of characteristic p and $[\varrho] \in M(K)$.

(b) $\implies$ (c): this is obvious.

(c) $\implies$ (a): Let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ with K a field of characteristic p such that $[\varrho] \in M(K)$. Then, the image $\varrho(\mathrm{Im}[\pi_1(Z_o^{\mathrm{norm}}) \to \pi_1(X)])$ is finite by our assumption, and is thus bounded. By the property in Theorem 1.2, $s_\varrho(Z)$ is a point. By Definition 2.6, $s_M(Z)$ is also a point. □

Remark 2.8. — We remark that in the case of characteristic 0, i.e. meaning that M is a Zariski closed subset of $M_{\mathrm{B}}(X,N)$ defined over $\bar{\mathbb{Q}}$, the reduction map $s_M : X \to S_M$ constructed in [DYK23] is the same as Definition 2.6: it is obtained through the simultaneous Stein factorization of the reductions $\{s_\tau : X \to S_\tau\}_{[\tau]\in M(K)}$, where $\tau : \pi_1(X) \to \mathrm{GL}_N(K)$ ranges over all reductive representations with K a local field of *characteristic* 0 such that $[\tau] \in M(K)$ and $s_\tau : X \to S_\tau$ is the reduction map defined in Theorem 1.2. The interested readers can refer to [DYK23, §3.1] for further details.

2.4. Construction of the Shafarevich morphism. —

Theorem 2.9. — *Let X be a quasi-projective normal variety and $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a linear representation, where K is a field of characteristic $p > 0$. Then the Shafarevich morphism $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$ exists. That is, for any connected Zariski closed subset $Z \subset X$, the following properties are equivalent:*

(a) $\mathrm{sh}_\varrho(Z)$ *is a point;*
(b) $\varrho(\mathrm{Im}[\pi_1(Z) \to \pi_1(X)])$ *is finite;*
(c) *for each irreducible component Z_o of Z, $\varrho^{ss}(\mathrm{Im}[\pi_1(Z_o^{\mathrm{norm}}) \to \pi_1(X)])$ is finite.*

Proof. — *Step 1.* In this step, we will assume that X is smooth. Define

$$M := \bigcap_{f:Y\to X} j_f^{-1}\{[1]\}, \tag{2.1}$$

where 1 stands for the trivial representation, and $f : Y \to X$ ranges over all proper morphisms from positive dimensional quasi-projective normal varieties such that $f^*\varrho = 1$. Here $j_f : M_{\mathrm{B}}(X,N)_{\mathbb{F}_p} \to M_{\mathrm{B}}(Y,N)_{\mathbb{F}_p}$ is a morphism of affine $\mathbb{F}_p$-scheme induced by f. Then M is a Zariski closed subset. We apply Theorem 2.7 to construct the Shafarevich morphism $s_M : X \to S_M$ associated with M. It is a dominant morphism with general fibers connected. Let $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$ be $s_M : X \to S_M$ and we will prove that it satisfies the properties in the theorem.

(a) $\Rightarrow$ (b): this follows from the fact that $[\varrho] \in M(K)$ and Theorem 2.7.

(b) $\Rightarrow$ (c): obvious.

(c) $\Rightarrow$ (a): We take a finite étale cover $Y \to Z_o^{\mathrm{norm}}$ such that $f^*\varrho^{ss}(\pi_1(Y))$ is trivial, where we denote by $f : Y \to X$ the natural proper morphism. Let $\tau : \pi_1(X) \to \mathrm{GL}_N(L)$ be any linear representation such that $[\tau] \in M(L)$ where L is any field of characteristic p. Then $[f^*\tau] = [1]$ by (2.1). Thanks to Lemma 2.1, $f^*\tau(\pi_1(Y))$ is finite, and it follows that $\tau(\mathrm{Im}[\pi_1(Z_o^{\mathrm{norm}}) \to \pi_1(X)])$ is finite as $\mathrm{Im}[\pi_1(Y) \to \pi_1(Z_o^{\mathrm{norm}})]$ is a finite index subgroup of $\pi_1(Z_o^{\mathrm{norm}})$. According to Theorem 2.7, $s_M(Z)$, and thus $\mathrm{sh}_\varrho(Z)$ is a point.

Step 2. We now do not assume that X is smooth. Let $\mu : X_1 \to X$ be a resolution of singularities. Then the Shafarevich morphism $\mathrm{sh}_{\mu^*\varrho} : X_1 \to \mathrm{Sh}_{\mu^*\varrho}(X_1)$ exists and satisfies the properties in the theorem. Since X is normal, each fiber F of μ is compact and connected. Since $\mu^*\varrho(\mathrm{Im}[\pi_1(F) \to$

$\pi_1(X_1)]) = \{1\}$, it implies that $\mathrm{sh}_{\mu^*\varrho}(F)$ is a point. Hence there exists a morphism $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_{\mu^*\varrho}(X_1)$ such that $\mathrm{sh}_\varrho \circ \mu = \mathrm{sh}_{\mu^*\varrho}$.

(a) $\Rightarrow$ (b): Let $W := \mu^{-1}(Z)$, which is a connected Zariski closed subset of X_1. Then $\mathrm{sh}_{\mu^*\varrho}(W)$ is a point, which implies that $\mu^*\varrho(\mathrm{Im}[\pi_1(W) \to \pi_1(X_1)])$ is finite. By [DYK23, Lemma 3.45], $\pi_1(W) \to \pi_1(Z)$ is surjective. It follows that $\varrho(\mathrm{Im}[\pi_1(Z) \to \pi_1(X)]) = \mu^*\varrho(\mathrm{Im}[\pi_1(W) \to \pi_1(X_1)])$ is finite.

(b) $\Rightarrow$ (c): obvious.

(c) $\Rightarrow$ (a): Let W be an irreducible component of $\mu^{-1}(Z_o)$ which is surjective onto Z_o. This implies that $\mu^*\varrho^{ss}(\mathrm{Im}[\pi_1(W^{\mathrm{norm}}) \to \pi_1(X_1)])$ is finite. Since $[(\mu^*\varrho)^{ss}] = [\mu^*\varrho^{ss}]$, by Lemma 2.1, $(\mu^*\varrho)^{ss}(\mathrm{Im}[\pi_1(W^{\mathrm{norm}}) \to \pi_1(X_1)])$ is also finite. Hence $\mathrm{sh}_\varrho(Z_o) = \mathrm{sh}_{\mu^*\varrho}(W)$ is a point. Since Z is connected, we conclude that $\mathrm{sh}_\varrho(Z)$ is a point.

Letting $\mathrm{Sh}_\varrho(X) := \mathrm{Sh}_{\mu^*\varrho}(X_1)$, we prove the theorem. □

Lemma 1.6 and Theorems 2.7 and 2.9 yield the following result.

Corollary 2.10. — *Let X be a smooth quasi-projective variety and $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a linear representation, where K is a field of characteristic $p > 0$. Then there exist*

— *a local field K' of* char $K' = p$,
— *a positive integer $N' > 0$,*
— *a reductive representation $\sigma : \pi_1(X) \to \mathrm{GL}_{N'}(K')$*

such that the Katzarkov-Eyssidieux reduction map $s_\sigma : X \to S_\sigma$ of σ is the Shafarevich morphism $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$ *of ϱ.*

Proof. — By Theorems 2.7 and 2.9, $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$ is obtained through the simultaneous Stein factorization of the reduction maps $\{s_\tau : X \to S_\tau\}_{[\tau]\in M(K)}$, where $\tau : \pi_1(X) \to \mathrm{GL}_N(K)$ ranges over all reductive representations with K a local field of characteristic p, and M is defined in (2.1). By Lemma 1.6, there exist a local field K of char $K = p$, a positive integer N' and a reductive representation $\sigma : \pi_1(X) \to \mathrm{GL}_{N'}(K)$ such that $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$ coincides with $s_\sigma : X \to S_\sigma$. □

Remark 2.11. — For any quasi-projective normal variety X, in [Kol93] Kollár constructed the *H-Shafarevich map* $\mathrm{sh}_X^H : X \dashrightarrow \mathrm{Sh}^H(X)$ associated with any normal subgroup $H \triangleleft \pi_1(X)$. It is a dominant rational map satisfying the following properties:

(i) the indeterminacy locus of sh_X^H does not dominate $\mathrm{Sh}^H(X)$;
(ii) the general fibers of sh_X^H are connected;
(iii) for any closed subvariety Z of X containing a very general point of X, $\mathrm{sh}_X^H(Z)$ is a point if and only if $\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)/H]$ is finite.

If $H = \{1\}$, then we simply write $\mathrm{sh}_X : X \dashrightarrow \mathrm{Sh}(X)$ for the H-Shafarevich map. In [Cam94], Campana also constructed the Shafarevich map for compact Kähler manifolds (which is also called Γ-reduction). The proofs of their theorems are based on cycle theoretic methods. Therefore, the work [Cam94, Kol93] do not give the precise structure of the Shafarevich morphism as described in the proof of Theorem 2.7, and thus cannot help in studying the geometric (e.g. hyperbolicity or holomorphic convexity) property of the Shafarevich variety.

3. Hyperbolicity via linear representation in positive characteristic

The structure of the Shafarevich morphism, as presented in the proof of Theorem 2.9, is related to the hyperbolicity of algebraic varieties. We will prove Theorems C to E in this section.

3.1. On the generalized Green-Griffiths-Lang conjecture. —

Theorem 3.1. — *Let X be a quasi-projective normal variety. Let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a big representation where K is a field of positive characteristic. Then the following properties are equivalent:*

(i) *X is of log general type;*
(ii) *X is pseudo Picard hyperbolic;*
(iii) *X is pseudo Brody hyperbolic;*
(iv) *X is strongly of log general type.*

It's worth noting that the conjectural four equivalent properties mentioned in Theorem 3.1 constitute the statement of the generalized Green-Griffiths-Lang conjecture, as presented in [CDY22].

Proof of Theorem 3.1. — By replacing X with a desingularization and ϱ with the pullback on this birational model, we can assume that X is smooth. Let $\overline{X}$ be a smooth projective compactification of X such that $D := \overline{X}\backslash X$ is a simple normal crossing divisor. By Theorem 2.9, the Shafarevich morphism $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$ exists. By Corollary 2.10, the Shafarevich morphism $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$ coincides with a Katzarkov-Eyssidieux reduction map $s_\tau : X \to S_\tau$, where $\tau : \pi_1(X) \to \mathrm{GL}_{N'}(K')$ is a reductive representation with some local field K' of char $K' = \mathrm{char}\, K > 0$. By the construction of s_τ in [CDY22, Proof of Theorem H], there exists a finite (ramified) Galois cover $\pi : \overline{X^{\mathrm{sp}}} \to \overline{X}$ with Galois group H such that

(a) there exists a set of forms $\{\eta_j\}_{j=1,\ldots,m} \subset H^0(\overline{X^{\mathrm{sp}}}, \pi^*\Omega_{\overline{X}}(\log D))$ which are invariant under H;
(b) π is étale outside

$$R := \{x \in \overline{X^{\mathrm{sp}}} \mid \exists \eta_j \neq \eta_\ell \text{ with } (\eta_j - \eta_\ell)(x) = 0\}. \tag{3.1}$$

(c) There exists a morphism $a : X^{\mathrm{sp}} \to A$ to a semi-abelian variety A with H acting on A such that a is H-equivariant.
(d) The reduction map $s_\tau : X \to S_\tau$ is the quasi-Stein factorization of the quotient $X \to A/H$ of a by H.

Claim 3.2. — *We have* $\dim X^{\mathrm{sp}} = \dim a(X^{\mathrm{sp}})$.

Proof. — Since ϱ is big, it follows that sh_ϱ, hence s_τ is birational. Thus $\dim X^{\mathrm{sp}} = \dim a(X^{\mathrm{sp}})$ by Item (d). □

Now we will apply the techniques and results in [CDY22] to prove the theorem.
(i) $\Rightarrow$ (ii): We first recall some notions in Nevanlinna theory used in [CDY22, §4.1]. Let Y be a connected Riemann surface with a proper surjective holomorphic map $p : Y \to \mathbb{C}_{>\delta}$, where $\mathbb{C}_{>\delta} := \{z \in \mathbb{C} \mid \delta < |z|\}$ with some fixed positive constant $\delta > 0$. For $r > 2\delta$, define $Y(r) = p^{-1}\big(\mathbb{C}_{>2\delta}(r)\big)$ where $\mathbb{C}_{>2\delta}(r) = \{z \in \mathbb{C} \mid 2\delta < |z| < r\}$. In the following, we assume that $r > 2\delta$. The *ramification counting function* of the covering $p : Y \to \mathbb{C}_{>\delta}$ is defined by

$$N_{\mathrm{ram}\, p}(r) := \frac{1}{\deg p} \int_{2\delta}^{r} \left[\sum_{y \in Y(t)} \mathrm{ord}_y \mathrm{ram}\, p \right] \frac{dt}{t}, \tag{3.2}$$

where $\mathrm{ram}\, p \subset Y$ is the ramification divisor of $p : Y \to \mathbb{C}_{>\delta}$.

For any holomorphic map $f : \mathbb{C}_{>\delta} \to X$ whose image is not contained in $\pi(R)$, there exists a surjective finite holomorphic map $p : Y \to \mathbb{C}_{>\delta}$ from a connected Riemann surface Y to $\mathbb{C}_{>\delta}$ and a holomorphic map $g : Y \to X^{\mathrm{sp}}$ satisfying the following diagram:

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "p"] & X^{\mathrm{sp}} \arrow[d, "\pi"] \\ \mathbb{C}_{>\delta} \arrow[r, "f"] & X \end{tikzcd} \tag{3.3}$$

By [CDY22, Proposition 6.9], there exists a proper Zariski closed subset $E \subsetneq X$ such that for any holomorphic map $f : \mathbb{C}_{>\delta} \to X$ whose image not contained in E, one has

$$N_{\mathrm{ram}\, p}(r) = o(T_g(r, L)) + O(\log r)\|, \tag{3.4}$$

where $g : Y \to X^{\mathrm{sp}}$ is the induced holomorphic map in (3.3), L is an ample line bundle on $\overline{X^{\mathrm{sp}}}$ equipped with a smooth hermitian metric and $T_g(r, L)$ is the order function defined by

$$T_g(r, L) := \frac{1}{\deg p} \int_{2\delta}^{r} \left[\int_{Y(t)} g^* c_1(L, h_L) \right] \frac{dt}{t}. \tag{3.5}$$

Note that X^{sp} is of log general type as we assume that X is of log general type and $\pi : X^{\mathrm{sp}} \to X$ is a Galois cover. We apply [CDY22, Theorem 4.1] to conclude that there exists a proper Zariski closed set $\Xi \subsetneq X^{\mathrm{sp}}$ such that an extension $\overline{g} : \overline{Y} \to \overline{X^{\mathrm{sp}}}$ of g exists provided $g(Y) \not\subset \Xi$, where $\overline{Y}$ is a Riemann surface such that $p : Y \to \mathbb{C}_{>\delta}$ extends to a proper map $\overline{p} : \overline{Y} \to \mathbb{C}_{>\delta} \cup \{\infty\}$. This induces an extension $\bar{f} : \mathbb{C}_{>\delta} \cup \{\infty\} \to \overline{X}$. Hence, X is pseudo Picard hyperbolic.

(ii) $\Rightarrow$ (iii): It is obvious that pseudo Picard hyperbolicity implies pseudo Brody hyperbolicity (cf. [CDY22, Lemma 4.3]).

(iii) $\Rightarrow$ (iv): Since $\pi : X^{\mathrm{sp}} \to X$ is a finite surjective morphism, X^{sp} is also pseudo Brody hyperbolic. By [CDY22, Corollary 4.2], we conclude that there exists a proper Zariski closed subset $Z \subsetneq X^{\mathrm{sp}}$ such that any positive dimensional closed subvariety V is of log general type if it is not contained in Z. Then the rest of the proof is basically the same as that of [CDY22, Theorem 6.3] and let us explain it for the sake of completeness.

By the proof of [CDY22, Theorem 6.3], there exists a Zariski closed subset $\Xi \subset X$ such that we have the following properties:

(1) $\pi(Z \cup R) \subset \Xi$, where R is defined in eq. (3.1);
(2) Let $V \subset X$ be any closed subvariety such that $V \not\subset \Xi$. Let $W \to V$ be a smooth modification and let $\overline{W}$ be a smooth projective compactification of W such that $D_{\overline{W}} := \overline{W} - W$ is a simple normal crossing divisor and $(\overline{W}, D_{\overline{W}}) \to (\overline{X}, D)$ is a log morphism. Let $\overline{S}$ be a normalization of an irreducible component of $\overline{W} \times_{\overline{X}} \overline{X^{\mathrm{sp}}}$.

$$\begin{array}{ccc} \overline{S} & \xrightarrow{g} & \overline{X^{\mathrm{sp}}} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\ \overline{W} & \longrightarrow & \overline{X} \end{array}$$

Then by [CDY22, Claim 6.5] the finite morphism $p : \overline{S} \to \overline{W}$ is a Galois morphism with the Galois group $H' \subset H$. The morphism $g : \overline{S} \to \overline{X^{\mathrm{sp}}}$ is H'-equivariant.
(3) Define $\psi_j := g^*\eta_j \in H^0(\overline{S}, p^*\Omega_{\overline{W}}(\log D_{\overline{W}}))$ for $j = 1, \ldots, m$. Let I be the set of all (i, j) such that

— $\eta_i - \eta_j \neq 0$;
— the image of $S \to X^{\mathrm{sp}}$ intersects with $\{z \in \overline{X^{\mathrm{sp}}} \mid (\eta_i - \eta_j)(z) = 0\}$.

Then by by [CDY22, Claim 6.6], for $(i, j) \in I$, $\psi_i - \psi_j \neq 0$ in $H^0(\overline{S}, p^*\Omega_{\overline{W}}(\log D_{\overline{W}}))$.

We set

$$R' := \{z \in \overline{S} \mid \exists (i, j) \in I \text{ with } (\psi_i - \psi_j)(z) = 0\}.$$

Then R' is a proper Zariski closed subset of $\overline{S}$. Denote by R_0 the ramification locus of $p : \overline{S} \to \overline{W}$. By the purity of branch locus of finite morphisms, we know that R_0 is a (Weil) divisor, and thus $p(R_0)$ is also a divisor. Moreover $R_0 = p^{-1}(p(R_0))$ since p is Galois with Galois group H'. Denote by E the sum of prime components of $p(R_0)$ which intersect with W. One observes that $S - p^{-1}(E) \to W - E$ is finite étale.

Since $\pi : \overline{X^{\mathrm{sp}}} \to \overline{X}$ is étale over $\overline{X^{\mathrm{sp}}} - R$, it follows that p is étale over $\overline{S} - g^{-1}(R)$ and thus $R_0 \cap S \subset R' \cap S$.

Since $\overline{S} \to \overline{X^{\mathrm{sp}}}$ is H'-equivariant, it follows that any $h \in H'$ acts on $\{\psi_i - \psi_j\}_{(i,j)\in I} \subset H^0(\overline{S}, p^*\Omega_{\overline{W}}(\log D_{\overline{W}}))$ as a permutation by our choice of I. Define a section

$$\sigma := \prod_{h\in H'} \prod_{(i,j)\in I} h^*(\psi_i - \psi_j) \in H^0(\overline{S}, \mathrm{Sym}^\ell p^*\Omega_{\overline{W}}(\log D_{\overline{W}})),$$

which is non-zero and vanishes at R' by our choice of I. Then it is invariant under the H'-action and thus descends to a section

$$\sigma^{H'} \in H^0(\overline{W}, \mathrm{Sym}^\ell \Omega_{\overline{W}}(\log D_{\overline{W}}))$$

so that $p^*\sigma^{H'} = \sigma$. Since $R_0 \cap S \subset R' \cap S$ and $p^{-1}(p(R_0)) = R_0$, $\sigma^{H'}$ vanishes at the divisor E. This implies that there is a non-trivial morphism

$$\mathcal{O}_{\overline{W}}(E) \to \mathrm{Sym}^\ell \Omega_{\overline{W}}(\log D_{\overline{W}}). \tag{3.6}$$

Recall that S is of log general type by Item (1) and the choice of Z. Since $S - p^{-1}(E) \to W - E$ is finite étale, $W\backslash E$ is also of log general type. By [NWY13, Lemma 3], $K_{\overline{W}} + E + D_{\overline{W}}$ is big. Together with (3.6) we can apply [CP19, Corollary 8.7] to conclude that $K_{\overline{W}} + D_{\overline{W}}$ is big. Hence W, so V is of log general type.

(iv) $\Rightarrow$ (i): This is obvious. Thus we have completed the proof of Theorem 3.1. □

Remark 3.3. — From the proof of (i) $\Rightarrow$ (ii) in Theorem 3.1, we note that the weaker condition of X^{sp} being of log general type is sufficient to establish the pseudo Picard hyperbolicity of X. This observation holds significant importance in the proof of Theorem 3.7 below.

3.2. Comparison of special subsets.

— This subsection is devoted to the proof of Theorem D. We first prove a lemma on the special subset of the big representation in positive characteristic.

Lemma 3.4. — *Let X be a smooth quasi-projective variety. Let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a big representation where K is a field of positive characteristic. Then the special subset* $\mathrm{Sp}(\varrho)$ *defined in Definition 0.1 is a proper Zariski closed subset of X.*

Proof. — Note that the Shafarevich morphism $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$ exists, which is dominant morphism with general fibers connected (cf. Theorem 2.9). Since ϱ is big, it follows that sh_ϱ is birational. Therefore, we have a Zariski dense open set $X^\circ \subset X$ such that $\mathrm{sh}_\varrho : X^\circ \to \mathrm{sh}_\varrho(X^\circ)$ is an isomorphism. By Theorem 2.9, for any positive dimensional closed subvariety $Z \subset X$ not contained in $X \setminus X^\circ$, $\varrho(\mathrm{Im}[\pi_1(Z) \to \pi_1(X)])$ is infinite. Hence $\mathrm{Sp}(\varrho) \subset X \setminus X^\circ$. This concludes that $\mathrm{Sp}(\varrho)$ is a proper Zariski closed subset of X. □

Before proceeding to the proof of Theorem D, we will first prove the following result.

Proposition 3.5. — *Let X be a smooth quasi-projective variety. Let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a big representation where K is a field of positive characteristic. Assume that there exists a holomorphic map $f : \mathbb{D}^* \to X$ such that f has essential singularity at the origin and that the image $f(\mathbb{D}^*)$ is Zariski dense. Then* $\mathrm{Sp}_{\mathrm{sab}}(X) = X$.

Proof. — We will maintain the same notations as introduced in the proof of Theorem 3.1. Let $\pi : X^{\mathrm{sp}} \to X$ be the Galois covering, as defined therein. By the assumption, X is not pseudo Picard hyperbolic. Hence X^{sp} is not of log general type by Remark 3.3. Recall that in the proof of Theorem 3.1, we have a morphism $a : X^{\mathrm{sp}} \to A$, where each A is a semiabelian variety, and $\dim X^{\mathrm{sp}} = \dim a(X^{\mathrm{sp}})$. Note that $\bar{\kappa}(X^{\mathrm{sp}}) \geq 0$. We denote by $j : X^{\mathrm{sp}} \dashrightarrow J(X^{\mathrm{sp}})$ the logarithmic Iitaka fibration of X^{sp}, whose general fibers are positive dimensional as X^{sp} is not of log general type. After replacing X^{sp} be a proper birational model, we might assume that X^{sp} is smooth and j is regular. Then for a very general fiber F of j, we have $\bar{\kappa}(F) = 0$ and $\dim F = \dim a(F) > 0$. By [CDY22, Lemma 3.5], we have $\mathrm{Sp}_{\mathrm{sab}}(F) = F$. Consequently, $\mathrm{Sp}_{\mathrm{sab}}(X^{\mathrm{sp}}) = X^{\mathrm{sp}}$. Since $\pi : X^{\mathrm{sp}} \to X$ is surjective, it follows that $\mathrm{Sp}_{\mathrm{sab}}(X) = X$. The proposition is proved. □

Theorem 3.6. — *Let X be a quasi-projective normal variety. Let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a big representation where K is a field of positive characteristic. Then we have*

$$\mathrm{Sp}_{\mathrm{sab}}(X) \setminus \mathrm{Sp}(\varrho) = \mathrm{Sp}_{\mathrm{alg}}(X) \setminus \mathrm{Sp}(\varrho) = \mathrm{Sp}_{\mathrm{p}}(X) \setminus \mathrm{Sp}(\varrho) = \mathrm{Sp}_{\mathrm{h}}(X) \setminus \mathrm{Sp}(\varrho).$$

We have $\mathrm{Sp}_\bullet(X) \subsetneq X$ if and only if X is of log general type, where $\mathrm{Sp}_\bullet$ denotes any of $\mathrm{Sp}_{\mathrm{sab}}$, $\mathrm{Sp}_{\mathrm{alg}}$, Sp_{h} or Sp_{p}.

Proof. — By [CDY22, Lemma 4.3] one has

$$\mathrm{Sp}_{\mathrm{sab}}(X) \subseteq \mathrm{Sp}_{\mathrm{h}}(X) \subseteq \mathrm{Sp}_{\mathrm{p}}(X). \tag{3.7}$$

Step 1. Let $f : \mathbb{D}^* \to X$ be a holomorphic map with essential singularity at the origin such that $f(\mathbb{D}^*) \not\subset \mathrm{Sp}(\varrho)$. Let Z be a desingularization of the Zariski closure of $f(\mathbb{D}^*)$. By the definition of $\mathrm{Sp}(\varrho)$ in Definition 0.1, we note that the natural morphism $\iota : Z \to X$ induces a big representation $\iota^*\varrho$. Since $\mathrm{Sp}_{\mathrm{p}}(Z) = Z$, Theorem 3.1 implies that Z is not of log general type. Hence $\iota(Z) \subset \mathrm{Sp}_{\mathrm{alg}}(X)$, which implies

$$\mathrm{Sp}_{\mathrm{p}}(X) \setminus \mathrm{Sp}(\varrho) \subseteq \mathrm{Sp}_{\mathrm{alg}}(X) \setminus \mathrm{Sp}(\varrho). \tag{3.8}$$

By Proposition 3.5, $\mathrm{Sp}_{\mathrm{sab}}(Z) = Z$. It follows that

$$\mathrm{Sp}_{\mathrm{p}}(X) \setminus \mathrm{Sp}(\varrho) \subseteq \mathrm{Sp}_{\mathrm{sab}}(X) \setminus \mathrm{Sp}(\varrho).$$

Combining this with (3.7), we get

$$\mathrm{Sp}_{\mathrm{sab}}(X) \setminus \mathrm{Sp}(\varrho) = \mathrm{Sp}_{\mathrm{h}}(X) \setminus \mathrm{Sp}(\varrho) = \mathrm{Sp}_{\mathrm{p}}(X) \setminus \mathrm{Sp}(\varrho).$$

Thus it remains to prove $\mathrm{Sp}_{\mathrm{p}}(X) \setminus \mathrm{Sp}(\varrho) = \mathrm{Sp}_{\mathrm{alg}}(X) \setminus \mathrm{Sp}(\varrho)$.

Step 2. Let Y be a closed subvariety of X that is not of log general type. Assume that $Y \not\subseteq \mathrm{Sp}(\varrho)$. Let $\iota : Z \to Y$ be a desingularization. Then $\iota^*\varrho$ is a big representation. By Theorem 3.1, we have $\mathrm{Sp}_{\mathrm{p}}(Y) = Y$. Hence $Y \subset \mathrm{Sp}_{\mathrm{p}}(X)$, which implies

$$\mathrm{Sp}_{\mathrm{alg}}(X) \setminus \mathrm{Sp}(\varrho) \subseteq \mathrm{Sp}_{\mathrm{p}}(X) \setminus \mathrm{Sp}(\varrho).$$

Together with (3.8), we obtain $\mathrm{Sp}_{\mathrm{p}}(X) \setminus \mathrm{Sp}(\varrho) = \mathrm{Sp}_{\mathrm{alg}}(X) \setminus \mathrm{Sp}(\varrho)$. We have proved our theorem. □

3.3. A characterization of hyperbolicity via fundamental groups. —

Theorem 3.7. — *Let X be a quasi-projective normal variety. Let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a big representation where K is a field of positive characteristic. If the Zariski closure G of $\varrho(\pi_1(X))$ is a semisimple algebraic group, then $\mathrm{Sp}_\bullet(X) \subsetneq X$ where $\mathrm{Sp}_\bullet$ denotes any of $\mathrm{Sp}_{\mathrm{sab}}$, $\mathrm{Sp}_{\mathrm{alg}}$, Sp_{h} or Sp_{p}.*

Proof. — We may assume that K is algebraically closed. Replacing X by a desingularization, we may assume that X is smooth. We will still maintain the same notations as introduced in the proof of Theorem 3.1. Let $\pi : X^{\mathrm{sp}} \to X$ be the Galois covering defined therein. Consider the representation $\pi^*\varrho : \pi_1(X^{\mathrm{sp}}) \to G(K)$, which is Zariski dense as $\mathrm{Im}[\pi_1(X^{\mathrm{sp}}) \to \pi_1(X)]$ is a finite index subgroup of $\pi_1(X)$. By the proof of Theorem 3.1, there exists a morphism $a : X^{\mathrm{sp}} \to A$ where A is a semiabelian variety such that $\dim X^{\mathrm{sp}} = \dim a(X^{\mathrm{sp}})$. Hence we have $\bar{\kappa}(X^{\mathrm{sp}}) \geq 0$.

Claim 3.8. — *X^{sp} is of log general type.*

Proof. — Let $\mu : Y \to X^{\mathrm{sp}}$ be a desingularization such that the logarithmic Iitaka fibration $j : Y \to J(Y)$ is regular. For a very general fiber F of j, we have $\bar{\kappa}(F) = 0$ and $\dim F = \dim a(F)$. By [CDY22, Lemma 3.3], we have $\pi_1(F)$ is abelian.

We write $\tau = (\pi \circ \mu)^*\varrho : \pi_1(Y) \to G(K)$. Notably, $\tau(\pi_1(Y))$ is Zariski dense in G. By [CDY22, Lemma 2.2], $\mathrm{Im}[\pi_1(F) \to \pi_1(Y)]$ is a normal subgroup of $\pi_1(Y)$. Consequently, the Zariski closure N of $\tau(\mathrm{Im}[\pi_1(F) \to \pi_1(Y)])$ is a normal subgroup of G. It's worth noting that the connected component N° of N is a tori since $\tau(\mathrm{Im}[\pi_1(F) \to \pi_1(Y)])$ is commutative. Therefore, N° must be trivial since G is assumed to be semisimple. Consequently, $\tau(\mathrm{Im}[\pi_1(F) \to \pi_1(Y)])$ is finite.

Given our assumption that ϱ is big, we conclude that τ is also big. Therefore, we arrive at the conclusion that $\dim F = 0$, leading us to deduce that both Y and, consequently, X^{sp} are of log general type. □

We can carry out the same proof as the step (i)⇒(ii) in the proof of Theorem 3.1 to conclude that X is pseudo Picard hyperbolic. It's essential to emphasize that in that proof, the condition of X being of log general type is only used to show that X^{sp} is of log general type. See Remark 3.3. We now apply Theorem 3.6 to conclude that $\mathrm{Sp}_\bullet \subsetneq X$ where $\mathrm{Sp}_\bullet$ denotes any of $\mathrm{Sp}_{\mathrm{sab}}$, $\mathrm{Sp}_{\mathrm{alg}}$, Sp_{h} or Sp_{p}. □

Remark 3.9. — Note that the condition in Theorem 3.7 is sharp. For example, the representation

$$\begin{aligned}\mathbb{Z} \simeq \pi_1(\mathbb{C}^*) &\to \mathrm{GL}_1(\mathbb{F}_p(t))\\ n &\mapsto t^n\end{aligned}$$

is a big and Zariski dense representation. However, $\mathbb{C}^*$ is not of general type and contains a Zariski dense entire curve. This example demonstrates that the semisimplicity of G is necessary for Theorem 3.7 to hold.

Furthermore, the condition of bigness in Theorem 3.7 is also indispensable. Indeed, it's worth noting that for any complex algebraic variety X, we have $\pi_1(X) \simeq \pi_1(X \times \mathbb{P}^1)$. The requirement of ϱ being big effectively excludes the case of uniruled varieties $X \times \mathbb{P}^1$, which is apparently non-hyperbolic.

In the next three sections, we will provide some applications of Theorem 3.7. From those applications we will see the significance of the stronger notion of pseudo Picard hyperbolicity in comparison to the pseudo Brody one.

4. Algebraic varieties with compactifiable universal cover

In the paper [CHK13, CH13], Claudon, Höring and Kollár proposed the following intriguing conjecture:

Conjecture 4.1. — *Let X be a complex projective manifold with infinite fundamental group $\pi_1(X)$. Suppose that the universal cover $\widetilde{X}$ is quasi-projective. Then after replacing X by a finite étale cover, there exists a locally trivial fibration $X \to A$ with simply connected fiber F onto a complex torus A. In particular we have $\widetilde{X} \simeq F \times \mathbb{C}^{\dim A}$.*

It's worth noting that assuming abundance conjecture, Claudon, Höring and Kollár proved this conjecture in [CHK13], thereby providing unconditional proof for Conjecture 4.1 in cases where $\dim X \leqslant 3$. Additionally, Claudon and Höring, in [CH13], proved Conjecture 4.1 when $\pi_1(X)$ is virtually abelian, a result essential for the proof of Theorem 4.7.

As the first application of Theorem E, in this section we establish a linear version of Conjecture 4.1 without relying on the abundance conjecture.

4.1. A factorization result (I). — In this subsection, we will prove that the any linear representation factors through its Shafarevich morphism after passing to a finite étale cover. It will be used in the proof of Theorems 4.7 and 7.15.

Lemma 4.2. — *Let $g : X \to S$ be a proper surjective map between quasi-projective normal varieties X and S such that all fibers of g are connected. Let $\varrho : \pi_1(X) \to \mathrm{GL}_n(K)$ be a representation, where K is any field. Assume that for every fiber F of g, the image $\varrho(\mathrm{Im}[\pi_1(F) \to \pi_1(X)])$ is trivial. Then there exists a representation $\tau : \pi_1(S) \to \mathrm{GL}_n(K)$ such that $g^*\tau = \varrho$.*

Proof. — By [Kol95, 2.10.(2)], $\pi_1(X) \to \pi_1(S)$ is surjective. Hence it is enough to show that $\varrho(\ker[\pi_1(X) \to \pi_1(S)])$ is trivial. We take $\gamma \in \ker[g_* : \pi_1(X) \to \pi_1(S)]$. We shall show $\varrho(\gamma) = 1$. Since $\varrho(\pi_1(X))$ is residually finite, it is enough to show $\varrho(\gamma) \in N$ for every finite indexed normal subgroup $N \subset \varrho(\pi_1(X))$. We fix such N. Let $\mu : X_N \to X$ be a Galois covering corresponding to $\varrho^{-1}(N) \subset \pi_1(X)$. Then $\mathrm{Gal}(X_N/X) = \varrho(\pi_1(X))/N$. Set $l = \#\mathrm{Gal}(X_N/X)$. Let $X_N \to S_N \to S$ be the quasi-Stein factorization of the composite $X_N \xrightarrow{\mu} X \xrightarrow{g} S$. Then

$\varphi : X_N \to S_N$ is proper, for the composite $g \circ \mu : X_N \to S$ is proper. In particular, $\varphi : X_N \to S_N$ is surjective. Hence the action $\mathrm{Gal}(X_N/X) \curvearrowright X_N$ descends to $\mathrm{Gal}(X_N/X) \curvearrowright S_N$. Since we are assuming that the image $\varrho(\mathrm{Im}[\pi_1(F) \to \pi_1(X)])$ is trivial for every fiber F of g, there exist exactly ℓ connected components of $\mu^{-1}(F) \subset X_N$. Note that $\mathrm{Gal}(X_N/X)$ permutes these ℓ connected components and these connected components correspond to ℓ distinct points in S_N. Thus the action $\mathrm{Gal}(X_N/X) \curvearrowright S_N$ does not have fixed points. Since S is normal, the finite map $S_N \to S$ is also Galois covering with $\mathrm{Gal}(S_N/S) = \mathrm{Gal}(X_N/X)$. Thus there exists a finite indexed normal subgroup $N_0 \subset \pi_1(S)$ such that $(g_*)^{-1}(N_0) = \varrho^{-1}(N)$.

Now by $\gamma \in \ker[g_* : \pi_1(X) \to \pi_1(S)]$, we have $\gamma \in (g_*)^{-1}(N_0)$. Thus $\gamma \in \varrho^{-1}(N)$, hence $\varrho(\gamma) \in N$. The proof is completed. □

Lemma 4.3. — *Let X be a projective normal variety. Let $F \subset X$ be a connected Zariski closed subset. Then there exists a connected open neighbourhood $U \subset X$ of F such that* $\mathrm{Im}[\pi_1(F) \to \pi_1(X)] = \mathrm{Im}[\pi_1(U) \to \pi_1(X)]$.

Proof. — The construction of U is as follows. For each $t \in F$, we take a connected open neighbourhood $\Omega_t \subset X$ of t such that

(a) any loop in Ω_t is null homotopic in X,
(b) $F \cap \Omega_t$ is connected.

We take a connected open neighbourhood W_t of t such that $\overline{W_t} \subset \Omega_t$. Since X is projective, F is compact. We may take $t_1, \ldots, t_l \in F$ such that $F \subset W_{t_1} \cup \cdots \cup W_{t_l}$. For each $s \in F$, we take a connected open neighbourhood U_s of s such that for each $i = 1, \ldots, l$, if $s \in W_{t_i}$, then $U_s \subset W_{t_i}$, if $s \in \overline{W_{t_i}}$ then $U_s \subset \Omega_{t_i}$, and if $s \notin \overline{W_{t_i}}$, then $U_s \cap \overline{W_{t_i}} = \emptyset$. We set $U = \cup_{s\in F} U_s$. Note that U is open and $F \subset U$. Since F and U_s are connected, U is connected. We shall show that U satisfies the property of our lemma.

Let $\pi : \widetilde{X} \to X$ be the universal covering space. Set $\Gamma = \pi_1(X)$. The action $\Gamma \curvearrowright \widetilde{X}$ is free. We fix a connected component $F' \subset \widetilde{X}$ of $\pi^{-1}(F)$. Define $H \subset \Gamma$ by $\gamma \in H$ iff $\gamma F' = F'$. Note that $F \subset \widetilde{X}$ is an analytic set, hence locally path connected. Hence $F' \subset \widetilde{X}$ is path connected analytic set. Thus $H = \mathrm{Im}[\pi_1(F) \to \pi_1(X)]$.

For each $t \in F$, we take $t' \in F'$ such that $t = \pi(t')$. Then $F' \cap \pi^{-1}(t) = \{\gamma t'; \gamma \in H\}$. For each $i = 1, \ldots, l$, we denote by $\Omega'_{t'_i} \subset \widetilde{X}$ the connected component of $\pi^{-1}(\Omega_{t_i})$ which contains t'_i. Since any loop in Ω_{t_i} is null homotopic in X, the natural map $\Omega'_{t'_i} \to \Omega_{t_i}$ is isomorphic. Then $\pi^{-1}(\Omega_{t_i}) = \sqcup_{\gamma\in\Gamma} \gamma\Omega'_{t'_i}$ is disjoint union. Since $F \cap \Omega_{t_i}$ is connected, $\Omega'_{t'_i}$ does not intersect with $\gamma F'$ for all $\gamma \in \Gamma - H$. Thus

$$F' \cap (\cup_{\gamma\in\Gamma-H} \gamma\Omega'_{t'_i}) = \emptyset. \tag{4.1}$$

Let $W'_{t'_i} \subset \Omega'_{t'_i}$ be the inverse image of W_{t_i} under $\Omega'_{t'_i} \to \Omega_{t_i}$. Then $\pi^{-1}(\overline{W_{t_i}}) = \cup_{\gamma\in\Gamma} \gamma\overline{W'_{t'_i}}$, which is a closed subset of $\widetilde{X}$. Set $C_i = \cup_{\gamma\in\Gamma-H} \gamma\overline{W'_{t'_i}}$. By $C_i = \pi^{-1}(\overline{W_{t_i}}) \cap (\cup_{\gamma\in H} \gamma\Omega'_{t'_i})^c$, we conclude that C_i is a closed set. By (4.1), we have $F' \cap C_i = \emptyset$. We set $C = C_1 \cup \cdots \cup C_l$. Then $C \subset \widetilde{X}$ is a closed subset and $F' \cap C = \emptyset$.

Now for each $t \in F$, we denote by $U'_{t'} \subset \widetilde{X}$ the connected component of $\pi^{-1}(U_t)$ which contains t'. We claim that if $\tau \in H$, then $\tau U'_{t'} \cap C = \emptyset$, and if $\tau \in \Gamma - H$, then $\tau U'_{t'} \subset \mathrm{Int}(C)$, where $\mathrm{Int}(C)$ is the interior of C. To prove this, we first suppose $\tau \in H$. To show $\tau U'_{t'} \cap C = \emptyset$, it is enough to show that $\tau U'_{t'} \cap \gamma\overline{W'_{t'_i}} = \emptyset$ for each $\gamma \in \Gamma - H$ and $i = 1, \ldots, l$. Indeed, if $t \in \overline{W_{t_i}}$, then $U_t \subset \Omega_{t_i}$. By $\gamma \neq \tau$, we have $\gamma\Omega'_{t'_i} \cap \tau\Omega'_{t'_i} = \emptyset$. Hence $\tau U'_{t'} \cap \gamma\overline{W'_{t'_i}} = \emptyset$. If $t \notin \overline{W_{t_i}}$, then $U_t \cap \overline{W_{t_i}} = \emptyset$. Thus $\tau U'_{t'} \cap \gamma\overline{W'_{t'_i}} = \emptyset$. We have proved that $\tau U'_{t'} \cap C = \emptyset$, if $\tau \in H$. Next suppose $\tau \in \Gamma - H$. We may take i such that $t \in W_{t_i}$. Then by $U_t \subset W_{t_i}$, we have $\tau U'_{t'} \subset \tau W'_{t'_i} \subset \mathrm{Int}(C)$.

Now we set $U' = \cup_{t\in F} \cup_{\gamma\in H} \gamma U'_{t'}$. Then we have $F' \subset U'$. Since F' and $\gamma U'_{t'}$ are connected, U' is connected. Note that $U' \cap C = \emptyset$ and $\gamma U' = U'$ for all $\gamma \in H$. On the other hand, for all $\gamma \in \Gamma - H$, we have $\gamma U' \subset \mathrm{Int}(C)$. By $\pi^{-1}(U) = \cup_{\gamma\in\Gamma} \gamma U'$, we conclude that U' is a connected component of $\pi^{-1}(U)$ and $\varrho(\mathrm{Im}[\pi_1(U) \to \pi_1(X)]) = H$. The proof is completed. □

Remark 4.4. — A stronger result holds as a consequence of [Hof09, Theorem 4.5]. Namely there exists a connected open neighbourhood $U \subset X$ of F such that the induced map $\pi_1(F) \to \pi_1(U)$ is an isomorphism. Here we give a direct proof of Lemma 4.3 for the sake of convenience.

Lemma 4.5. — *Let $f : X \to S$ be a proper surjective map between projective normal varieties X and S such that all fibers of f are connected. Let $\varrho : \pi_1(X) \to \mathrm{GL}_n(K)$ be a representation, where K is any field. Assume that for every fiber F of f, the image $\varrho(\mathrm{Im}[\pi_1(F) \to \pi_1(X)])$ is finite. Then there exists a finite étale covering $\mu : X' \to X$ with the following property: Let $X' \to S' \to S$ be the Stein factorization of the composite $X' \to X \to S$. Then for every fiber G of $g : X' \to S'$, the image $\mu^*\varrho(\mathrm{Im}[\pi_1(G) \to \pi_1(X')])$ is trivial.*

Proof. — Let $N \subset \varrho(\pi_1(X))$ be a finite indexed normal subgroup. We define $Z_N \subset S$ by $s \in Z_N$ iff $\varrho(\mathrm{Im}[\pi_1(f^{-1}(s)) \to \pi_1(X)]) \cap N \neq \{1\}$.

We claim that $Z_N \subset S$ is a Zariski closed subset. We prove this. Let $M \subset \varrho(\pi_1(X))$ be a finite indexed normal subgroup such that $M \subset N$. Let $\mu : X_M \to X$ be the Galois covering associated to $\varrho^{-1}(M) \subset \pi_1(X)$. Let $X_M \to S_M \to S$ be the quasi-Stein factorization of the composite $X_M \to X \to S$. Then since $X \to S$ is proper, $X_M \to S_M$ is proper. In particular, $X_M \to S_M$ is surjective. Hence $\mathrm{Gal}(X_M/X) \curvearrowright X_M$ descends to $\mathrm{Gal}(X_M/X) \curvearrowright S_M$. We define $E_M \subset S_M$ by $s \in E_M$ iff there exists $\sigma \in N/M$ such that $\sigma \neq 1$ and $\sigma(s) = s$. Then $E_M \subset S_M$ is Zariski closed. Let $T_M \subset S$ be the image of E_M under the finite map $S_M \to S$. Then $T_M \subset S$ is Zariski closed set. We note that $s \in T_M$ if and only if $\varrho(\mathrm{Im}[\pi_1(f^{-1}(s)) \to \pi_1(X)]) \cap N \neq \{1\}$ and $\varrho(\mathrm{Im}[\pi_1(f^{-1}(s)) \to \pi_1(X)]) \cap N \not\subset M$. Hence we have the following two properties:

(a) $T_M \subset Z_N$.
(b) If $M' \subset M$, then $T_M \subset T_{M'}$.

We take $s \in S$. We may take $M_s \subset N$ such that $\varrho(\mathrm{Im}[\pi_1(f^{-1}(s)) \to \pi_1(X)]) \cap M_s = \{1\}$. We take a connected open neighborhood $U \subset X$ of $f^{-1}(s)$ as in Lemma 4.3. Then $\varrho(\mathrm{Im}[\pi_1(U) \to \pi_1(X)]) \cap M_s = \{1\}$. We note that $f : X \to S$ is proper. Hence, we may take an open neighborhood $V_s \subset S$ of $s \in S$ such that $f^{-1}(V_s) \subset U$. Then for $t \in V_s$, we have $f^{-1}(t) \subset U$, hence $\varrho(\mathrm{Im}[\pi_1(f^{-1}(t)) \to \pi_1(X)]) \cap M_s = \{1\}$. Hence if $t \in Z_N$, then $t \in T_{M_s}$. This shows that $T_{M_s} \cap V_s = Z_N \cap V_s$. Since S is compact, we may take finite points $s_1, \ldots, s_k \in S$ such that $S = V_{s_1} \cup \cdots \cup V_{s_k}$. Set $M = M_{s_1} \cap \cdots \cap M_{s_k}$. Then we have $Z_N = T_M$. Thus Z_N is Zariski closed.

Now if $Z_N \neq \emptyset$, we take $s \in Z_N$. We chose $N' \subset N$ such that $\varrho(\mathrm{Im}[\pi_1(f^{-1}(s)) \to \pi_1(X)]) \cap N' = \{1\}$. Then $s \notin Z_{N'}$. Thus we have $Z_{N'} \subsetneq Z_N$. Starting from $N_0 = \varrho(\pi_1(X))$, we take $N_1, N_2, \ldots$ so that $N_{i+1} = N_i'$, whenever $Z_{N_i} \neq \emptyset$. By the Noetherian property, this sequence should terminate. Hence there exists $N = N_k$ such that $Z_N = \emptyset$. We set $X' = X_N$ to conclude the proof. □

Proposition 4.6. — *Let X be a projective normal variety and let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a representation where K is any field of positive characteristic. Then there exists a finite étale cover $\mu : \widehat{X} \to X$ and a large representation $\tau : \pi_1(\mathrm{Sh}_{\mu^*\varrho}(\widehat{X})) \to \mathrm{GL}_N(K)$ such that $(\mathrm{sh}_{\mu^*\varrho})^*\tau = \mu^*\varrho$, where $\mathrm{sh}_{\mu^*\varrho} : \widehat{X} \to \mathrm{Sh}_{\mu^*\varrho}(\widehat{X})$ is the Shafarevich morphism of $\mu^*\varrho$.*

Proof. — By Theorem 2.9, the Shafarevich morphism $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$ exists, and for each fiber F of sh_ϱ, $\varrho(\mathrm{Im}[\pi_1(F) \to \pi_1(X)])$ is finite. By virtue of Lemma 4.5, there exists a finite étale cover $\mu : \widehat{X} \to X$ such that considering the Stein factorization $\widehat{X} \xrightarrow{g} S \to \mathrm{Sh}_\varrho(X)$ of the composite $\mathrm{sh}_\varrho \circ \mu$, for every fiber G of g, the image $\mu^*\varrho(\mathrm{Im}[\pi_1(G) \to \pi_1(\widehat{X})])$ is trivial. Applying Lemma 4.2, we conclude that there exists a representation $\tau : \pi_1(S) \to \mathrm{GL}_N(K)$ such that $g^*\tau = \mu^*\varrho$.

We note that for each fiber G of g, $g(G)$ is a point if and only if $\mu^*\varrho(\mathrm{Im}[\pi_1(G^{\mathrm{norm}}) \to \pi_1(\widehat{X})])$ is finite. By the unicity of the Shafarevich morphism, $g : \widehat{X} \to S$ is identified with the Shafarevich morphism $\mathrm{sh}_{\mu^*\varrho} : \widehat{X} \to \mathrm{Sh}_{\mu^*\varrho}(\widehat{X})$ of $\mu^*\varrho$.

Let us prove that τ is large. Let $Z \subset S$ be a positive dimensional closed subvariety. Let $W \subset g^{-1}(Z)$ be an irreducible component that is dominant over Z. It follows that $\mu^*\varrho(\mathrm{Im}[\pi_1(W^{\mathrm{norm}}) \to \pi_1(\widehat{X})])$ is infinite by Theorem 2.9. Since $g^*\tau = \mu^*\varrho$, it follows that

$$\mu^*\varrho(\mathrm{Im}[\pi_1(W^{\mathrm{norm}}) \to \pi_1(\widehat{X})]) \subset \tau(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(S)])$$

is infinite. This implies that τ is a large representation. □

4.2. On the conjecture by Claudon-Höring-Kollár. — Let us state and prove the main result of this section.

Theorem 4.7. — *Let X be a smooth projective variety with an infinite fundamental group $\pi_1(X)$, such that its universal covering $\widetilde{X}$ is a Zariski open subset of some compact Kähler manifold $\overline{\widetilde{X}}$. If there exists a faithful representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$, where K is any field, then the Albanese map of X is (up to finite étale cover) locally isotrivial with simply connected fiber F. In particular we have $\widetilde{X} \simeq F \times \mathbb{C}^{q(X)}$ with $q(X)$ the irregularity of X.*

The above theorem therefore confirms Conjecture 4.1 for projective varieties whose fundamental groups are linear in any characteristic.

Proof of Theorem 4.7. — We may assume that K is algebraically closed. Let G be the Zariski closure of $\varrho(\pi_1(X))$. After replacing X by a finite étale cover, we may assume that G is connected. Let $R(G)$ be the radical of G.

Step 1: we prove that G is solvable. Let $H := G/R(G)$, which is semisimple. Then ϱ induces a Zariski dense representation $\sigma : \pi_1(X) \to H(K)$. It is noteworthy that the Shafarevich morphism $\mathrm{sh}_\sigma : X \to \mathrm{Sh}_\sigma(X)$ of σ also exists in the case where $\mathrm{char}\, K = 0$ and is algebraic by the work [Eys04, DYK23]. By the property of the Shafarevich morphism in Theorem 2.9 for K positive characteristic and [DYK23, Theorem A] for K characteristic zero, each fiber F of sh_σ is connected and $\sigma(\mathrm{Im}[\pi_1(F) \to \pi_1(X)])$ is finite.

By Proposition 4.6, after we replace X by a finite étale cover, σ factors through its Shafarevich morphism. Namely, there exists a *large* representation $\tau : \pi_1(\mathrm{Sh}_\sigma(X)) \to \mathrm{GL}_N(K)$ such that $\sigma = \mathrm{sh}^*\tau$.

Claim 4.8. — *The group H is trivial.*

Proof. — Given that G is connected, it follows that $G/R(G)$ is also connected. Consequently, to prove that H is trivial, it suffices to show that $\dim H = 0$. Assume for the sake of contradiction that $\dim H > 0$.

Since $\sigma(\pi_1(X))$ is Zariski dense in H, $\sigma(\pi_1(X))$ is infinite, and thus $\dim \mathrm{Sh}_\sigma(X) > 0$. By Theorem 3.7 for positive characteristic K and [DYK23, Theorem 0.1] for K characteristic zero, we conclude that $\mathrm{Sh}_\sigma(X)$ is pseudo Picard hyperbolic.

Consider the surjective holomorphic map $h : \widetilde{X} \to \mathrm{Sh}_\sigma(X)$, which is the composition of $\mathrm{sh}_\sigma : X \to \mathrm{Sh}_\sigma(X)$ with the universal covering $\pi : \widetilde{X} \to X$. Given that $\widetilde{X}$ is a Zariski open subset of a compact Kähler manifold $\overline{\widetilde{X}}$, according to the property of pseudo Picard hyperbolicity proven in [Den23, Proposition 4.2], h can be extended to a meromorphic map $\bar{h} : \overline{\widetilde{X}} \dashrightarrow \mathrm{Sh}_\sigma(X)$. By blowing up the boundary $\overline{\widetilde{X}} \backslash \widetilde{X}$, we can assume that $\bar{h}$ is holomorphic.

(4.2)
$$
\begin{tikzcd}
\overline{\widetilde{X}} \arrow[drr, bend right, "\bar{h}"'] & \widetilde{X} \arrow[l, hook'] \arrow[r, "\pi"] \arrow[dr, "h"'] & X \arrow[d, "\mathrm{sh}_\sigma"] \\
& & \mathrm{Sh}_\sigma(X)
\end{tikzcd}
$$

Now, consider a general fiber F given by $\mathrm{sh}_\sigma^{-1}(y)$ with $y \in \mathrm{Sh}_\sigma(X)$, which is smooth and connected. As $\sigma(\mathrm{Im}[\pi_1(F) \to \pi_1(X)])$ is trivial, and $\sigma(\pi_1(X))$ is infinite, it implies that $\mathrm{Im}[\pi_1(F) \to \pi_1(X)]$ has infinite index in $\pi_1(X)$. Therefore, $\pi_0(\pi^{-1}(F))$ is infinite.

We note that $\pi^{-1}(F) = \widetilde{X} \cap \bar{h}^{-1}(y)$. Since $\overline{\widetilde{X}}$ is compact, we can deduce that $\pi_0(\pi^{-1}(F))$ is finite, leading to the contradiction. Hence H is trivial. □

By Claim 4.8, it follows that $G = R(G)$ is solvable.

Step 2: we prove that $\pi_1(X)$ is virtually abelian in cases where $\mathrm{char}\, K = 0$.

Claim 4.9. — *The Albanese map is surjective for every finite étale cover of X.*

Proof. — We replace X by any finite étale cover and would like to prove that its Albanese map $a : X \to A$ is surjective. Assume for the sake of contradiction that a is not surjective. By the universal property of the Albanese map, $a(X)$ is not a translation of abelian subvariety and thus the Kodaira dimension $\kappa(a(X)) > 0$. Let $B \subset A$ be the stabilizer of $a(X)$. We consider the morphism $c : X \to C = A/B$ which is the composition of a and the quotient $A \to A/B$. Then $Y = c(X) \subsetneq C$ is general type. Hence Y is pseudo Picard hyperbolic by [Nog81, Theorem 4.5].

Denote by $f : \widetilde{X} \to Y$ the composition of the universal covering $\pi : \widetilde{X} \to X$ and $c : X \to Y$. We claim that for each $y \in Y$, the number of connected components of $f^{-1}(y)$ is infinite. Indeed, f factors as $\widetilde{X} \xrightarrow{\tilde{f}} \widetilde{Y} \xrightarrow{\pi'} Y$, where $\pi' : \widetilde{Y} \to Y$ is the universal covering of Y, and $\tilde{f} : \widetilde{X} \to \widetilde{Y}$ is the holomorphic map lifting f. Since $\pi_1(Y)$ is infinite, $\pi'^{-1}(y)$ is infinite. Since $c : X \to Y$ is a surjective morphism with connected fibers, $c_* : \pi_1(X) \to \pi_1(Y)$ is surjective. Hence the fiber product $X \times_Y \widetilde{Y}$ is connected by [DYK23, Claim 3.44]. Since $\tilde{f}$ factors as $\widetilde{X} \to X \times_Y \widetilde{Y} \to \widetilde{Y}$, it implies that $\tilde{f}$ is surjective. Hence $\tilde{f}^{-1}(\pi'^{-1}(y))$ has infinitely many connected component. Consequently,

$$\pi_0(f^{-1}(y)) = \pi_0((\pi' \circ \tilde{f})^{-1}(y)),$$

is an infinite set.

Since Y is pseudo Picard hyperbolic, f extends to a meromorphic map $g : \overline{\widetilde{X}} \dashrightarrow Y$. We can assume that g is holomorphic after we blow-up the boundary $\overline{\widetilde{X}} \backslash \widetilde{X}$. We note that $f^{-1}(y) = \widetilde{X} \cap g^{-1}(y)$. Since $\overline{\widetilde{X}}$ is compact, we can deduce that $\pi_0(\pi^{-1}(F))$ is finite, leading to the contradiction. Hence the Albanese map $a : X \to A$ is surjective. □

By Claim 4.9, we apply [Cam04, Theorem 7.4] by Campana to conclude that every linear solvable quotient of $\pi_1(X)$ in characteristic zero is virtually abelian. Since ϱ is faithful and G is solvable, it follows that, up to some étale cover, the image $\varrho(\pi_1(X))$, hence $\pi_1(X)$ is abelian.

Step 3: we prove that $\pi_1(X)$ is virtually abelian when $\operatorname{char} K > 0$. Since G is solvable and ϱ is faithful, it follows that $\pi_1(X)$ is solvable. By a theorem of Delzant [Del10, Théorème 1.4], $\pi_1(X)$ is virtually nilpotent. Thanks to Lemma 4.10 below, we conclude that $\varrho(\pi_1(X))$, hence $\pi_1(X)$ is virtually abelian.

Step 4. Completion of the proof. By Step 2 for $\operatorname{char} K = 0$ and Step 3 for $\operatorname{char} K > 0$, $\pi_1(X)$ is virtually abelian. By [CH13, Theorem 1.5], replacing X by a suitable finite étale cover, its Albanese map is a locally trivial fibration with simply connected fiber. We accomplish the proof of the theorem. □

Lemma 4.10. — *Let $\Gamma \subset \mathrm{GL}_N(K)$ be a finitely generated subgroup where K is an algebraically closed field of positive characteristic. If Γ is virtually nilpotent, then it is virtually abelian.*

Proof. — After replacing Γ by a suitable finite index subgroup, we can assume that the Zariski closure G of Γ in GL_N is connected and nilpotent. Hence we have $\mathscr{D}G \subset R_u(G)$, where $R_u(G)$ is the unipotent radical of G and $\mathscr{D}G$ is the the derived group of G. Consequently, We have

$$[\Gamma, \Gamma] \subset [G(K), G(K)] \subset R_u(G)(K).$$

Note that $R_u(G)$ is unipotent. It is thus a successive extension of $\mathbb{G}_{a,K}$.

Since Γ is finitely generated and nilpotent, it follows that $[\Gamma, \Gamma]$ is also finitely generated by [ST00, Corollary 5.45]. From this fact we conclude that $[\Gamma, \Gamma]$ is thus a successive extension of finitely generated subgroups of $\mathbb{G}_{a,K}$. Since finitely generated p-groups are finite, it implies that $[\Gamma, \Gamma]$ is a finite group. Since Γ is residually finite by Malcev's theorem, it has a finite index normal subgroup Γ_1 such that $\Gamma_1 \cap [\Gamma, \Gamma] = \{e\}$. Consequently, $[\Gamma_1, \Gamma_1] \subset [\Gamma, \Gamma] \cap \Gamma_1 = \{e\}$. Hence Γ_1 is abelian. We conclude that Γ is virtually abelian. □

5. On Campana's abelianity conjecture

As another application of Theorem E, in this section we prove Campana's abelianity conjecture in the context of representations in positive characteristic.

5.1. Special and h-special varieties: properties and conjectures. — We first recall the definition of special varieties by Campana [Cam04, Cam11].

***Definition 5.1* (Campana's specialness). —** Let X be a quasi-projective normal variety.

(i) X is *weakly special* if for any finite étale cover $\widehat{X} \to X$ and any proper birational modification $\widehat{X}' \to \widehat{X}$, there exists no dominant morphism $\widehat{X}' \to Y$ with connected general fibers such that Y is a positive-dimensional quasi-projective variety of log general type.
(ii) X is *special* if for any proper birational modification $X' \to X$ there is no dominant morphism $X' \to Y$ with connected general fibers over a positive-dimensional quasi-projective variety Y such that the *Campana orbifold base* (or simply orbifold base) is of log general type.
(iii) X is *Brody special* if it contains a Zariski dense entire curve.

Campana defined X to be *H-special* if X has vanishing Kobayashi pseudo-distance. Motivated by [Cam11, 11.3 (5)], in [CDY22, Definition 1.11] we introduce the following definition.

***Definition 5.2* (h-special). —** Let X be a smooth quasi-projective variety. We define the equivalence relation $x \sim y$ of two points $x, y \in X$ iff there exists a sequence of holomorphic maps $f_1, \ldots, f_l : \mathbb{C} \to X$ such that letting $Z_i \subset X$ to be the Zariski closure of $f_i(\mathbb{C})$, we have

$$x \in Z_1, Z_1 \cap Z_2 \neq \emptyset, \ldots, Z_{l-1} \cap Z_l \neq \emptyset, y \in Z_l.$$

We set $R = \{(x, y) \in X \times X; x \sim y\}$. We define X to be *hyperbolically special* (h-special for short) iff $R \subset X \times X$ is Zariski dense.

By definition, rationally connected projective varieties are h-special without referring to a theorem of Campana and Winkelmann [CW16], who proved that all rationally connected projective varieties contain Zariski dense entire curves. It also has the following properties.

***Lemma 5.3* ([CDY22, Lemmas 10.2 & 10.3 & 10.4]). —** (i) *If a smooth quasi-projective variety X is Brody special, then it is h-special.*
(ii) *Let X be an h-special quasi-projective variety. Let S be a quasi-projective variety and let $p : X \to S$ be a dominant morphism. Then S is h-special.*
(iii) *Let X be an h-special smooth quasi-projective variety, and let $p : X' \to X$ be a finite étale morphism or proper birational morphism from a quasi-projective variety X'. Then X' is h-special.* □

***Proposition 5.4* ([CDY22, Proposition 11.14]). —** *If a quasi-projective smooth variety X is special or h-special, the quasi-albanese map $a : X \to A$ of X is π_1-exact, i.e., we have the following exact sequence:*

$$\pi_1(F) \to \pi_1(X) \to \pi_1(A) \to 1,$$

where F is a general fiber of a. □

In [Cam04, Cam11], Campana proposed the following tantalizing abelianity conjecture.

***Conjecture 5.5* (Campana). —** *A special smooth projective variety has virtually abelian fundamental group.*

In [CDY22] we observed that Conjecture 5.5 fails for quasi-projective variety. As illustrated in [CDY22, Example 11.26], we constructed a smooth quasi-projective variety such that it is both special and Brody special, yet it has nilpotent fundamental group that is not virtually abelian. Consequently, within the quasi-projective context, we revised Conjecture 5.5 as follows.

***Conjecture 5.6* ([CDY22, Conjecture 1.14]). —** *A special or h-special smooth quasi-projective variety has virtually nilpotent fundamental group.*

In [CDY22], we confirm Conjecture 5.6 for quasi-projective varieties with linear fundamental groups in characteristic zero.

***Theorem 5.7* ([CDY22, Theorem E]). —** *Let X be a special or h-special smooth quasiprojective variety. Let $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ be a linear representation. Then $\varrho(\pi_1(X))$ is virtually nilpotent.* □

By [CDY22, Example 11.26], Theorem 5.7 is shown to be sharp. Surprisingly, in the context of representations in positive characteristic, we can obtain a stronger result.

5.2. A factorization result (II). — In this subsection we prove another factorization result, which does not require the variety to be compact.

Proposition 5.8. — *Let $f : X \to Y$ be a dominant morphism from a smooth quasi-projective variety X to a normal quasi-projective variety Y such that general fibers of f are connected. Let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ where K is any field. Assume that for a general smooth fiber F of f, $\varrho(\mathrm{Im}[\pi_1(F) \to \pi_1(X)])$ is finite. Then there exists*

— *a generically finite proper surjective morphism $\mu : X_1 \to X$ obtained by the composition of birational modifications and finite étale Galois covers;*
— *a dominant morphism $f_1 : X_1 \to Y_1$ onto a smooth quasi-projective variety Y_1 with connected general fibers;*
— *a generically finite dominant morphism $\nu : Y_1 \to Y$;*
— *a representation $\tau : \pi_1(Y_1) \to \mathrm{GL}_N(K)$,*

such that $f_1^*\tau = \mu^*\varrho$ and we have the following commutative diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\mu} & X \\ \big\downarrow{\scriptstyle f_1} & & \big\downarrow{\scriptstyle f} \\ Y_1 & \xrightarrow{\nu} & Y \end{array}$$

Proof. — Step 1: we can assume that $\varrho(\mathrm{Im}[\pi_1(F) \to \pi_1(X)])$ is trivial. Let Y° be the Zariski open subset of Y such that it is smooth and f is smooth over Y°. Denote by $X^\circ := f^{-1}(Y^\circ)$. We take a fiber $F := f(y)$ with $y \in Y^\circ$, which is smooth and connected. Since $\Gamma := \varrho(\pi_1(X))$ is residually finite by Malcev's theorem, we can find a finite index normal subgroup $N \triangleleft \Gamma$ such that

$$N \cap \varrho(\mathrm{Im}[\pi_1(F) \to \pi_1(X)]) = \{e\}.$$

Let $\mu : X_1 \to X$ be a finite étale cover such that

$$\mu^*\varrho(\pi_1(X_1)) = N.$$

Let $X_1 \xrightarrow{f_1} Y_1 \xrightarrow{\nu} Y$ be the quasi-Stein factorization of $f \circ \mu$.

$$\begin{array}{ccc} X_1 & \xrightarrow{\mu} & X \\ \big\downarrow{\scriptstyle f_1} & & \big\downarrow{\scriptstyle f} \\ Y_1 & \xrightarrow{\nu} & Y \end{array}$$

Then f_1 is smooth over $Y_1^\circ := \nu^{-1}(Y^\circ)$. Take any point $y_1 \in \nu^{-1}(y)$. Then the fiber $F_1 := f^{-1}(y_1)$ is smooth and

$$\mu^*\varrho(\mathrm{Im}[\pi_1(F_1) \to \pi_1(X_1)]) \subset N \cap \varrho(\mathrm{Im}[\pi_1(F) \to \pi_1(X)]) = \{e\}$$

as $\mu(F_1) \subset F$.

In the following, to lighten the notation we will replace X, Y, f and ϱ by X_1, Y_1, f_1 and $\mu^*\varrho$ respectively, and assume that $\varrho(\mathrm{Im}[\pi_1(F) \to \pi_1(X)])$ is trivial for a general smooth fiber F of f.

Step 2. Compactifications and first reduction step. We take a partial smooth compactification $\overline{X}$ of X such that f extends to a projective surjective morphism $\bar{f} : \overline{X} \to Y$ with connected fibers.

Claim 5.9. — *We may assume that $\bar{f} : \overline{X} \to Y$ is equidimensional.*

Indeed, by Hironaka-Gruson-Raynaud's flattening theorem, there is a birational proper morphism $Y_1 \to Y$ from a quasi-projective manifold Y_1 such that for the irreducible component T of $\overline{X} \times_Y Y_1$ which dominates Y_1, the induced morphism $f_T := T \to Y_1$ is surjective, proper and flat. In particular, the fibers of f_T are equidimensional. Consider the normalization map $\nu : \overline{X}_1 \to T$. Then the induced morphism $f_1 : \overline{X}_1 \to Y_1$ still has equidimensional fibers. Write $\mu : \overline{X}_1 \to \overline{X}$ for the induced proper birational morphism, and let $X_1 := \mu^{-1}(X)$. Note that $\pi_1(X_1) \to \pi_1(X)$ is an isomorphism.

Then one has a diagram

$$\begin{array}{ccc} X_1 & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y_1 & \longrightarrow & Y \end{array}$$

where the horizontal maps are proper birational, and the two spaces on the left satisfy the hypotheses of the proposition if we take the representation induced on $\pi_1(X_1)$. Clearly, it suffices to show the result where X (resp. Y) is replaced by X_1 (resp. Y_1). In the following, we may also replace $\overline{X}$ (resp. Y) by $\overline{X}_1$ and Y (resp. Y_1).

Step 3. Induced representation on an open subset of Y. Consider a Zariski open set $Y^\circ \subset Y$ such that $X^\circ := f^{-1}(Y^\circ)$ is a topologically locally trivial fibration over Y°. We may assume that $Y \backslash Y^\circ$ is simple normal crossing. Then we have a short exact sequence

$$\pi_1(F) \to \pi_1(X^\circ) \to \pi_1(Y^\circ) \to 0$$

where F is a general fiber of f over Y°. By Step 1, $\varrho(\mathrm{Im}[\pi_1(F) \to \pi_1(X)])$ is trivial. Hence we can pass to the quotient, which yields a representation $\tau_1 : \pi_1(Y^\circ) \to \mathrm{GL}_N(K)$ so that $\varrho|_{\pi_1(X^\circ)} = f^*\tau_1$.

Step 4. Reducing Y, we may assume that all divisorial components of $Y - Y^\circ$ intersect $f(X)$. Denote by E the sum of prime divisors of Y contained in the complement $Y \backslash Y^\circ$. We decompose $E = E_1 + E_2$ so that E_1 is the sum of prime divisors of E that do not intersect $f(X)$. We replace Y by $Y \backslash E_1$. Then for any prime divisor P contained in $Y \backslash Y^\circ$, $f^{-1}(P) \cap X \neq \varnothing$.

Step 5. Second use of Malcev's theorem. Recall that $Y \backslash Y^\circ$ is a simple normal crossing divisor $D := \sum_{i \in I} D_i$ and $f^{-1}(D_i) \neq \varnothing$ for each $i \in I$. Since $\bar{f} : \overline{X} \to Y$ is equidimensional, then for any prime component P of $f^{-1}(D_i)$, the morphism $f|_P : P \to D_i$ is dominant. Also, since X is normal, X is smooth at the general points of P.

This allows to find a smooth point x in P (resp. $y \in D_i \backslash \bigcup_{j \in I; j \neq i} D_j$) with local coordinates $(z_1, \ldots z_m)$ (resp. $(w_1, \ldots, w_n)$) around x (resp. y), such that around x (resp. around y) we have $P = (z_1 = 0)$ (resp. $D = (w_1 = 0)$) and $f^*(w_1) = z_1^k$ for some $k \geq 1$. Hence the small meridian loop γ around the general point of P is mapped to η_i^k where η_i is the small meridian loop around D_i. On the other hand, since γ is trivial in $\pi_1(X)$, it follows that

$$0 = \varrho(\gamma) = \tau_1(\eta_i^k).$$

Hence $\tau_1(\eta_i)$ is a torsion element. Let T be the finite subgroup of $\tau_1(\pi_1(Y^\circ))$ generated by $\{\tau_1(\eta_i)\}_{i \in I}$.

By Malcev's theorem again, we know that $\Gamma := \tau_1(\pi_1(Y^\circ)) = \varrho(\pi_1(X))$ is residually finite. Then it has a normal subgroup N with finite index such that $N \cap T = \{e\}$. Let $\mu : X_1 \to X$ be the finite Galois étale cover such that $\mu^*\varrho(\pi_1(X_1)) = N$.

Step 6. Completion of the proof. Let $X_1 \xrightarrow{f_1} Y_1 \xrightarrow{\nu} Y$ be the quasi-Stein factorization of $f \circ \mu$. Let $Y_1^\circ := \nu^{-1}(Y^\circ)$ and $X_1^\circ := f_1^{-1}(Y_1^\circ)$. Since f is smooth over Y°, it follows that f_1 is smooth over Y_1°. We note that for the representation $\nu^*\tau_1 : \pi_1(Y_1^\circ) \to \mathrm{GL}_N(K)$, its image

$$\nu^*\tau_1(\pi_1(Y_1^\circ)) = \mu^*\varrho(\pi_1(X_1)) = N.$$

Let $a : \mathbb{D} \to Y_1$ be any disk such that $a^{-1}(D) = \{0\}$. Then $a_*\pi_1(\mathbb{D}^*)$ lies in some conjugation class of T. Since $N \cap T = \{e\}$ and given that N is a normal subgroup of Γ, it implies that $(\nu \circ a)^*\tau_1(\pi_1(\mathbb{D}^*)) = \{e\}$. Consequently, $\nu := \nu^*\tau_1$ extends to a representation $\tau : \pi_1(Y_1) \to \mathrm{GL}_N(K)$. Note that $f_1^*\nu = \mu^*\varrho$. We now blow-up X_1 and Y_1 such that we can assume that both X_1 and Y_1 are smooth and f_1 is still a morphism. The proposition is proved. □

***Remark 5.10.* —** Proposition 5.8 is proven in [CDY22, Proposition 2.5] in cases where $\mathrm{char}\, K = 0$ and the proof utilizes Selberg's lemma: a finitely generated linear group in characteristic zero is virtually torsion free.

5.3. On the Abelianity conjecture. —

Theorem 5.11. — *Let X be a smooth quasi-projective variety, and let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be any representation with K any field of positive characteristic. If X is special or h-special, then $\varrho(\pi_1(X))$ is virtually abelian.*

Proof. — *Step 1. We prove that $\varrho(\pi_1(X))$ is solvable.* We may assume that K is algebraically closed. Let G be the Zariski closure of $\varrho(\pi_1(X))$. Note that any finite étale cover of a special (resp. h-special) variety is still special (resp. h-special). After replacing X by a finite étale cover, we may assume that G is connected. Let $R(G)$ be the radical of G. Let $H := G/R(G)$, which is semisimple. If $\dim H > 0$, then ϱ induces a Zariski dense representation $\sigma : \pi_1(X) \to H(K)$. Let $\mathrm{sh}_\sigma : X \to \mathrm{Sh}_\sigma(X)$ be the Shafarevich morphism of σ. By the property of the Shafarevich morphism in Theorem 2.9, a general fiber F of sh_σ is connected and $\sigma(\mathrm{Im}[\pi_1(F) \to \pi_1(X)])$ is finite. We apply Proposition 5.8 to conclude that there exist

(i) a generically finite proper surjective morphism $\mu : X_1 \to X$ from a smooth quasi-projective variety obtained by the composition of birational modifications and finite étale Galois covers;
(ii) a generically finite dominant morphism $\nu : Y_1 \to \mathrm{Sh}_\sigma(X)$;
(iii) a dominant morphism $f_1 : X_1 \to Y_1$ with Y_1 a smooth quasi-projective variety with connected general fibers;
(iv) a representation $\tau : \pi_1(Y_1) \to H(K)$

such that we have following commutative diagram

$$\begin{tikzcd} X_1 \arrow[r, "\mu"] \arrow[d, "f_1"] & X \arrow[d, "\mathrm{sh}_\sigma"] \\ Y_1 \arrow[r, "\nu"] & \mathrm{Sh}_\sigma(X) \end{tikzcd}$$

and $\mu^*\sigma = f_1^*\tau$. We can show that τ is a big representation. Thanks to Theorem 3.7, Y_1 is of log general type and pseudo Picard hyperbolic. This leads to a contradiction since X is special (thus weakly special by [Cam11]) or h-special. Hence $G = R(G)$.

Step 2. We prove that $\varrho(\pi_1(X))$ is virtually abelian. Note that any finite étale cover of a special (resp. h-special) variety is still special (resp. h-special) by [Cam04] and Lemma 5.3. Replacing X by a finite étale cover, we may assume that $\pi_1(X)^{ab} \to \pi_1(A)$ is an isomorphism, where $\pi_1(X)^{ab} := \pi_1(X)/[\pi_1(X), \pi_1(X)]$. Since X is special or h-special, by Proposition 5.4, the quasi-albanese map $a : X \to A$ of X is π_1-exact, i.e., we have the following exact sequence:

$$\pi_1(F) \to \pi_1(X) \to \pi_1(A) \to 1,$$

where F is a general fiber of a. Hence $[\pi_1(X), \pi_1(X)]$ is the image of $\pi_1(F) \to \pi_1(X)$, which is thus finitely generated. It implies that $[\varrho(\pi_1(X)), \varrho(\pi_1(X))] = \varrho([\pi_1(X), \pi_1(X)])$ is also finitely generated. By Step 1, G is solvable. Hence we have $\mathcal{D}G \subset R_u(G)$, where $R_u(G)$ is the unipotent radical of G and $\mathcal{D}G$ is the the derived group of G. Consequently, we have

$$[\varrho(\pi_1(X)), \varrho(\pi_1(X))] \subset [G(K), G(K)] \subset R_u(G)(K).$$

Note that every subgroup of finite index in $[\pi_1(X), \pi_1(X)]$ is also finitely generated (cf. [ST00, Proposition 4.17]. By the same arguments in Lemma 4.10, we conclude that $[\varrho(\pi_1(X)), \varrho(\pi_1(X))]$ is finite. Hence $\varrho(\pi_1(X))$ is virtually abelian.

□

5.4. A characterization of semiabelian varieties. — Theorem 5.11 allows us to give a characterization of semiabelian varieties.

Proposition 5.12. — *Let Y be a smooth quasi-projective variety, and let $\varrho : \pi_1(Y) \to \mathrm{GL}_N(K)$ be a big representation where K is a field of positive characteristic.*

(i) *If Y is special or h-special, then there exists a finite étale cover X of Y, such that its Albanese map $\alpha : X \to A$ is birational and $\alpha_* : \pi_1(X) \to \pi_1(A)$ is an isomorphism.*

(ii) *If the logarithmic Kodaira dimension $\bar{\kappa}(Y) = 0$, then there exists a finite étale cover X of Y, such that its Albanese map $\alpha : X \to A$ is birational and proper in codimension one, i.e. there exists a Zariski closed subset $Z \subset A$ of codimension at least two such that α is proper over $A \backslash Z$.*

The proof closely follows that of [CDY22, Proposition 12.7], and we present it here for the sake of completeness.

Proof. — Proof of (i). By Theorem 5.11, there is a finite étale cover X of Y such that $G := \varrho(\pi_1(X))$ is abelian and torsion free. It follows that $\varrho|_{\pi_1(X)} : \pi_1(X) \to \mathrm{GL}_N(K)$ factors through $H_1(X,\mathbb{Z})/\text{torsion}$. By [CDY22, Lemma 11.5], α is dominant with connected general fibers. Since $\alpha_* : H_1(X,\mathbb{Z})/\text{torsion} \to H_1(A,\mathbb{Z})$ is isomorphic, ϱ further factors through $H_1(A,\mathbb{Z})$.

(5.1)
$$\begin{tikzcd}
 & \pi_1(X) \arrow[r, "\rho|_{\pi_1(X)}"] \arrow[d] \arrow[dl, bend right] & G \\
 & H_1(X,\mathbb{Z})/\text{torsion} \arrow[ur] \arrow[d, "\alpha_*"'] & \\
\pi_1(A) \arrow[r, "="] & H_1(A,\mathbb{Z}) \arrow[uur, bend right, "\beta"'] &
\end{tikzcd}$$

From the above diagram for every fiber F of α, $\varrho(\mathrm{Im}[\pi_1(F) \to \pi_1(X)])$ is trivial. Since ϱ is big, the general fiber of α is thus a point. Hence α is birational. Since $\alpha : X \to A$ is π_1-exact by Proposition 5.4, it follows that $\alpha_* : \pi_1(X) \to \pi_1(A)$ is an isomorphism.

Proof of (ii). If $\bar{\kappa}(Y) = 0$, then Y is special by [Cam11, Corollary 5.6] and for any finite étale cover X of Y, we have $\bar{\kappa}(X) = 0$ by [CDY22, Lemma 6.7]. By Proposition 5.12.(i), there is a finite étale cover X of Y such that its Albanese map $\alpha : X \to A$ is birational. We apply [CDY22, Lemma 3.2] to conclude that α is proper in codimension one. □

6. A structure theorem: on a conjecture by Kollár

In [Kol95, Conjecture 4.18], Kollár raised the following conjecture on the structure of varieties with big fundamental group.

Conjecture 6.1. — *Let X be a smooth projective variety with big fundamental group such that $0 < \kappa(X) < \dim X$. Then X has a finite étale cover $p : X' \to X$ such that X' is birational to a smooth family of abelian varieties over a projective variety of general type Z which has big fundamental group.*

In this section we address Conjecture 6.1. Our theorem is the following:

Theorem 6.2. — *Let X be a quasi-projective normal variety and let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a big representation where K is a field of positive characteristic. Then*

(i) *the logarithmic Kodaira dimension satsifies $\bar{\kappa}(X) \geq 0$.*
(ii) *There is a proper Zariski closed subset Ξ of X such that each non-constant morphism $\mathbb{A}^1 \to X$ has image in Ξ.*
(iii) *If $0 < \bar{\kappa}(X) < \dim X$, after replacing X by a finite étale cover and a birational modification, there are a semi-abelian variety A, a quasi-projective manifold V and a birational morphism $a : X \to V$ such that the following commutative diagram holds:*

$$\begin{tikzcd}
X \arrow[rr, "a"] \arrow[dr, "j"'] & & V \arrow[dl, "h"] \\
 & J(X) &
\end{tikzcd}$$

where j is the logarithmic Iitaka fibration of X and $h : V \to J(X)$ is a locally trivial fibration with fibers isomorphic to A. Moreover, for a general fiber F of j, $a|_F : F \to A$ is proper in codimension one.

In our previous work [CDY22, Theorem 12.5], we established Theorem 6.2 for the cases where $\operatorname{char} K = 0$ and ϱ is big and reductive. The proof of Theorem 6.2 closely follows that of [CDY22, Theorem 12.5]

Proof. — We may assume that K is algebraically closed. To prove the theorem we are free to replace X by a birational modification and by a finite étale cover since the logarithmic Kodaira dimension will remain unchanged. We replace ϱ by its semisimplification, which is still big by Lemma 2.1. Hence we might assume that ϱ is big and semisimple. Consequently, after replacing X by a finite étale cover, the Zariski closure G of ϱ is reductive and connected. Let $\mathcal{D}G$ be the derived group of G, which is semisimple. Then $T := G/\mathcal{D}G$ is a torus and the natural morphism $G \to \mathcal{D}G \times T$ is a central isogeny. The induced representation $\varrho' : \pi_1(X) \to \mathcal{D}G(K) \times T(K)$ by ϱ is also big. Consider the representation $\sigma : \pi_1(X) \to \mathcal{D}G(K)$, obtained by composing ϱ' with the projection $\mathcal{D}G \times T \to \mathcal{D}G$. Then $\sigma(\pi_1(X))$ is Zariski dense. Let $\mathrm{sh}_\sigma : X \to \mathrm{Sh}_\sigma(X)$ be the Shafarevich morphism of σ.

Like Step 1 of the proof of Theorem 5.11, we apply Proposition 5.8 to conclude that there exist

(i) a generically finite proper surjective morphism $\mu : X_1 \to X$ from a smooth quasi-projective variety obtained by the composition of birational modifications and finite étale Galois covers;
(ii) a generically finite dominant morphism $\nu : Y_1 \to \mathrm{Sh}_\sigma(X)$;
(iii) a dominant morphism $f_1 : X_1 \to Y_1$ with Y_1 a smooth quasi-projective variety with connected general fibers;
(iv) a big representation $\tau : \pi_1(Y_1) \to \mathcal{D}G(K)$;

such that we have following commutative diagram

$$\begin{tikzcd} X_1 \arrow[r, "\mu"] \arrow[d, "f_1"] & X \arrow[d, "\mathrm{sh}_\sigma"] \\ Y_1 \arrow[r, "\nu"] & \mathrm{Sh}_\sigma(X) \end{tikzcd}$$

and $\mu^*\sigma = f_1^*\tau$. It is straightforward to show that τ is a big representation. Thanks to Theorem 3.7, the special loci $\mathrm{Sp}_{\mathrm{alg}}(Y_1)$ and $\mathrm{Sp}_{\mathrm{p}}(Y_1)$ are both proper Zariski closed subset of Y_1. In particular, Y_1 is of log general type. Note that Y_1 can be a point.

Consider the morphism

$$\begin{aligned} g : X_1 &\to A \times Y_1 \\ x &\mapsto (\alpha(x), f_1(x)). \end{aligned}$$

where $\alpha : X_1 \to A$ is the quasi Albanese map of X_1.

Claim 6.3. — *We have* $\dim X_1 = \dim g(X_1)$.

Proof. — For a general smooth fiber F of f_1, $\mu^*\sigma(\operatorname{Im}[\pi_1(F) \to \pi_1(X)])$ is trivial. Since $\mu^*\varrho' : \pi_1(X_1) \to \mathcal{D}G(K) \times T(K)$ is big, by the construction of σ, we conclude that the representation $\eta : \pi_1(F) \to T(K)$ obtained by

$$\pi_1(F) \to \pi_1(X_1) \overset{\mu^*\varrho'}{\to} \mathcal{D}G(K) \times T(K) \to T(K)$$

is big. Since $T(K)$ is commutative, similar to (5.1), η factors through $\pi_1(F) \to \pi_1(A) \to T(K)$. This implies that $\dim F = \dim \alpha(F)$. Hence $\dim X_1 = \dim g(X_1)$. □

Let us prove Theorem 6.2.(i). Thanks to Claim 6.3, for a general smooth fiber F of f_1, we have $\dim F = \dim \alpha(F)$. Hence $\bar{\kappa}(F) \geq 0$. Since Y_1 is of log general type, by the subadditivity of the logarithmic Kodaira dimension proven in [Fuj17, Theorem 1.9], we obtain

$$\bar{\kappa}(X_1) \geq \bar{\kappa}(Y_1) + \bar{\kappa}(F) \geq \bar{\kappa}(Y_1) = \dim Y_1 \geq 0.$$

Hence $\bar{\kappa}(X) = \bar{\kappa}(X_1) \geq 0$. The first claim is proved.

Let us prove Theorem 6.2.(ii). By Lemma 3.4, the special subset $\mathrm{Sp}(\varrho)$ defined in Definition 5.1 is a proper closed subset of X. Let $\gamma : \mathbb{A}^1 \to X$ be a non-constant algebraic morphism. Then $\gamma^*\varrho(\pi_1(\mathbb{A}^1)) = \{1\}$. By the definition of $\mathrm{Sp}(\varrho)$, we have $\gamma(\mathbb{A}^1) \subset \mathrm{Sp}(\varrho)$.

Finally Theorem 6.2.(iii) follows from [CDY22, Theorem 12.1]. □

7. On the holomorphic convexity of universal covering

In this section we will prove Theorem B.

7.1. Partial Albanese morphism. — For this subsection we refer the readers to [CDY22, Definition 5.19] for details. Let X be a smooth projective variety. Let $\{\eta_1, \ldots, \eta_k\} \subset H^0(X, \Omega_X^1)$ be a set of holomorphic one forms. Consider the Albanese map $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$. Then there exist holomorphic one forms $\{\omega_1, \ldots, \omega_k\} \subset H^0(\mathrm{Alb}(X), \Omega^1_{\mathrm{Alb}(X)})$ such that $\mathrm{alb}_X^*\omega_i = \eta_i$ for each i. Let B be the largest abelian subvariety in $\mathrm{Alb}(X)$ such that $\omega_i|_B \equiv 0$ for each i. Denote by $A := \mathrm{Alb}(X)/B$ the quotient which is also an abelian variety. Then the morphism $a : X \to A$ that is the composite of alb_X and the quotient map $q : \mathrm{Alb}(X) \to A$ is called the *partial Albanese morphism* induced by $\{\eta_1, \ldots, \eta_k\}$.

We remark that there exist holomorphic 1-forms $\{\omega_1', \ldots, \omega_k'\} \subset H^0(A, \Omega_A^1)$ such that $q^*\omega_i' = \omega_i$ for each i. They satisfy the following property:

Lemma 7.1. — *For any positive-dimensional closed subvariety Z of $a(X)$, there exists some ω_i' such that $\omega_i'|_Z \not\equiv 0$.*

Proof. — Since $a : X \to a(X)$ is surjective, there exists an irreducible component W in $a^{-1}(Z)$ that is dominant over Z. By the property of the partial Albanese map proved in [CDY22, Lemma 1.1], there exists some ω_i such that $\omega_i|_W \not\equiv 0$. Since $\omega_i = a^*\omega_i'$, it follows that $\omega_i'|_Z \not\equiv 0$. □

7.2. A recollection of spectral covering and canonical currents. — Let X be a smooth projective variety. Let $\tau : \pi_1(X) \to \mathrm{GL}_N(K)$ be a reductive representation where K is a non-archimedean local field. According to Theorem 1.2, the *Katzarkov-Eyssidieux reduction map* $s_\tau : X \to S_\tau$ of τ exists, fulfilling the properties outlined therein. Since X is projective, $s_\tau : X \to S_\tau$ is unique. We will outline the construction of certain *canonical* positive closed $(1,1)$-currents over S_τ.

We first recall some facts about *spectral forms* as we have already seen in Item (a) in the proof of Theorem 3.1. We refer the readers to [CDY22,DYK23] for more details. Let $\tau : \pi_1(X) \to \mathrm{GL}_N(K)$ be a reductive representation where K is a non-archimedean local field. By the work of Gromov-Schoen [GS92] (see also [BDDM22] in cases where X is non-compact), there exists a τ-equivariant harmonic mapping $u : \widetilde{X} \to \Delta(G)$ where $\Delta(G)$ is the (enlarged) Bruhat-Tits building of G (see [KP23, Definition 4.3.2] for the definition). Such a harmonic map is pluriharmonic, and the $(1,0)$-part of the complexified differentials gives rise to a set of multivauled holomorphic 1-forms over a dense open set of X whose complement has Hausdorff codimension at least two. We can take some finite (ramified) Galois covering $\pi : X^{\mathrm{sp}} \to X$ (so-called *spectral covering*) with the Galois group H and such that the pullback of these multivalued one forms becomes single valued ones $\{\eta_1, \ldots, \eta_k\} \subset H^0(X^{\mathrm{sp}}, \pi^*\Omega_X^1)$, that are called *spectral 1-forms* of τ. Consequently, the Stein factorization of the *partial Albanese morphism* $a : X^{\mathrm{sp}} \to A$ of $\{\eta_1, \ldots, \eta_k\}$ leads to the (Katzarkov-Eyssidieux) reduction map $s_{\pi^*\tau} : X^{\mathrm{sp}} \to S_{\pi^*\tau}$ of $\pi^*\tau$. This map $s_{\pi^*\tau}$ is H-equivariant and its quotient by H gives rise to the reduction map $s_\tau : X \to S_\tau$ by τ. More precisely, we have the following commutative diagram:

$$
\begin{tikzcd}
X^{\mathrm{sp}} \arrow[r, "\pi"] \arrow[d, "s_{\pi^*\tau}"] \arrow[dd, bend right, "a"'] & X \arrow[d, "s_\tau"] \\
S_{\pi^*\tau} \arrow[r, "\sigma_\pi"] \arrow[d, "b"] & S_\tau \\
A &
\end{tikzcd}
$$

Here σ_π is also a finite ramified Galois cover with Galois group H. Note that there are one forms $\{\eta_1', \ldots, \eta_m'\} \subset H^0(A, \Omega_A^1)$ such that $a^*\eta_i' = \eta_i$. We define a smooth positive closed $(1,1)$-form $T_{\pi^*\tau} := b^* \sum_{i=1}^m i\eta_i' \wedge \overline{\eta_i}'$ on $S_{\pi^*\tau}$. Note that $T_{\pi^*\tau}$ is invariant under the Galois action H. Therefore, there is a positive closed $(1,1)$-current T_τ defined on S_τ with continuous local potential such that $\sigma_\pi^* T_\tau = T_{\pi^*\tau}$.

***Definition 7.2* (Canonical current).** — The closed positive $(1,1)$-current T_τ on S_τ is called the *canonical current* of τ.

Lemma 7.3. — *$\{T_\tau\}$ is strictly nef. Namely, for any irreducible curve $C \subset S_\tau$, we have $\{T_\tau\} \cdot C > 0$.*

Proof. — Let $C' \subset \sigma_\pi^{-1}(C)$ be an irreducible component which is dominant over C. Consider its image $b(C')$. By Lemma 7.1, there exists some $\eta_i' \in H^0(A, \Omega_A^1)$ such that $\eta_i'|_{b(C')} \neq 0$. Hence $i\eta_i' \wedge \overline{\eta_i'}|_{b(C')}$ is strictly positive at general points. Consequently, $\{T_\tau\} \cdot C > 0$. □

The canonical current T_ϱ will serve as a lower bound for the complex hessian of plurisubharmonic functions constructed by the method of harmonic mappings.

***Proposition 7.4* ([Eys04, Proposition 3.3.6, Lemme 3.3.12]).** — *Let X be a projective normal variety and let $\varrho : \pi_1(X) \to G(K)$ be a Zariski dense representation where K is a non archimedean local field and G is a reductive group. Let $x_0 \in \Delta(G)$ be an arbitrary point. Let $u : \widetilde{X} \to \Delta(G)$ be the associated harmonic mapping, where $\widetilde{X}$ is the universal covering of X. The function $\phi : \widetilde{X} \to \mathbb{R}_{\geq 0}$ defined by*

$$\phi(x) = 2d^2\big(u(x), u(x_0)\big)$$

satisfies the following properties:

(a) ϕ descends to a function ϕ_ϱ on $\widetilde{X_\varrho} = \widetilde{X}/\ker(\varrho)$.

(b) Let Σ be a normal complex space and $r : \widetilde{X}_\varrho \to \Sigma$ a proper holomorphic fibration such that $s_\varrho \circ \pi : \widetilde{X}_\varrho \to S_\varrho$ factorizes via a morphism $\nu : \Sigma \to S_\varrho$. The function ϕ_ϱ is of the form $\phi_\varrho = \phi_\varrho^\Sigma \circ r$ with ϕ_ϱ^Σ being a continuous plurisubharmonic function on Σ;

(c) $\mathrm{dd^c}\phi_\varrho^\Sigma \geq \nu^* T_\varrho$. □

We require the following criterion for the Stein property of an infinite topological Galois covering of a compact complex normal space.

***Proposition 7.5* ([Eys04, Proposition 4.1.1]).** — *Let S be a compact complex normal space and let $\nu : \Sigma \to S$ be some infinite topological Galois covering. Let T be a closed positive $(1,1)$-current on S with continuous potential such that $\{T\}$ is a Kähler class. Assume that there exists a continuous plurisubharmonic function $\phi : \Sigma \to \mathbb{R}_{\geq 0}$ such that $\mathrm{dd^c}\phi \geq \nu^* T$. Then Σ is a Stein space.* □

7.3. Some lemmas in algebraic groups. — We start with the following lemma.

Lemma 7.6. — *Let Γ_1 be a finite index subgroup of Γ_2. Let G be a linear algebraic group over a field K. Assume that $\Gamma_2 \subset G(K)$ is Zariski dense. Then for the Zariski closure H of Γ_1 in G, we have $H^o = G^o$, where H^o and G^o are the identity components of H and G respectively.*

Proof. — We may assume that K is algebraically closed. Replacing Γ_1 and Γ_2 by some finite index subgroups, we may assume that their Zariski closures are H^o and G^o respectively. Let $x_1\Gamma_1, \ldots, x_k\Gamma_1$ be left cosets of Γ_1 in Γ_2. It follows that $x_i\Gamma_1$ is contained in the Zariski closed subset x_iH. Hence G^o/H^o is finite. Since G^o and H^o are both connected, it implies that $G^o = H^o$. The lemma is proved. □

The following lemma plays a crucial role in the proof of Theorem 7.15. It is a variant of [DYK23, Lemma 4.8] and its proof is based on [CDY22, Lemma 5.3].

Lemma 7.7. — *Let G be a semisimple algebraic group over the non-archimedean local field K. Let $\Gamma \subset G(K)$ be a finitely generated subgroup which is Zariski dense in G. If its derived group $\mathcal{D}\Gamma$ is* bounded, *then Γ is also bounded.*

Proof. — We can replace Γ by a finite index subgroup and assume that G is connected. Since G is semisimple, according to the decomposition theorem [Mil17, Theorem 21.51] there are finitely many almost K-simple normal subgroups $H_1, \ldots, H_k$ of G, such that $H_1 \times \cdots \times H_k \to G$ is a central isogeny. Hence each quotient $G_i := G/H_1 \cdots H_{i-1}H_{i+1} \cdots H_k$ is an almost K-simple algebraic group. Let Γ_i be the image of Γ under the homomorphism $q_i : G(K) \to G_i(K)$. Then

Γ_i is Zariski dense in G_i, and we have $\mathcal{D}\Gamma_i = q_i(\mathcal{D}\Gamma)$. Since $\mathcal{D}\Gamma$ is bounded, by [KP23, Fact 2.2.4], $\mathcal{D}\Gamma_i$ is also bounded.

To show that Γ_i is bounded, we assume contrary that Γ_i is unbounded. Since $\mathcal{D}\Gamma_i$ is bounded and G_i is almost K-simple, we apply [CDY22, Lemma 5.3] to conclude that $\mathcal{D}\Gamma_i$ is finite. Let Γ_i' by a finite index subgroup of Γ_i such that $\mathcal{D}\Gamma_i'$ is trivial. Then Γ_i' is commutative. Hence the Zariski closure of $\Gamma_i' \subset G_i(K)$ is commutative. Since Γ_i is Zariski dense in G_i, by Lemma 7.6 we conclude that the identity component G_i° of G_i is commutative. This contradicts with that G_i is almost K-simple. Hence Γ_i is bounded for each i.

Note that the natural morphism $G \to G_1 \times \cdots \times G_k$ is also an isogeny. As a result, Γ is bounded. □

7.4. Holomorphic convexity of universal covering. — First we prove the following lemma.

***Lemma 7.8.* —** *Let X be a quasi-projective normal variety and let $a : X \to A$ be a finite morphism to a semi-abelian variety A. Then the universal covering $\widetilde{X}$ of X is a Stein space.*

Proof. — Let $\pi_A : \widetilde{A} \to A$ be the universal covering map and let X' be a connected component of $X \times_A \widetilde{A}$. Then we have the following commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\mu} & X \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle a} \\ \widetilde{A} & \xrightarrow{\pi_A} & A \end{array}$$

where $\mu : X' \to X$ and $f : X' \to \widetilde{A}$ are the induced maps. Then μ is étale and f is finite. Hence the image $f(X')$ is a closed subvariety of $\widetilde{A} \simeq \mathbb{C}^{\dim A}$, which is thus a Stein space. Since $f : X' \to f(X')$ is finite, it follows that X' is also Stein. Note that any unramified covering of a Stein space is Stein. Therefore, $\widetilde{X}$ is a Stein space. □

***Theorem 7.9.* —** *Let X_0 be a complex projective normal surface and let p be a fixed prime and N be a fixed positive integer. We assume that $M_{\mathrm{B}}(X_0, N)_{\mathbb{F}_p}$ is large, meaning that for any positive dimensional closed subvariety Z of X_0, there exists a linear representation $\varrho : \pi_1(X_0) \to \mathrm{GL}_N(K)$ with* char $K = p$ *such that $\varrho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X_0)])$ is infinite. Then the universal covering of X_0 is Stein.*

Proof. — Let $\mu : X \to X_0$ be a desingularization. It induces a morphism $\iota : M_{\mathrm{B}}(X_0, N)_{\mathbb{F}_p} \hookrightarrow M_{\mathrm{B}}(X, N)_{\mathbb{F}_p}$ between affine $\mathbb{F}_p$-schemes of finite type which is a closed immersion. Let $M := \iota(M_{\mathrm{B}}(X_0, N)_{\mathbb{F}_p})$ which is a Zariski closed subset of $M_{\mathrm{B}}(X, N)_{\mathbb{F}_p}$. By Theorem 2.7, the Shafarevich morphism $\mathrm{sh}_M : X \to \mathrm{Sh}_M(X)$ of M exists.

***Claim 7.10.* —** *The Shafarevich morphism $\mathrm{sh}_M : X \to \mathrm{Sh}_M(X)$ coincides with $\mu : X \to X_0$.*

Proof. — Let Z be any closed subvariety of X and let $W := \mu(Z)$. If $\mathrm{sh}_M(Z)$ is a point, then $\mu^*\sigma(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)])$ is finite for any $[\sigma] \in M_{\mathrm{B}}(X_0, N)_{\mathbb{F}_p}(L)$. Therefore, $\sigma(\mathrm{Im}[\pi_1(W^{\mathrm{norm}}) \to \pi_1(X_0)])$ is finite. Hence W is a point since $M_{\mathrm{B}}(X_0, N)_{\mathbb{F}_p}$ is assumed to be large.

On the other hand, assume that $\mu(Z)$ is a point. Let $\sigma : \pi_1(X) \to \mathrm{GL}_N(L)$ be any representation where L is a field of characteristic p such that $[\sigma] \in M(L)$. Then there exists a representation $\sigma' : \pi_1(X_0) \to \mathrm{GL}_N(\bar{L})$ such that $[\mu^*\sigma'] = [\sigma]$. By Lemma 2.1, for any subgroup $\Gamma \subset \pi_1(X)$, $\mu^*\sigma'(\Gamma)$ is finite if and only if $\sigma(\Gamma)$ is finite. Since $\mu^*\sigma'(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)])$ is trivial, it follows that $\sigma(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)])$ is finite. By the properties of sh_M proven in Theorem 2.7, $\mathrm{sh}_M(Z)$ is a point. This proves that $\mathrm{sh}_M = \mu$. □

Let $H_0 := \cap_\varrho \ker \varrho$ where $\varrho : \pi_1(X_0) \to \mathrm{GL}_N(L)$ ranges over all linear representation and L is any field with char $L = p$. Denote by $H := \mu^* H_0$. Let $\pi_H : \widetilde{X}_H \to X$ be the Galois covering with the Galois group $\pi_1(X)/H$ and $\pi_0 : \widetilde{X}_0 \to X_0$ be that with $\pi_1(X_0)/H_0$. Consequently, we have the following commutative diagram

$$\begin{tikzcd} \widetilde{X}_H \arrow[r, "\pi_H"] \arrow[d, "f"] & X \arrow[d, "\mu"] \arrow[dr, "\mathrm{sh}_M"] & \\ \widetilde{X}_0 \arrow[r, "\pi_0"] & X_0 \arrow[r, equal] & \mathrm{Sh}_M(X) \end{tikzcd} \tag{7.1}$$

where $f : \widetilde{X}_H \to \widetilde{X}_0$ is a holomorphic proper fibration.

Claim 7.11. — *Let $\sigma : \pi_1(X) \to \mathrm{GL}_N(L)$ be a linear representation such that $[\sigma] \in M(L)$, where L is a field of characteristic p. Then $H \subset \ker(\sigma^{ss})$.*

Proof. — There exists a linear representation $\sigma' : \pi_1(X_0) \to \mathrm{GL}_N(\bar{L})$ such that $[\mu^*\sigma'] = [\sigma]$. Then $(\mu^*\sigma')^{ss}$ is conjugate to σ^{ss}. Hence $\ker(\mu^*\sigma') \subset \ker((\mu^*\sigma')^{ss}) = \ker(\sigma^{ss})$. We have $H \subset \mu^*\ker(\sigma') = \ker(\mu^*\sigma')$. Thus $H \subset \ker(\sigma^{ss})$. □

Consider the affine $\mathbb{F}_p$-scheme $R_{\mathbb{F}_p}$ and the GIT quotient $\pi : R_{\mathbb{F}_p} \to M_{\mathrm{B}}(X,N)_{\mathbb{F}_p}$, as defined in § 2.1. Define $R := \pi^{-1}(M)$, which is an affine $\overline{\mathbb{F}_p}$-scheme of finite type. According to the structure of the Shafarevich morphism described in Theorem 2.7 and Definition 2.6, the Shafarevich morphism $\mathrm{sh}_M : X \to \mathrm{Sh}_M(X)$ of M is obtained through the simultaneous Stein factorization of the reductions $\{s_\tau : X \to S_\tau\}_{[\tau]\in M(K)}$. Here $\tau : \pi_1(X) \to \mathrm{GL}_N(K)$ ranges over all reductive representations with K a local field of characteristic p such that $[\tau] \in M(K)$ and $s_\tau : X \to S_\tau$ is the reduction map defined in Theorem 1.2.

By Lemma 1.6, there exists a reductive representation $\boldsymbol{\tau} : \pi_1(X) \to \mathrm{GL}_{N'}(K')$ with K' a local field of characteristic p such that the Katzarkov-Eyssidieux reduction map $s_{\boldsymbol{\tau}} : X \to S_{\boldsymbol{\tau}}$ of $\boldsymbol{\tau}$ coincides with $\mathrm{Sh}_M : X \to \mathrm{Sh}_M(X)$. Moreover by Lemma 1.6 and Claim 7.11, we have

$$H \subset \ker(\boldsymbol{\tau}). \tag{7.2}$$

Case 1: The spectral 1-forms have rank 2. Assume that the spectral one forms $\{\eta_1,\ldots,\eta_k\} \subset H^0(X^{\mathrm{sp}},\pi^*\Omega_X^1)$ with respect to $\boldsymbol{\tau} : \pi_1(X) \to \mathrm{GL}_{N'}(K')$ has rank 2.

Let $s_{\boldsymbol{\tau}} : X \to S_{\boldsymbol{\tau}}$ be the reduction map of $\boldsymbol{\tau}$, which coincides with $\mathrm{sh}_M : X \to \mathrm{Sh}_M(X)$ (and thus $\mu : X \to X_0$). Let $T_{\boldsymbol{\tau}}$ be the canonical current on $X_0 = S_{\boldsymbol{\tau}}$ defined in Definition 7.2. Then $T_{\boldsymbol{\tau}}$ is strictly positive at general points since the spectral forms associated with $\boldsymbol{\tau}$ has rank 2. Since $T_{\boldsymbol{\tau}}$ has continuous local potentials, it follows that $T_{\boldsymbol{\tau}}$ is a big and nef class.

On the other hand, by Lemma 7.3, we conclude that $T_{\boldsymbol{\tau}}$ is strictly nef. We now apply a theorem by Demailly-Păun [DP04] (or [Lam99]) to conclude that $\{T_{\boldsymbol{\tau}}\}$ is a Kähler class.

According to Proposition 7.4 and (7.2), there exists a continuous plurisubharmonic function $\phi : \widetilde{X}_0 \to \mathbb{R}_{\geq 0}$ such that $\mathrm{dd^c}\phi \geq \pi_0^* T_{\boldsymbol{\tau}}$. We can apply Proposition 7.5 to conclude that $\widetilde{X}_0$ is a Stein space. Note that any unramified covering of a Stein space is Stein. Therefore, the universal covering of X_0 is a Stein space.

Case 2: The spectral 1-forms have rank 1: Assume that the spectral covering $X^{\mathrm{sp}} \to X$ with respect to $\boldsymbol{\tau} : \pi_1(X) \to \mathrm{GL}_{N'}(K')$ has rank 1; i.e., $\eta_i \wedge \eta_j \equiv 0$ for every η_i and η_j, where $\{\eta_1,\ldots,\eta_\ell\} \subset H^0(X^{\mathrm{sp}},\pi^*\Omega_X^1)$ is the spectral forms associated with $\boldsymbol{\tau} : \pi_1(X) \to \mathrm{GL}_{N'}(K')$, and $\pi : X^{\mathrm{sp}} \to X$ is the spectral covering associated with $\boldsymbol{\tau} : \pi_1(X) \to \mathrm{GL}_{N'}(K')$.

Case 2.1: The dimension of spectral 1-forms is at least two: Suppose $\dim_{\mathbb{C}}\mathrm{Span}\{\eta_1,\ldots,\eta_\ell\} \geq 2$.

Without loss of generality, we may assume that $\eta_1 \wedge \eta_2 \equiv 0$ and $\eta_1 \notin \{\mathbb{C}\eta_2\}$. According to the Castelnuovo-De Franchis theorem (cf. [ABC+96, Theorem 2.7]), there exists a proper fibration $h : X^{\mathrm{sp}} \to C$ over a smooth projective curve C such that $\{\eta_1,\eta_2\} \subset h^*H^0(C,\Omega_C^1)$. Since $\mathrm{sh}_M = \mu$ is birational, we can choose a general fiber F of h, which is irreducible and such that $\mathrm{sh}_M \circ \pi(F) = s_{\boldsymbol{\tau}} \circ \pi(F)$ is not a point. There exists some i such that $\eta_i|_F \neq 0$ (cf. Lemma 7.1). Given that $\eta_1|_F \equiv 0$, this implies that $\eta_i \wedge \eta_1 \neq 0$. It contradicts with our assumption that the spectral 1-forms have rank 1. Therefore, this case cannot occur.

Case 2.2: The dimension of spectral 1-forms is 1: We have $\dim_{\mathbb{C}}\mathrm{Span}\{\eta_1, \ldots, \eta_\ell\} = 1$.

Let G be the Zariski closure of $\boldsymbol{\tau}(\pi_1(X))$, which is reductive. Consider the isogeny $g : G \to G/Z \times G/\mathcal{D}G$ where Z is the central torus of G and $\mathcal{D}G$ is the derived group of G. As a result, $G' := G/Z$ is semisimple and $G'' := G/\mathcal{D}G$ is a torus. Let $\tau' : \pi_1(X) \to G'(\overline{K'})$ be the composite of $\boldsymbol{\tau}$ with the projection $G \to G'$, and $\tau'' : \pi_1(X) \to G''(\overline{K'})$ be the composite of $\boldsymbol{\tau}$ with the projection $G \to G''$. Then τ' and τ'' are both Zariski dense representations.

Let $\nu : Y \to X^{\rm sp}$ be a desingularization and denote $\eta := \nu^*\eta_1$. Consider the partial Albanese morphism $a : Y \to A$ induced by η. Then there exists a one form $\eta' \in H^0(A, \Omega_A^1)$ such that $a^*\eta' = \eta$. If $\dim a(Y) = 1$, then the Stein factorization $h : Y \to C$ of a is a proper holomorphic fibration over a smooth projective curve C such that $\eta_1 \in h^*H^0(C, \Omega_C^1)$. We are now in a situation akin to Case 2.1, and we can apply the same arguments to reach a contradiction. Hence $\dim a(Y) = 2$. Let $\pi_A : \widetilde{A} \to A$ denote the universal covering map. We denote by $Y' := Y \times_{\widetilde{A}} A$ a connected component of the fiber product and let $\pi' : Y' \to Y$ be the induced étale cover. It's worth noting that $\pi'^*\eta$ is exact. Consequently, we can define the following holomorphic map:

$$\begin{aligned} h : Y' &\to \mathbb{C} \\ y &\mapsto \int_{y_0}^{y} \pi'^*\eta. \end{aligned}$$

We then have the following commutative diagram:

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "p"] \arrow[rr, bend left, "\pi_Y"] & Y' \arrow[r, "\pi'"] \arrow[d] \arrow[dd, bend right, "h"'] & Y \arrow[d, "a"] \\ & \widetilde{A} \arrow[r, "\pi_A"] \arrow[d] & A \\ & \mathbb{C} & \end{tikzcd}$$

The holomorphic map $\widetilde{A} \to \mathbb{C}$ in the above diagram is defined by the linear 1-form $\pi_A^*\eta'$ on $\widetilde{A}$. By Simpson's Lefschetz theorem [Sim93a], for any $t \in \mathbb{C}$, $h^{-1}(t)$ is connected and we have the surjectivity $\pi_1(h^{-1}(t)) \twoheadrightarrow \pi_1(Y')$. By the definition of h, $\pi_Y^*\eta|_Z \equiv 0$ where Z is any connected component of $p^{-1}(h^{-1}(t))$. Here $p : \widetilde{Y} \to Y'$ is the natural covering map.

Consider the Zariski dense representation $\tau' : \pi_1(X) \to G'\big(\overline{K'}\big)$ as defined previously. Let L be a finite extension of K' such that G' is defined on L and $\tau' : \pi_1(X) \to G'(L)$. We denote by $\sigma : \pi_1(Y) \to G'\big(L\big)$ the pullback of τ' via the morphism $Y \to X$. The existence of a σ-equivariant harmonic mapping $u : \widetilde{Y} \to \Delta(G')$ is guaranteed by [GS92], where $\Delta(G')$ is the Bruhat-Tits building of G'.

We note that $\pi_Y^*\eta$ is the (1,0)-part of the complexified differential of the harmonic mapping u at general points of $\widetilde{Y}$, with $\pi_Y : \widetilde{Y} \to Y$ denoting the universal covering. For any connected component Z of $p^{-1}(h^{-1}(t))$ for a general $t \in \mathbb{C}$, since $\pi_Y^*\eta|_Z \equiv 0$, and all the spectral forms are assumed to be $\mathbb{C}$-linearly equivalent, it follows that $u(Z)$ is constant. Since u is σ-equivariant, it follows that $\pi'^*\sigma(\mathrm{Im}[\pi_1(h^{-1}(t)) \to \pi_1(Y')])$ is contained in the subgroup of $G'(L)$ fixing the point $u(Z)$. Recall that $\pi_1(h^{-1}(t)) \to \pi_1(Y')$ is surjective. Hence $\pi'^*\sigma(\pi_1(Y'))$ is a bounded subgroup of $G'(L)$. Additionally, note that $\mathcal{D}\pi_1(Y) \subset \mathrm{Im}[\pi_1(Y') \to \pi_1(Y)]$, and it follows that $\sigma(\mathcal{D}\pi_1(Y))$ is bounded. Since τ' is Zariski dense, and $\mathrm{Im}[\pi_1(Y) \to \pi_1(X)]$ is a finite index subgroup of $\pi_1(X)$, according to Lemma 7.6 the Zariski closure of $\sigma(\pi_1(Y))$ contains the identity component of G', and it is also semisimple. We apply Lemma 7.7 to conclude that $\sigma(\pi_1(Y))$ is bounded.

Since $\sigma(\pi_1(Y))$ is a finite index subgroup of $\tau'(\pi_1(X))$, it follows that $\tau'(\pi_1(X))$ is also bounded. Then the reduction map $s_{\tau'}$ is the constant map. This implies that the reduction map $s_{\boldsymbol{\tau}}$ is identified with $s_{\tau''}$. Recall that G'' is a tori. By [CDY22, Step 6 in Proof of Theorem H], we know that there exists a morphism $a : X \to A$ with A an abelian variety such that $s_{\tau''}$ is the Stein factorization of a.

***Claim 7.12.* —** *There exists a morphism $b : X_0 \to A$ such that $b \circ \mu = a$.*

Proof. — Recall that $[\tau] \in M(K')$. Let F be any fiber of the birational morphism $\mu : X \to X_0$. Then $\tau(\mathrm{Im}[\pi_1(F) \to \pi_1(X)])$ is finite since $\mathrm{sh}_M : X \to \mathrm{Sh}_M$ coincides with μ. It follows that $\tau(\mathrm{Im}[\pi_1(F) \to \pi_1(X)])$ is a finite group. By the definition of τ'', we conclude that $\tau''(\mathrm{Im}[\pi_1(F) \to \pi_1(X)])$ is also finite, and is thus bounded. According to the property of the reduction map Theorem 1.2, $s_{\tau''}(F)$ is a point. Hence there exists a morphism $b : X_0 \to A$ such that $b \circ \mu = a$. □

Consider the map $b : X_0 \to A$. Note that this map is finite. We apply Lemma 7.8 to conclude that the universal covering of X_0 is a Stein space. □

***Remark 7.13.* —** The proof of Theorem 7.9 utilizes techniques similar to those used in the proof of [DYK23, Theorem C]. In order to extend Theorem 7.9 to any projective normal variety, we have to establish Simpson's theory [Sim93b] on *absolutely constructible subsets* for character varieties of representations in positive characteristic. We plan to explore this problem in our future work.

We recall the following definition by Campana [Cam94].

***Definition 7.14* (Γ-dimension). —** Let X be a projective normal variety. The Γ-dimension of X is defined to be $\dim \mathrm{Sh}(X)$, where $\mathrm{sh}_X : X \dashrightarrow \mathrm{Sh}(X)$ is the Shafarevich map constructed by Campana [Cam94] and Kollár [Kol93] (see Remark 2.11).

***Theorem 7.15.* —** *Let X be a projective normal variety and let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a faithful representation where K is a field of positive characteristic. If the Γ-dimension of X is at most two (e.g. when $\dim X \leq 2$), then the universal covering $\widetilde{X}$ of X is holomorphically convex.*

Proof. — By Proposition 4.6, after we replace X by a suitable finite étale cover, there exists a large representation $\tau : \pi_1(\mathrm{Sh}_\varrho(X)) \to \mathrm{GL}_N(K)$ such that $(\mathrm{sh}_\varrho)^*\tau = \varrho$, where $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$ is the Shafarevich morphism of ϱ. Since ϱ is faithful, it follows that τ is also faithful since the homomorphism $(\mathrm{sh}_\varrho)_* : \pi_1(X) \to \pi_1(\mathrm{sh}_\varrho(X))$ is surjective. As we assume that the Γ-dimension of X is at most two, therefore, $\dim \mathrm{Sh}_\varrho(X) \leq 2$. We apply Theorem 7.9 to conclude that the universal covering S of $\dim \mathrm{Sh}_\varrho(X)$ is Stein. Note that $(\mathrm{sh}_\varrho)_* : \pi_1(X) \to \pi_1(\mathrm{sh}_\varrho(X))$ is an isomorphism. Hence there exists a proper holomorphic fibration $\widetilde{X} \to S$ between the universal coverings of X and $\mathrm{Sh}_\varrho(X)$ that lifts sh_ϱ. It follows that $\widetilde{X}$ is holomorphically convex. □

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