ON THE NILPOTENT ORBIT THEOREM OF COMPLEX VARIATION OF HODGE STRUCTURES

by

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Abstract. — We prove some results on the nilpotent orbit theorem for complex variation of Hodge structures.

Contents

0. Introduction

The nilpotent and $SL(2)$-orbit theorems of Schmid for integral variation of Hdoge structures plays a fundamental role in the study of degeneration of Hodge structures. Their full generalization to complex variation of Hodge structures seems unproven. In this paper we will prove some results on Schmid's nilpotent orbit theorem for complex variation of Hodge structures. The first result is indeed the main part of the nilpotent orbit theorem.

***Theorem A* (=Theorem 2.5).** — *Let X be a complex manifold and let $D = \sum_{i=1}^{\ell} D_i$ be a simple normal crossing divisor on X. Let $(V, \nabla, F^\bullet, Q)$ be a complex polarized variation of Hodge structures on $X - D$. Then for any $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_\ell) \in \mathbb{R}^\ell$, $F^p_{\boldsymbol{\alpha}} = j_* F^p \cap V^{Del}_{\boldsymbol{\alpha}}$ and $F^p_{\boldsymbol{\alpha}}/F^{p+1}_{\boldsymbol{\alpha}}$ are both locally free sheaves. Here $V^{Del}_{\boldsymbol{\alpha}}$ is the Deligne extension of the flat bundle (V, ∇) with the eigenvalues of the residue of ∇ over D_i lying in $[-\alpha_i, -\alpha_i + 1)$.*

We prove moreover that the grading $\oplus_{p=0}^m F^p_{\boldsymbol{\alpha}}/F^{p+1}_{\boldsymbol{\alpha}}$ is naturally identified with $\oplus_{p=0}^m \mathcal{P}_{\boldsymbol{\alpha}} E_p$ where $\oplus_{p=0}^m \mathcal{P}_{\boldsymbol{\alpha}} E_p$ is the prolongation of the Hodge bundles $\oplus_{p=0}^m E_p =: \oplus_{p=0}^m F^{m-p}/F^{m-p+1}$ in terms of the norm growth of the Hodge metric (see § 1.5 for the definition).

Based on Theorem A, we can generalize main parts of Schmid's nilpotent orbit theorem to complex polarized variation of Hodge structures.

Theorem B. — *Let $(V, \nabla, F^\bullet, Q)$ be a complex polarized variation of Hodge structures on $(\Delta^*)^p \times \Delta^q$. Denote by $\Phi : \mathbb{H}^p \times \Delta^q \to \mathscr{D}$ its period mapping, where $\mathscr{D}$ is the period domain and $\mathbb{H} = \{z \in \mathbb{C} \mid \Re z < 0\}$. Let us denote by $2\pi i R_i$ is the logarithm of the monodromy operator associated to the counter-clockwise generator of the fundamental group of the i-th*

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copy of Δ^* *in* $(\Delta^*)^p$*, whose eigenvalues lie in* $(2\pi i(\alpha_i-1), 2\pi i\alpha_i]$ *for some* $\boldsymbol{\alpha}\in\mathbb{R}^p$*. Then for the holomorphic mapping* $\Psi:(\Delta^*)^p\times\Delta^q\to\check{\mathscr{D}}$ *induced by* $\tilde{\Psi}:=\exp(\sum_{i=1}^p z_iR_i)\circ\Phi(z,w)$*,*

(i) Ψ *extends holomorphically to* Δ^{p+q}*;*

(ii) *the holomorphic mapping*

$$\vartheta:\mathbb{H}^p\times\Delta^q\to\check{\mathscr{D}}$$

$$(z,w)\mapsto\exp(-\sum_{i=1}^p z_iR_i)\circ a(w)$$

is horizontal, where $a(w):=\Psi(0,w)$*, and* $\check{\mathscr{D}}$ *is the compact dual of the period domain* $\mathscr{D}$*.*

(iii) *In the one variable case,* $\exp(-zR)\circ a$ *lies in* $\mathscr{D}$ *when* $\Re z\le -C$ *for some* $C>0$*. Moreover, we have the distance estimate*

$$d_{\mathscr{D}}(\exp(-zR)\circ a,\Phi(z))\le C'|\Re z|^\beta e^{\delta\Re z}\quad\text{for some}\quad C',\delta,\beta>0$$

if $\Re z\le -C$*.*

When (V,∇) has quasi-unipotent monodromies around D, Theorems A and B are contained in Schmid's nilpotent orbit theorem [Sch73]. Under this monodromy assumption he proved Theorem B.(iii) for the cases of several variables.

Theorems A and B were also proved by Sabbah and Schnell [SS22] for the cases of one variable in a different way. Their methods can be extended to prove the general cases.

Our proof of Theorem A is based on Mochizuki's work on the prolongation of acceptable bundles [Moc11] and methods in L^2-estimates. The proof of Theorems B.(ii) and B.(iii) essentially follows Schmid's method.

1. Preliminary

1.1. Complex polarized variation of Hodge structures. — Introduced by Simpson in [Sim88], a complex polarized variation of Hodge structures $(V,\nabla,F^\bullet,Q)$ on a complex manifold U consists of the following data. (V,∇) is a flat bundle, and we equip V the natural holomorphic structure induced by $\nabla^{0,1}$. $F^\bullet=\{V=F^0\supsetneq\cdots\supsetneq F^m\}$ is the Hodge filtration, *i.e.* $F^\bullet$ is a filtration of holomorphic subbundles of V so that the Griffiths transversality $\nabla:F^p\to F^{p-1}\otimes\Omega_U^1$ holds. Q is the polarization for the $\mathbb{C}$-VHS, *i.e.* it is a non-definite hermitian form for V which is ∇-parallel so that for the $\mathscr{C}^\infty$-bundle $V_p:=F^{m-p}\cap(F^{m-p+1})^\perp$, one has

$$Q(V_p,V_q)=0\quad\text{for}\quad p\neq q,$$
$$(-1)^pQ(u,u)>0\quad\text{for}\quad u\in V_p.$$

Here $(F^{m-p+1})^\perp$ is the Q-orthogonal complement of F^{m-p+1}. We denote by h_p the hermitian metric $(-1)^pQ(\bullet,\bullet)$ over V_p.

By the above construction $V\overset{\mathscr{C}^\infty}{=}\oplus_{p=0}^mV_p$. We equip it with the hermitian metric $h=\oplus_{p=0}^m h_p$. Then $\nabla|_{V^p}:=\theta_p+D_p+\theta_{p-1}^\dagger$, where

$$\theta_p:V_p\to A^{1,0}(V_{p+1})\quad\text{and}\quad\theta_{p-1}^\dagger:V_p\to A^{0,1}(V_{p-1}),\tag{1.1.1}$$

and D_p is a connection of V_p with $D_p^2\in A^{1,1}(\mathrm{End}(V_p))$. One can prove that $\theta_{p-1}^\dagger$ is the adjoint of θ_{p-1} with respect to the metric h. There is a natural $\mathscr{C}^\infty$-isomorphism $E_p:=F^{m-p}/F^{m-p+1}\to V_p$, and we abusively write h_p the induced metric on E_p via this isomorphism. The induced holomorphic structure on V_p by this $\mathscr{C}^\infty$-isomorphism is indeed given by $D_p^{0,1}$. By the Griffiths transversality ∇ induces a $\mathcal{O}_U$-morphism

$$\theta_p:E_p\to E_{p+1}\otimes\Omega_U^1$$

and it follows from the fact $\nabla^2=0$ that $\theta_{p+1}\wedge\theta_p=0$. Denote by $(E,\theta)=(\oplus_{p=0}^mE_p,\oplus_{p=0}^m\theta_p)$, which is called the *system of Hodge bundles* associated to $(V,\nabla,F^\bullet,Q)$.

For the hermitian metric $h=\oplus_{p=0}^m h_p$ of E, the connection $D_h+\theta+\theta_h^\dagger$ is flat by the above construction. Here D_h is the Chern connection of (E,h).

1.2. Deligne extension. — Let X be a complex manifold and let D be a simple normal crossing divisor on X. For the flat bundle (V,∇) defined on $U:=X-D$, Deligne introduced a way to extend it across D. For any point $x\in D$, we choose an admissible coordinate $(\Omega;z_1,\ldots,z_n)$ so that $\Omega\simeq\Delta^n$ and $D\cap\Omega=(z_1\ldots z_p=0)$ (see Definition 1.1 for the definition). Write $q=n-p$. The fundamental group $\pi_1\big((\Delta^*)^p\times\Delta^q\big)$ is generated by elements $\gamma_1,\ldots,\gamma_p$, where γ_j may be identified with the counter-clockwise generator of the fundamental group of the j-th copy of Δ^* in $(\Delta^*)^p$. We denote by V^∇ the space of multivalued flat sections of (V,∇), which is a finite dimensional $\mathbb{C}$-vector space. Set T_j to be the monodromy transformation with respect to γ_j, which pairwise commute and is an endomorphism of V^∇; that is, for any multivalued section $v(t_1,\ldots,t_{p+q})\in V^\nabla$, one has

$$v(t_1,\ldots,e^{2\pi i}t_j,\ldots,t_{p+q})=(T_jv)(t_1,\ldots,t_{p+q})$$

and $[T_j,T_k]=0$ for any $j,k=1,\ldots,p$. Let us write $Sp(T_j)$ the set of eigenvalues of T_j, and for any $\lambda_j\in Sp(T_j)$, we denote by $\mathbb{E}(T_j,\lambda_j)\subset V^\nabla$ the correspoding generalized eigenspace. We know that all $\lambda_j\in Sp(T_j)$ has norm 1 (see *e.g.* [Sch73, Lemma 4.5]). Write $Sp:=\prod_{i=1}^p Sp(T_j)$. For $\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_p)$, we define

$$\mathbb{E}_{\boldsymbol{\lambda}}:=\cap_{j=1}^p\mathbb{E}(T_j,\lambda_j)$$

Since T_j pairwise commute, one has

$$V^\nabla=\oplus_{\boldsymbol{\lambda}\in Sp}\mathbb{E}_{\boldsymbol{\lambda}},$$

and $\mathbb{E}_{\boldsymbol{\lambda}}$ is an invariant subspace of T_j for any $\boldsymbol{\lambda}\in Sp$ and any j.

Let us fix a p-tuple $\boldsymbol{\alpha}:=(\alpha_1,\ldots,\alpha_p)\in\mathbb{R}^p$. Then for $\boldsymbol{\lambda}\in Sp$, there exists unique $\beta_i\in(\alpha_i-1,\alpha_i]$ so that $\exp(2\pi i\beta_i)=\lambda_i$. Since $\lambda_i^{-1}T_i|_{\mathbb{E}_{\boldsymbol{\lambda}}}$ is unipotent, its logarithm can be defined as

$$\log(\lambda_i^{-1}T_i|_{\mathbb{E}_{\boldsymbol{\lambda}}}):=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(\lambda_i^{-1}T_i|_{\mathbb{E}_{\boldsymbol{\lambda}}}-I)^k}{k}.$$

We denote $N_i:=\frac{\log(\lambda_i^{-1}T_i|_{\mathbb{E}_{\boldsymbol{\lambda}}})}{2\pi i}$. Then for any $v\in\mathbb{E}_{\boldsymbol{\lambda}}$, we define

$$\tilde{v}(t):=\exp\big(-\sum_{i=1}^p(\beta_iI+N_i)\cdot\log t_i\big)v(t)=\prod_{i=1}^p t_i^{-\beta_i}\exp\big(-\sum_{i=1}^p N_i\cdot\log t_i\big)v(t).\tag{1.2.1}$$

One can check that $\tilde{v}$ is single valued, and that $\nabla^{0,1}\tilde{v}=0$. We now fix a basis $v_1,\ldots,v_r$ of V^∇ so that each v_i belongs to some $\mathbb{E}_{\boldsymbol{\lambda}}$. Then the holomorphic sections $\tilde{v}_1,\ldots,\tilde{v}_r$ of V defines a prolongation of V over X which we denoted by $V_{\boldsymbol{\alpha}}^{Del}$. One can check that this construction does not depend on our choice of the basis. This is called the *Deligne extension* of the flat bundle (V,∇) with the eigenvalues of the residue of ∇ over D_i lying in $[-\alpha_i,-\alpha_i+1)$. Note that it is defined for any flat bundle (V,∇) (not necessarily complex variation of Hodge structures).

If (V,∇) underlies a complex polarized variation of Hodge structures $(V,\nabla,F^\bullet,Q)$, we define $F_{\boldsymbol{\alpha}}^p:=j_*F^p\cap V_{\boldsymbol{\alpha}}^{Del}$. It is called the extension of Hodge filtration.

1.3. Acceptable bundles. —

Definition 1.1. — (Admissible coordinate) Let X be a complex manifold and let D be a simple normal crossing divisor. Let x be a point of X, and assume that $\{D_j\}_{j=1,\ldots,\ell}$ be components of D containing p. An *admissible coordinate* around x is the tuple $(\Omega;z_1,\ldots,z_n;\varphi)$ (or simply $(\Omega;z_1,\ldots,z_n)$ if no confusion arises) where

- Ω is an open subset of X containing x.
- there is a holomorphic isomorphism $\varphi:\Omega\to\Delta^n$ so that $\varphi(D_j)=(z_j=0)$ for any $j=1,\ldots,\ell$.

We shall write $\Omega^*:=\Omega-D$, $\Omega(r):=\{z\in\Omega\mid|z_i|<r,\forall i=1,\ldots,n\}$ and $\Omega^*(r):=\Omega(r)\cap\Omega^*$.

We define a (incomplete) Poincaré-type metric ω_P on $(\Delta^*)^\ell \times \Delta^{n-\ell}$ by

$$\omega_P = \sum_{j=1}^{\ell} \frac{\sqrt{-1}dz_j \wedge d\bar{z}_j}{|z_j|^2(\log |z_j|^2)^2} + \sum_{k=\ell+1}^{n} \sqrt{-1}dz_k \wedge d\bar{z}_k. \tag{1.3.1}$$

Note that

$$\omega_P = i\partial\bar{\partial}\log\big(\prod_{j=1}^{\ell}(-\log|z_j|^2)^{-1}\cdot\prod_{k=\ell+1}^{n}\exp(|z_k|^2)\big).$$

For any system of Hodge bundles (E, θ, h), we have the following crucial norm estimate for its Higgs field θ. The one dimensional case is due to Simpson [Sim90, Theorem 1] and the general case was proved by Mochizuki in [Moc02, Proposition 4.1]. Its proof relies on a clever use of Ahlfors-Schwarz lemma.

Theorem 1.2. — *Let (E, θ, h) be a system of Hodge bundle on $X - D$. Then for any point $x \in D$, it has an admissible coordinate $(\Omega; z_1, \ldots, z_n)$ so that the norm*

$$|\theta|_{h,\omega_P} \leq C$$

holds over Ω^* for some constant $C > 0$. □

Here we also recall the following definition in [Moc07, Definition 2.7].

***Definition 1.3* (Acceptable bundle).** — Let (E, h) be a hermitian vector bundle over $X - D$. We say that (E, h) is an acceptable at $p \in D$, if the following holds: there is an admissible coordinate $(\Omega; z_1, \ldots, z_n)$ around p, so that the norm $|R(E, h)|_{h,\omega_P} \leq C$ for some $C > 0$. When $(E, \bar{\partial}_E, h)$ is acceptable at any point p of D, it is called acceptable.

Hodge filtrations and Hodge bundles endowed with the Hodge metric are all acceptable.

Lemma 1.4. — *Let $(V, \nabla, F^\bullet, Q)$ be a complex polarized variation of Hodge structures on $X - D$. Let h be the hermitian metric on V introduced in § 1.1. For the hermitian metric $\tilde{h}_p := h|_{F^p}$, $(F^p, \tilde{h}_p)$ is acceptable. The bundle (E_p, h_p) is also acceptable.*

Proof. — For the hermitian bundle $(F^p, \tilde{h}_p)$, its curvature is

$$R_{\tilde{h}_p}(F^p) = -2\theta_0^\dagger \wedge \theta_0 + 2\sum_{i=1}^{m-p}(-\theta_i^\dagger \wedge \theta_i - \theta_{i-1} \wedge \theta_{i-1}^\dagger) + \theta_{m-p}^\dagger \wedge \theta_{m-p}.$$

The curvature of the bundle (E_p, h_p) is

$$R_{h_p}(E_p) = -\theta_p^\dagger \wedge \theta_p - \theta_{p-1} \wedge \theta_{p-1}^\dagger.$$

By Theorem 1.2, for any point $x \in D$ there is an admissible coordinate $(\Omega; z_1, \ldots, z_n)$ around x so that the norm

$$\sum_i |\theta_i|_{h,\omega_P} = |\theta|_{h,\omega_P} \leq C$$

holds over Ω^* for some constant $C > 0$. Since $\theta_i^\dagger$ is the adjoint of θ_i with respect to h, one has $|\theta_i|_{h,\omega_P} \leq C$ for any p. It follows that $|R_{\tilde{h}_p}(F^p)|_{\tilde{h}_p,\omega_P} \leq C'$ and $|R_{h_p}(E_p)|_{h_p,\omega_P} \leq C'$ for some $C' > 0$. Hence $(F^p, \tilde{h}_p)$ and (E_p, h_p) are acceptable. □

1.4. Parabolic vector bundles. — In this subsection, we recall the notions of parabolic (vector) bundles. For more details we refer to [Moc06]. Let X be a complex manifold, $D = \sum_{i=1}^{\ell} D_i$ be a reduced simple normal crossing divisor, $U = X - D$ be the complement of D and $j : U \to X$ be the inclusion.

Definition 1.5. — A parabolic bundle $\mathcal{P}_*E$ on (X, D) is a holomorphic vector bundle E on U, together with an $\mathbb{R}^\ell$-indexed filtration $\mathcal{P}_{\boldsymbol{\alpha}}E$ (*parabolic structure*) by locally free sheaves of j_*E such that

(i) $\boldsymbol{\alpha} \in \mathbb{R}^\ell$ and $\mathcal{P}_{\boldsymbol{\alpha}}E|_U = E$.
(ii) $\mathcal{P}_{\boldsymbol{\alpha}}E \subset \mathcal{P}_{\boldsymbol{\beta}}E$ if $\alpha_i \leq \beta_i$ for all i.
(iii) For $1 \leq i \leq \ell$, $\mathcal{P}_{\boldsymbol{\alpha}+\mathbf{1}_i}E = \mathcal{P}_{\boldsymbol{\alpha}}E \otimes \mathcal{O}_X(D_i)$, where $\mathbf{1}_i = (0, \ldots, 1, \ldots, 0)$ with 1 in the i-th component.

(iv) $\mathcal{P}_{\boldsymbol{\alpha}+\boldsymbol{\varepsilon}}E = \mathcal{P}_{\boldsymbol{\alpha}}E$ for any vector $\boldsymbol{\epsilon} = (\epsilon, \ldots, \epsilon)$ with $0 < \epsilon \ll 1$.
(v) The set of *weights* $\{\boldsymbol{\alpha} \mid \mathcal{P}_{\boldsymbol{\alpha}}E/\mathcal{P}_{<\boldsymbol{\alpha}}E\} \neq 0$ is discrete in $\mathbb{R}^\ell$.

1.5. Prolongation via norm growth. — Let X be a complex manifold, $D = \sum_{i=1}^{\ell} D_i$ be a simple normal crossing divisor, $U = X - D$ be the complement of D and $j : U \to X$ be the inclusion. Let (E, h) be a hermitian vector bundle on U. For any $\boldsymbol{\alpha} = (a_1, \ldots, a_\ell) \in \mathbb{R}^\ell$, we can prolong E over X by a sheaf of $\mathcal{O}_X$-module $\mathcal{P}_{\boldsymbol{\alpha}}E$ as follows:

$$\mathcal{P}_{\boldsymbol{\alpha}}E(U) = \{\sigma \in \Gamma(U - D, E|_{U-D}) \mid |\sigma|_h \lesssim \prod_{i=1}^{\ell} |z_i|^{-\alpha_i - \varepsilon} \text{ for all } \varepsilon > 0\}.$$

In [Moc11, Theorem 21.3.1] Mochizuki proved that the prolongation of acceptable bundles defined above are parabolic bundles.

***Theorem 1.6* (Mochizuki).** — *Let (E, h) be an acceptable bundle over $X - D$. Then $\mathcal{P}_*E$ defined above is a parabolic bundle.* □

1.6. Period domain and period mapping. — In this subsection we quickly review the definitions of period domain and period mapping. We refer the readers to [CMP17,KKM11] for more details.

Let V be a finite dimensional $\mathbb{C}$-vector space equipped with a non-definite hermitian form Q. A complex Hodge structure on V is a decomposition $V = \oplus_{p=0}^{m} V_p$. It is called *polarized by* Q if $V_p \perp_Q V_q$ for $p \neq q$, and $(-1)^p Q$ is positively definite on V_p. The Hodge filtration is defined to be $F^p := \oplus_{i=0}^{m-p} V_i$. After fixing m and $\dim_{\mathbb{C}} F^p$, the set of all such filtration $F^\bullet$ is a complex flag manifold, which is denoted by $\check{\mathscr{D}}$. It is a closed submanifold of a product of Grassmannians, and is thus a projective manifold. The subset $\mathscr{D}$ of all complex polarized Hodge structures are charcterized by

1. $F^p = F^p \cap (F^{p+1})^\perp \oplus F^{p+1}$.
2. $(-1)^p Q$ is positively definite over $F^p \cap (F^{p+1})^\perp$.

It is an open submanifold of $\check{\mathscr{D}}$. We usually write F instead of $F^\bullet$ to lighten the notation. Since the groups $GL(V)$ and $G := U(V, Q)$ act transitively on $\check{\mathscr{D}}$ and $\mathscr{D}$, $\check{\mathscr{D}}$ and $\mathscr{D}$ are thus homogeneous spaces.

For any Hodge structure $F \in \check{\mathscr{D}}$, the holomorphic tangent space $T_{\check{\mathscr{D}},F}$ of $\check{\mathscr{D}}$ at F is identified with

$$\mathrm{End}(V)/\{A \in \mathrm{End}(V) \mid A(F^p) \subset F^p \text{ for all } p\}.$$

For any $A \in \mathrm{End}(V)$, we denote by $[A]_F$ its image in $T_{\check{\mathscr{D}},F}$.

A tangent vector $[A]_F$ in $T_{\check{\mathscr{D}},F}$ is called *horizontal* if $A(F^p) \subset F^{p-1}$ for all p. The subbundle of $T_{\check{\mathscr{D}}}$ consisting of horizontal vectors is denoted by $T_{\check{\mathscr{D}}}^{-1,1}$ and one can show that it is a holomorphic subbundle of $T_{\check{\mathscr{D}}}$. A holomorphic map $f : \Omega \to \check{\mathscr{D}}$ from a complex manifold Ω is called *horizontal* if $df : T_\Omega \to f^*T_{\check{\mathscr{D}}}$ factors through $f^*T_{\check{\mathscr{D}}}^{-1,1}$.

A complex (unpolarized) variation of Hodge structures $(V, \nabla, F^\bullet)$ over a complex manifold Ω induces a horizontal holomorphic map $\Phi : \tilde{\Omega} \to \check{\mathscr{D}}$ by the Griffiths transversality, where $\tilde{\Omega}$ is the universal cover of Ω. Here we choose the reference space of $\check{\mathscr{D}}$ to be the space of multivalued flat sections V^∇. Φ is called the period mapping associated to $(V, \nabla, F^\bullet)$. When this complex variation of Hodge structures is moreover polarized, Φ factors through $\mathscr{D}$.

2. Nilpotent orbit theorem

2.1. Two results of L^2-estimate. — Set $X = \Delta^n$ and $D = (z_1 \cdots z_\ell = 0)$. We equip the complement $U := X - D$ with the Poincaré metric ω_P defined in (1.3.1). Write

$$X(r) := \{z \in X \mid |z_i| < r \text{ for } i = 1, \ldots, \ell\} \quad \text{and} \quad U(r) = X(r) \cap U.$$

Lemma 2.1. — *Let (F, h_F) be a hermitian vector bundle on U such that $|R_{h_F}(F)| \leq C\omega_P$ for some constant $C > 0$. Then for any section $\eta \in \mathscr{C}^\infty(U, \Lambda^{0,1}T_U^* \otimes F)$ so that $|\eta|_{h_F,\omega_P} \lesssim \prod_{j=1}^{\ell} |z_j|^{\varepsilon}$ for some $\varepsilon > 0$, and $\bar{\partial}\eta = 0$, there exists $\sigma \in \mathscr{C}^\infty(U, F)$ so that $\bar{\partial}\sigma = \eta$ and*

$$\int_U |\sigma|_{h_F}^2 \prod_{j=1}^{\ell} (-\log |z_j|^2)^N dvol_{\omega_P} < \infty$$

for some $N \gg 1$.

Proof. — For the line bundle K_U^{-1} endowed with the natural metric g induced by ω_P, it is acceptable. Hence for the hermitian vector bundle $(E, h) := (K_U^{-1} \otimes F, g \cdot h_F)$, it is also acceptable. It follows from [DH19, Lemma 1.10] that one can choose $N \gg 1$ so that

$$iR_h(E) \geq_{Nak} -(N-1)\omega_P \otimes \mathrm{Id}_E,$$

where "$\geq_{Nak}$" stands for Nakano semipositive (see [Dem82, Définition 2.2]). For the function

$$\varphi := \log \Big(\prod_{j=1}^{\ell} (-\log |z_j|^2)^{-1} \cdot \prod_{k=\ell+1}^{n} \exp(|z_k|^2) \Big), \tag{2.1.1}$$

one has $i\partial\bar{\partial} \log \varphi = \omega_P$. For any $k \in \mathbb{Z}$ we define a new metric $h(k) = h \cdot e^{-k\varphi}$ for E. Therefore,

$$iR_{h(N)}(E) = iR_h(E) + N\omega_P \otimes \mathrm{Id}_E \geq_{Nak} \omega_P \otimes \mathrm{Id}_E.$$

Note that $\mathscr{C}^\infty(U, \Lambda^{n,1}T_U^* \otimes E) = \mathscr{C}^\infty(U, \Lambda^{0,1}T_U^* \otimes F)$ with $|\eta|_{h,\omega_P} = |\eta|_{h_F,\omega_P}$. Since $|\eta|_{h_F,\omega_P} \lesssim \prod_{j=1}^{\ell} |z_j|^{\varepsilon}$, $|\eta|_{h(N),\omega_P} \leq C'$ for some $C' > 0$. Hence $\|\eta\|_{h(N),\omega_P} < \infty$. By the Demailly-Hörmander L^2-estimate [Dem82, Théorème 4.1 and Remarque 4.2] there exists $\sigma \in \mathscr{C}^\infty(U, K_U \otimes E) = \mathscr{C}^\infty(U, F)$ so that

$$\bar{\partial}\sigma = \eta$$

and $\|\sigma\|_{h(N)} < \infty$. Here the smoothness of σ follows from the elliptic regularity of the Laplacian. The lemma is proved. □

Lemma 2.2. — *Let (E, h) be a hermitian vector bundle on U such that $|R_h(E)| \leq C\omega_P$ for some constant $C > 0$. Assume that $\sigma \in H^0(U, E)$ so that $\|\sigma\|_{h(N)} < \infty$ for some integer $N \geq 1$, where $h(N) := h \cdot e^{-N\varphi}$ with φ defined in (2.1.1), then over $U(\frac{1}{2})$, $|\sigma|_h \lesssim \prod_{j=1}^{\ell} |z_j|^{-\varepsilon}$ for any $\varepsilon > 0$.*

Proof. — Since $|R_h(E)| \leq C\omega_P$ for some constant $C > 0$, it follows from [DH19, Lemma 1.10] that $(E, h(-N'))$ is Griffiths semi-negative for some $N' \gg 1$, where $h(-N') := h \cdot e^{N'\varphi}$ with φ defined in (2.1.1). One can show that $\log |\sigma|_{h(-N')}^2$ is a plurisubharmonic function. For any $z \in U^*(\frac{1}{2})$, one has

$$\begin{aligned}
\log |\sigma(z)|_{h(-N')}^2 &\leqslant \frac{4^n}{\pi^n \prod_{i=1}^{\ell} |z_i|^2} \int_{\Omega_z} \log |\sigma(w)|_{h(-N')}^2 d\mathrm{vol}_g \\
&\leqslant \log \Big(\frac{4^n}{\pi^n \prod_{i=1}^{\ell} |z_i|^2} \cdot \int_{\Omega_z} |\sigma(w)|_{h(-N')}^2 d\mathrm{vol}_g \Big) \\
&\leqslant \log \Big(C \int_{\Omega_z} \frac{1}{\prod_{i=1}^{\ell} |w_i|^2} |\sigma(w)|_{h(-N')}^2 d\mathrm{vol}_g \Big) \\
&\leqslant C_1 + \log \int_{\Omega_z} |\sigma(w)|_{h(-N')}^2 \cdot |\prod_{i=1}^{\ell} (\log |w_i|^2)^2 | d\mathrm{vol}_{\omega_P} \\
&\leqslant C_2 + \log \int_{\Omega_z} |\sigma(w)|_{h(N)}^2 d\mathrm{vol}_{\omega_P} \\
&\leqslant C_2 + \log \|\sigma\|_{h(N)}^2
\end{aligned}$$

where $\Omega_z := \{w \in U^* \mid |w_i - z_i| \leq \frac{|z_i|}{2} \text{ for } i \leq \ell; |w_i - z_i| \leq \frac{1}{2} \text{ for } i > \ell\}$ and g is the Euclidean metric. C_1, C_2 are two positive constants which do not depend on $z \in U^*(\frac{1}{2})$. The first inequality is due to the (generalized) mean value inequality, and the second one follows from the Jensen inequality. It follows that

$$|\sigma(z)|_h = |\sigma(z)|_{h(-N')} \cdot \big(\prod_{j=1}^{\ell}(-\log|z_j|^2)^{\frac{N'}{2}} \cdot \prod_{k=\ell+1}^{n} \exp(|z_k|^2)^{-\frac{N'}{2}}\big)$$
$$\leq \exp(\frac{C_2}{2}) \cdot \|\sigma\|_{h(N)} \cdot \big(\prod_{j=1}^{\ell}(-\log|z_j|^2)^{\frac{N'}{2}} \cdot \prod_{k=\ell+1}^{n} \exp(|z_k|^2)^{-\frac{N'}{2}}\big) \lesssim \prod_{i=1}^{\ell}|z_i|^{-\varepsilon}$$

for any $\varepsilon > 0$. □

2.2. Proof of Theorem A. — We first prove that the Deligne extension of the flat bundle unerlying a complex variation of Hodge structures coincides with the prolongation defined in § 1.5.

Proposition 2.3. — *Let X be a complex manifold, $D = \sum_{i=1}^{\ell} D_i$ be a simple normal crossing divisor. For a complex variation of Hodge structures $(V, \nabla, F^\bullet, h)$ defined on $U := X - D$, one has $V^{Del}_{\boldsymbol{\alpha}} = \mathcal{P}_{\boldsymbol{\alpha}} V$, where $\mathcal{P}_{\boldsymbol{\alpha}} V$ is the prologation of V defined in § 1.5.*

Proof. — We first prove that $V^{Del}_{\boldsymbol{\alpha}} \subset \mathcal{P}_{\boldsymbol{\alpha}} V$. We will use the notation in § 1.2. Since this is a local problem, we can assume that $X = \Delta^n$ and $D = (t_1 \cdots t_p = 0)$. By the construction of $V^{Del}_{\boldsymbol{\alpha}}$ one can take a basis $v_1, \ldots, v_r$ of V^∇ with each $v_i \in \mathbb{E}_{\boldsymbol{\lambda}(v_i)}$ for some $\boldsymbol{\lambda}(v_i) \in Sp$ so that $\{\tilde{v}_1, \ldots, \tilde{v}_r\}$ defined in (1.2.1) forms a basis of $V^{Del}_{\boldsymbol{\alpha}}$. It thus suffices to estimate the norm

$$\tilde{v}(t) := \exp\big(-\sum_{i=1}^{p}(\beta_i I + N_i) \cdot \log t_i\big) v(t) = \prod_{i=1}^{p} t_i^{-\beta_i} \exp\big(-\sum_{i=1}^{p} N_i \cdot \log t_i\big) v(t)$$

for any $\boldsymbol{\lambda}$ and $v \in \mathbb{E}_{\boldsymbol{\lambda}}$. By the norm estimate in [Moc07], over a given sector of U one has the (weaker) norm estimate

$$(\prod_{i=1}^{p} |\log|t_i||)^{-M} \lesssim |v(t)|_h \lesssim (\prod_{i=1}^{p} |\log|t_i||)^{M}$$

for some $M > 0$. Since all N_i are nilpotent and pairwise commute,

$$\exp\big(-\sum_{i=1}^{p} N_i \cdot \log t_i\big) v(t) = \sum_{i=1}^{p}\sum_{k=0}^{N} \frac{1}{k!}(\log t_i)^k (N_i^k v)(t)$$

for some integer $N > 0$. Note that $N^k v \in \mathbb{E}_{\boldsymbol{\lambda}}$ for any $k \geq 0$. Since one can cover $X - D$ by finite sectors, this proves the norm estimate

$$(\prod_{i=1}^{p} |\log|t_i||)^{-M'} \lesssim |\exp\big(-\sum_{i=1}^{p} N_i \cdot \log t_i\big) v(t)|_h \lesssim (\prod_{i=1}^{p} |\log|t_i||)^{M'}$$

for some $M' > 0$. Hence

$$|\tilde{v}(t)|_h \lesssim \prod_{i=1}^{p} |t_i|^{-\beta_i - \varepsilon} \lesssim \prod_{i=1}^{p} |t_i|^{-\alpha_i - \varepsilon}$$

for any $\varepsilon > 0$. This proves the inclusion $V^{Del}_{\boldsymbol{\alpha}} \subset \mathcal{P}_{\boldsymbol{\alpha}} V$ by the very definition of $\mathcal{P}_{\boldsymbol{\alpha}} V$.

Now let us prove the inclusion $\mathcal{P}_{\boldsymbol{\alpha}} V \subset V^{Del}_{\boldsymbol{\alpha}}$. First we note that the decomposition $V^\nabla = \oplus_{\boldsymbol{\lambda} \in Sp} \mathbb{E}_{\boldsymbol{\lambda}}$ induces a decomposition of the flat bundle (V, ∇) into

(2.2.1) $$(V, \nabla) = \oplus_{\boldsymbol{\lambda} \in Sp} (V(\boldsymbol{\lambda}), \nabla|_{V(\boldsymbol{\lambda})}),$$

where $(V(\boldsymbol{\lambda}), \nabla|_{V(\boldsymbol{\lambda})})$ is the flat subbundle induced by $\mathbb{E}_{\boldsymbol{\lambda}}$. We fix a basis $(v_1, \ldots, v_r) \in V^\nabla$ so that $v_i \in \mathbb{E}_{\boldsymbol{\lambda}(v_i)}$ for some $\boldsymbol{\lambda}(v_i) \in Sp$. This means that such basis is compatible with the above decomposition (2.2.1); namely v_j is a mutivalued flat section of

$(V(\boldsymbol{\lambda}(v_j)), \nabla|_{V(\boldsymbol{\lambda}(v_j))})$. Consider the dual bundle V^* of V, and it can endowed with the natural connection ∇ (notion abusively) defined by

$$(\nabla\mu)v = d(\mu(v)) - \mu(\nabla(v))$$

for μ and v sections in V^* and V respectively. (V^*, ∇) is thus also a flat bundle. Moreover, the finite dimension $\mathbb{C}$-vector space $(V^*)^\nabla$ is the dual space of (V^∇). Consider the dual basis $(v_1^*, \ldots, v_r^*)$ of $(v_1, \ldots, v_r)$. Since

$$(\nabla v_i^*)v_j = d(v_i^*(v_j)) - v_i^*(\nabla v_j) = 0,$$

one has $v_i^* \in (V^*)^\nabla$. Recall that T_j is the monodromy transformation of (V, ∇) with respect to γ_j defined by

$$v(t_1, \ldots, e^{2\pi i}t_j, \ldots, t_{p+q}) = (T_j v)(t_1, \ldots, t_{p+q})$$

for any $v \in V^\nabla$. Let us denote by $\tilde{T}_j$ the monodromy transformation of (V, ∇) with respect to γ_j defined by

$$\mu(t_1, \ldots, e^{2\pi i}t_j, \ldots, t_{p+q}) = (\tilde{T}_j \mu)(t_1, \ldots, t_{p+q})$$

for any $\mu \in (V^*)^\nabla$. Then for any $v \in V^\nabla$ and any $\mu \in (V^*)^\nabla$ one has

$$\begin{aligned}\mu(t)(v(t)) &= \mu(t_1, \ldots, e^{2\pi i}t_j, \ldots, t_{p+q})(v(t_1, \ldots, e^{2\pi i}t_j, \ldots, t_{p+q})) \\ &= (\tilde{T}_j\mu(t))(T_j v(t)) = (\tilde{T}_j\mu)(T_j v) = (T_j^*\tilde{T}_j\mu)(v),\end{aligned}$$

where $T_j^* : (V^*)^\nabla \to (V^*)^\nabla$ is the adjoint of T_j. Hence

$$\tilde{T}_j = (T_j^*)^{-1}. \tag{2.2.2}$$

It follows that $Sp(\tilde{T}_j) = \{\lambda^{-1}\}_{\lambda \in Sp(T_i)}$. Set $\mathbb{E}(\tilde{T}_j, \lambda_j) \subset (V^*)^\nabla$ to be the corresponding generalized eigenspace of $\lambda_j \in Sp(\tilde{T}_j)$. We know that all $\lambda_j \in Sp(\tilde{T}_j)$ have norm 1 since (V^*, ∇) admits a complex variation of Hodge structures. For $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_p) \in Sp$, we define

$$\tilde{\mathbb{E}}_{\boldsymbol{\lambda}} := \cap_{j=1}^p \mathbb{E}(\tilde{T}_j, \lambda_j^{-1}) \subset (V^*)^\nabla$$

Since $T_j's$ are pairwise commute, one has

$$(V^*)^\nabla = \oplus_{\boldsymbol{\lambda} \in Sp} \tilde{\mathbb{E}}_{\boldsymbol{\lambda}},$$

and $\tilde{E}_{\boldsymbol{\lambda}}$ is an invariant subspace of $\tilde{T}_j$ for any $\boldsymbol{\lambda} \in Sp$ and any j.

By Lemma 2.4 below, one can show that for any $\mu \in \tilde{\mathbb{E}}_{\boldsymbol{\lambda}'}$ and $v \in \mathbb{E}_{\boldsymbol{\lambda}}$, $\mu(v) = 0$ if $\boldsymbol{\lambda} \neq \boldsymbol{\lambda}'$, which implies that $v_j^* \in \tilde{\mathbb{E}}_{\boldsymbol{\lambda}(v_j)}$.

For $\boldsymbol{\lambda} \in Sp$, there exists unique $\beta_i \in (\alpha_i - 1, \alpha_i]$ so that $\exp(2\pi i\beta_i) = \lambda_i$. Denote $N_i := \frac{\log(\lambda_i^{-1}T_i|_{\mathbb{E}_{\boldsymbol{\lambda}}})}{2\pi i}$. Recall that for any $v \in \mathbb{E}_{\boldsymbol{\lambda}}$, we define

$$\tilde{v}(t) := \exp\big(-\sum_{i=1}^p(\beta_i I + N_i)\cdot\log t_i\big)v(t) = \prod_{i=1}^p t_i^{-\beta_i}\exp\big(-\sum_{i=1}^p N_i\cdot\log t_i\big)v(t).$$

Likewise, since $\lambda_i\tilde{T}_i|_{\tilde{\mathbb{E}}_{\boldsymbol{\lambda}}}$ is unipotent, its logarithm can be defined as

$$\log(\lambda_i\tilde{T}_i|_{\tilde{\mathbb{E}}_{\boldsymbol{\lambda}}}) := \sum_{k=1}^\infty(-1)^{k+1}\frac{(\lambda_i\tilde{T}_i|_{\tilde{\mathbb{E}}_{\boldsymbol{\lambda}}} - I)^k}{k}.$$

Write $\tilde{N}_i := \frac{\log(\lambda_i\tilde{T}_i|_{\tilde{\mathbb{E}}_{\boldsymbol{\lambda}}})}{2\pi i}$. Then for any $\mu \in \tilde{\mathbb{E}}_{\boldsymbol{\lambda}}$, we define

$$\tilde{\mu}(t) := \exp\big(-\sum_{i=1}^p(-\beta_i I + \tilde{N}_i)\cdot\log t_i\big)\mu(t) = \prod_{i=1}^p t_i^{\beta_i}\exp\big(-\sum_{i=1}^p \tilde{N}_i\cdot\log t_i\big)\mu(t). \tag{2.2.3}$$

Since $\tilde{T}_i = (T_i^*)^{-1}$, one has $\tilde{N}_j = -N_j^*$. Therefore,

$$\begin{aligned}\tilde{\mu}(t)(\tilde{v}(t)) &= \exp\big(-\sum_{i=1}^{p}\tilde{N}_i\cdot\log t_i\big)\mu(t)\Big(\exp\big(-\sum_{i=1}^{p}N_i\cdot\log t_i\big)v(t)\Big)\\ &= \exp\big(\sum_{i=1}^{p}N_i^*\cdot\log t_i\big)\mu(t)\Big(\exp\big(-\sum_{i=1}^{p}N_i\cdot\log t_i\big)v(t)\Big)\\ &= \mu(v) = \text{constant}.\end{aligned}$$

This implies that $\tilde{v}_i^*(t)(\tilde{v}_j(t)) = v_i^*(v_j) \equiv \delta_{ij}$ if $\boldsymbol{\lambda} = \boldsymbol{\lambda}'$, where $\tilde{v}_i^*$ is defined in (2.2.3) in terms of $v_i^* \in \tilde{\mathbb{E}}_{\boldsymbol{\lambda}(v_i)}$.

If $\mu \in \tilde{\mathbb{E}}_{\boldsymbol{\lambda}}$ and $v \in \mathbb{E}_{\boldsymbol{\lambda}'}$ with $\boldsymbol{\lambda} \neq \boldsymbol{\lambda}'$, the above construction shows that $\tilde{\mu}$ and $\tilde{v}$ are holomorphic sections of $V^*(\boldsymbol{\lambda})$ and $V(\boldsymbol{\lambda}')$. Here $V(\boldsymbol{\lambda}')$ is the invariant flat subbundle of (V,∇) defined in (2.2.1), and $V^*(\boldsymbol{\lambda})$ is defined to be the invariant flat subbundle of (V^*,∇) generated by $\tilde{E}_{\boldsymbol{\lambda}}$. Hence $\tilde{v}_i^*$ and $\tilde{v}_j$ are holomorphic sections of $V^*(\boldsymbol{\lambda}(v_i))$ and $V(\boldsymbol{\lambda}(v_j))$ respectively. This shows that $\tilde{v}_i^*(t)(\tilde{v}_j(t)) \equiv 0$ for $\boldsymbol{\lambda} \neq \boldsymbol{\lambda}'$ by Lemma 2.4.

In conclusion, we prove that $\tilde{v}_1^*, \ldots, \tilde{v}_r^*$ is the dual frame of $\tilde{v}_1, \ldots, \tilde{v}_r$.

Recall that $v_j \in \mathbb{E}_{\boldsymbol{\lambda}(v_j)}$ for some $\boldsymbol{\lambda}(v_j) \in Sp$. There exists unique $\beta(v_j)_i \in (\alpha_i - 1, \alpha_i]$ so that $\exp(2\pi i\beta(v_j)_i) = \boldsymbol{\lambda}(v_j)_i$. Define a smooth section $\tilde{v}_j' = \tilde{v}_j \cdot \prod_{i=1}^{p}|t_i|^{\beta(v_j)_i}$. By the norm estimate in the first step, for all $\tilde{v}_j'$ one has the norm estimate

$$|\tilde{v}_j'|_h \lesssim (\prod_{i=1}^{p}|\log|t_i||)^M$$

for some $M > 0$. It follows that

$$H(h;\tilde{v}_1',\ldots,\tilde{v}_r') \lesssim (\prod_{i=1}^{p}|\log|t_i||)^M$$

Here $H(h;\tilde{v}_1',\ldots,\tilde{v}_r')$ is a $r\times r$-matrix function whose (i,j)-component is $h(\tilde{v}_i',\tilde{v}_j')$. On the other hand, we put $\mu_i' = \tilde{v}_i^* \cdot \prod_{j=1}^{p}|t_j|^{-\beta(v_i)_j}$. Since complex polarized variation of Hodge structures is functorial by taking dual, (V^*,∇) admits a complex polarized variation of Hodge structures whose Hodge metric is the dual metric h^* of the Hodge metric h for $(V,\nabla,F^\bullet,Q)$. In the same manner we obtain

$$|\mu_i'|_{h^*} \lesssim (\prod_{i=1}^{p}|\log|t_i||)^{M'}$$

for every μ_i' and some $M' > 0$. This implies that

$$H(h^*;\mu_1',\ldots,\mu_r') \lesssim (\prod_{i=1}^{p}|\log|t_i||)^{M'}$$

By our construction, $\mu_1',\ldots,\mu_r'$ is the dual of the smooth frame $\tilde{v}_1',\ldots,\tilde{v}_r'$. It follows that

$$(\prod_{i=1}^{p}|\log|t_i||)^{-M'} \lesssim H(h^*;\mu_1',\ldots,\mu_r')^{-1} = H(h;\tilde{v}_1',\ldots,\tilde{v}_r').$$

Hence

$$(\prod_{i=1}^{p}|\log|t_i||)^{-M'} \lesssim H(h;\tilde{v}_1',\ldots,\tilde{v}_r') \lesssim (\prod_{i=1}^{p}|\log|t_i||)^{M}. \tag{2.2.4}$$

Now we are ready to prove the inclusion $\mathcal{P}_{\boldsymbol{\alpha}}V \subset V_{\boldsymbol{\alpha}}^{Del}$. For any $s \in \mathcal{P}_{\boldsymbol{\alpha}}V(U)$, it can be written as $s = \sum_{i=1}^{r} f_i\tilde{v}_i$ where f_i is a holomorphic function on U. By (2.2.4) one has

$$\sum_{i=1}^{r}|f_i|^2\cdot\prod_{j=1}^{p}|t_j|^{-2\beta(v_i)_j}\cdot(\prod_{k=1}^{p}|\log|t_k||)^{-2M'} \lesssim |s|_h^2 \lesssim \prod_{i=1}^{p}|t_j|^{-2\alpha_j-\varepsilon}$$

for any $\varepsilon > 0$. Since $\beta(v_i)_j \in (\alpha_j - 1, \alpha_j]$, one can choose $\delta > 0$ so that $\beta(v_i)_j - \alpha + 1 > \delta$ for all v_i and every $j = 1, \ldots, p$. The above inequality implies that for every f_i,

$$|f_i| \lesssim \prod_{i=1}^{p} |t_j|^{-1+\delta}.$$

Hence all f_i extend to a holomorphic function over X. This proves that $s \in V_{\boldsymbol{\alpha}}^{Del}(X)$ since $\tilde{v}_1, \ldots, \tilde{v}_r$ is a holomorphic basis of $V_{\boldsymbol{\alpha}}^{Del}$ by the definition of Deligne extension in § 1.2. The inclusion $\mathcal{P}_{\boldsymbol{\alpha}} V \subset V_{\boldsymbol{\alpha}}^{Del}$ is proved. We complete the proof of the proposition. □

We leave the proof of the following lemma of linear algebra to the reader.

Lemma 2.4. — *Let $T : V \to V$ be an isomorphism of a finite dimensional $\mathbb{C}$-vector space V. Decompose $V = V_{\lambda_1} \oplus \ldots \oplus V_{\lambda_k}$ into generalized eigenspaces of T, where λ_i is a generalized eigenvalue of T and V_{λ_i} is the corresponding generalized eigenspace. Denote by V^* the dual vector space. Then for the isomorphism $(T^*)^{-1} : V^* \to V^*$, its generalized eigenvalues are $\lambda_1^{-1}, \ldots, \lambda_k^{-1}$ and its generalized eigenspace decomposition is $V^* = V^*_{\lambda_1^{-1}} \oplus \ldots \oplus V^*_{\lambda_k^{-1}}$, where $V^*_{\lambda_j^{-1}}$ is the corresponding generalized eigenspace of λ_j^{-1}. Moreover, one has $\mu(v) = 0$ if $\mu \in V^*_{\lambda_i^{-1}}$ and $v \in V_{\lambda_j}$ with $i \neq j$.* □

Let us prove Theorem A. By Lemma 1.4, $(F^p, \tilde{h}_p)$ and (E_p, h_p) are acceptable bundles for any p, where $\tilde{h}_p := h|_{F^p}$. It follows from Theorem 1.6 that the filtered bundle $\mathcal{P}_* F^p$ and $\mathcal{P}_* E_p$ defined in § 1.5 are parabolic ones. In particular, $\mathcal{P}_{\boldsymbol{\alpha}} F^p$ and $\mathcal{P}_{\boldsymbol{\alpha}} E_p$ are locally free sheaves. Note that

$$\mathcal{P}_{\boldsymbol{\alpha}} F^p = j_*(F^p) \cap \mathcal{P}_{\boldsymbol{\alpha}} V \overset{Proposition\ 2.3}{=} j_* F^p \cap V_{\boldsymbol{\alpha}}^{Del} := F^p_{\boldsymbol{\alpha}}. \tag{2.2.5}$$

To prove that $F^p_{\boldsymbol{\alpha}} / F^{p+1}_{\boldsymbol{\alpha}}$ is locally free, it is equivalent to show the local freeness of $\mathcal{P}_{\boldsymbol{\alpha}} F^p / \mathcal{P}_{\boldsymbol{\alpha}} F^{p+1}$.

Theorem 2.5. — *There is a natural exact sequence*

$$0 \to \mathcal{P}_{\boldsymbol{\alpha}} F^{p+1} \to \mathcal{P}_{\boldsymbol{\alpha}} F^p \to \mathcal{P}_{\boldsymbol{\alpha}} E_{m-p} \to 0. \tag{2.2.6}$$

In particular, $F^p_{\boldsymbol{\alpha}} / F^{p+1}_{\boldsymbol{\alpha}} = \mathcal{P}_{\boldsymbol{\alpha}} F^p / \mathcal{P}_{\boldsymbol{\alpha}} F^{p+1} = \mathcal{P}_{\boldsymbol{\alpha}} E_{m-p}$ is locally free for any p.

Proof. — It is easy to see

$$0 \to \mathcal{P}_{\boldsymbol{\alpha}} F^{p+1} \to \mathcal{P}_{\boldsymbol{\alpha}} F^p \xrightarrow{q} \mathcal{P}_{\boldsymbol{\alpha}} E_{m-p}.$$

To prove its right exactness, it suffices to prove that for any $x \in D$ and any section $s \in \mathcal{P}_{\boldsymbol{\alpha}} E_{m-p}(\Omega)$ where Ω is a neighborhood of x, there is a section $\tilde{s} \in \mathcal{P}_{\boldsymbol{\alpha}} F^p(\Omega')$ for some smaller neighborhood Ω' of x so that $q(\tilde{s}) = s|_{\Omega'}$. We shall construct such $\tilde{s}$ by the previous results on L^2-estimate, Lemmas 2.1 and 2.2.

Since this is a local problem, we can assume that $X = \Delta^n$ and $D = (z_1 \cdots z_\ell = 0)$, and we equip the complement $U := X - D$ with the Poincaré metric ω_P. By the semicontinuity of the parabolic bundle in Definition 1.5.(iv), we can choose $\boldsymbol{\beta} \in \mathbb{R}^\ell$ so that $\beta_i > \alpha_i$ and

$$\mathcal{P}_{\boldsymbol{\beta}} F^p = \mathcal{P}_{\boldsymbol{\alpha}} F^p. \tag{2.2.7}$$

Pick any $s \in \mathcal{P}_{\boldsymbol{\alpha}} E_{m-p}(X)$. Then $s \in H^0(U, E_{m-p})$ with $|s|_{h_{m-p}} \lesssim \prod_{i=1}^{\ell} |z_i|^{-\alpha_i - \varepsilon}$ for any $\varepsilon > 0$. We will construct a section $\tilde{s} \in H^0(U(r), F^p)$ for some $0 < r < 1$ so that $q(\tilde{s}) = s|_{U(r)}$ and $|\tilde{s}|_{\tilde{h}_p} \lesssim \prod_{i=1}^{\ell} |z_i|^{-\beta_i - \varepsilon}$ for any $\varepsilon > 0$. Note that there is an canonical $\mathscr{C}^\infty$ isomorphism (and isometry)

$$\Phi : (F^p, \tilde{h}_p) \to (F^{p+1}, \tilde{h}_{p+1}) \oplus (E_{m-p}, h_{m-p})$$

such that the holomorphic structure of F^p via Φ is defined by

$$\begin{bmatrix} \bar{\partial}_{F^{p+1}} & \theta^\dagger_{m-p-1} \\ 0 & \bar{\partial}_{E_{m-p}} \end{bmatrix},$$

where $\theta^\dagger_{m-p-1}$ is defined in (1.1.1). If $q(\tilde{s}) = s$, then $\Phi(\tilde{s}) = [\sigma, s]$ for some $\sigma \in \mathscr{C}^\infty(U, F^{p+1})$ so that

$$\begin{bmatrix} \bar{\partial}_{F^{p+1}} & \theta^\dagger_{m-p-1} \\ 0 & \bar{\partial}_{E_{m-p}} \end{bmatrix} \begin{bmatrix} \sigma \\ s \end{bmatrix} = 0$$

Hence $\bar{\partial}_{F^{p+1}}\sigma = -\theta^\dagger_{m-p-1}s$. We will solve this $\bar{\partial}$-equation with proper norm estimate.

By Theorem 1.2, $|\theta_{m-p-1}|_{h,\omega_P} \leq C$ over U (we can replace U by $U(r)$ for some $0 < r < 1$). It follows that $|\theta^\dagger_{m-p-1}|_{h,\omega_P} \leq C$ over U since $\theta^\dagger_{m-p-1}$ is the adjoint of θ_{m-p-1} with respect to h. Therefore,

$$|\theta^\dagger_{m-p-1}s|_{\tilde{h}_{p+1},\omega_P} \leq |\theta^\dagger_{m-p-1}|_{h,\omega_P} \cdot |s|_{h_{m-p}} \lesssim \prod_{i=1}^{\ell} |z_i|^{-\alpha_i-\varepsilon}$$

for any $\varepsilon > 0$. For every F^p we introduce a new metric

$$\tilde{h}_p(\boldsymbol{\beta}) := \tilde{h}_p \cdot \prod_{i=1}^{\ell} |z_i|^{\beta_i},$$

and thus

$$|\theta^\dagger_{m-p-1}s|_{\tilde{h}_{p+1}(\boldsymbol{\beta}),\omega_P} \lesssim \prod_{j=1}^{\ell} |z_j|^{\delta}$$

for some $\delta > 0$. Note that

$$\begin{aligned} \bar{\partial}_{F^{p+1}}(\theta^\dagger_{m-p-1}s) &= (\bar{\partial}_{E_{m-p-1}} + \theta^\dagger_{m-p})(\theta^\dagger_{m-p-1}s) \\ &= \bar{\partial}_{E_{m-p-1}}(\theta^\dagger_{m-p-1}s) \\ &= (D^{0,1}_h\theta^\dagger_{m-p-1})s - \theta^\dagger_{m-p-1}(\bar{\partial}_{E_{m-p}}s) = 0, \end{aligned}$$

where the second equality follows from $\theta^\dagger_{m-p-1} \wedge \theta^\dagger_{m-p} = 0$, and the last one follows from $D^{0,1}_h(\theta^\dagger) = 0$. Since $(F^{p+1}, \tilde{h}_{p+1}(\boldsymbol{\beta}))$ is also acceptable by Lemma 1.4, we can invoke Lemma 2.1 to find some $\sigma \in \mathscr{C}^\infty(U, F^{p+1})$ so that

$$\bar{\partial}_{F^{p+1}}(\sigma) = -\theta^\dagger_{m-p-1}s$$

and $\|\sigma\|_{\tilde{h}_{p+1}(\boldsymbol{\beta},N)} < \infty$. Here $\tilde{h}_{p+1}(\boldsymbol{\beta}, N)$ is a new metric for F^{p+1} define by

$$\tilde{h}_{p+1}(\boldsymbol{\beta}, N) = \tilde{h}_{p+1} \cdot \prod_{i=1}^{\ell} |z_i|^{\beta_i} \cdot e^{-N\varphi}$$

with φ defined in (2.1.1). Thus the section $\tilde{s} := \Phi^{-1}([\sigma, s])$ is a holomorphic section of F^p so that

$$\|\tilde{s}\|^2_{\tilde{h}_p(\boldsymbol{\beta},N)} = \|\sigma\|^2_{\tilde{h}_{p+1}(\boldsymbol{\beta},N)} + \|s\|^2_{h_{m-p}(\boldsymbol{\beta},N)} < \infty.$$

Here $h_{m-p}(\boldsymbol{\beta}, N)$ and $\tilde{h}_p(\boldsymbol{\beta}, N)$ are new metrics of E_{m-p} and F^p respectively defined by

$$h_{m-p}(\boldsymbol{\beta}, N) = h_{m-p} \cdot \prod_{i=1}^{\ell} |z_i|^{\beta_i} \cdot e^{-N\varphi}$$

and

$$\tilde{h}_p(\boldsymbol{\beta}, N) = \tilde{h}_p \cdot \prod_{i=1}^{\ell} |z_i|^{\beta_i} \cdot e^{-N\varphi} = \tilde{h}_p(\boldsymbol{\beta}) \cdot e^{-N\varphi}.$$

Since $(F^p, \tilde{h}_p(\boldsymbol{\beta}))$ is also acceptable by Lemma 1.4, by Lemma 2.2 over some $U(r)$ for $0 < r < 1$ we have $|\tilde{s}|_{\tilde{h}_p(\boldsymbol{\beta})} \lesssim \prod_{j=1}^{\ell} |z_j|^{-\varepsilon}$ for any $\varepsilon > 0$. Therefore, $|\tilde{s}|_{\tilde{h}_p} \lesssim \prod_{j=1}^{\ell} |z_j|^{-\beta_j-\varepsilon}$ for any $\varepsilon > 0$. This proves that

$$\tilde{s} \in \mathcal{P}_{\boldsymbol{\beta}}F^p(X(r)).$$

By (2.2.7) we conclude that $\tilde{s} \in \mathcal{P}_{\boldsymbol{\alpha}}F^p(X(r'))$ for some $0 < r' < 1$. This proves the right exactness of (2.2.6) since $q(\tilde{s}) = s$. The proof of the theorem is accomplished. □

2.3. On the nilpotent orbit theorem. — In this subsection we apply Theorem A to prove Theorem B following closely Schmid's original method [Sch73, p. 288-289]. We will use the notations and conventions in § 1.6.

Let $(V, \nabla, F^\bullet, Q)$ be a complex polarized variation of Hodge structures on $(\Delta^*)^p \times \Delta^q$. Denote by $\Phi : \mathbb{H}^p \times \Delta^q \to \mathscr{D}$ its period mapping, where we set

$$\mathbb{H}^p \times \Delta^q \to \Delta^n$$
$$(z, w) \mapsto (e^{z_1}, \ldots, e^{z_p}, w)$$

to be the uniformizing map. Let T_j be the monodromy transformation defined in § 1.2. For some fixed $\boldsymbol{\alpha} \in \mathbb{R}^p$, there exist $S_i, N_i \in \operatorname{End}(V^\nabla)$ so that

— $T_i = \exp(2\pi i(S_i + N_i))$;

— $[S_i, S_j] = 0$, $[S_i, N_j] = 0$, and $[N_i, N_j] = 0$;

— S_i is semisimple whose eigenvalues lying in $(\alpha_i - 1, \alpha_i]$ and N_i is nilpotent.

Let us define

$$\tilde{\Psi}(z, w) := \exp(\sum_{i=1}^{p}(S_i + N_i)z_i)\Phi(z, w),$$

which satisfies $\tilde{\Psi}(z_1, \ldots, z_i + 2\pi i, \ldots, z_p, w) = \tilde{\Psi}(z, w)$ for $i = 1, \ldots, p$. It thus descends to a single valued map $\Psi : (\Delta^*)^p \times \Delta^q \to \check{\mathscr{D}}$ so that $\Psi(e^{z_1}, \ldots, e^{z_p}, w) = \tilde{\Psi}(z, w)$. Theorem A implies that Ψ extends holomorphically to Δ^{p+q}, which thus proves Theorem B.(i). Write $a(w) := \Psi(0, w)$. In general it does not lie in $\mathscr{D}$.

The following well-known result follows from the fact that $GL(V^\nabla)$ acts transitively on $\check{\mathscr{D}}$.

Lemma 2.6. — *For any $g \in GL(V^\nabla)$, consider the left translation $L_g : \check{\mathscr{D}} \to \check{\mathscr{D}}$ with $L_g(F) := gF$. Then*

$$(L_g)_* : T^{-1,1}_{\check{\mathscr{D}},F} \xrightarrow{\sim} T^{-1,1}_{\check{\mathscr{D}},gF}.$$

□

Recall that for any $A \in \operatorname{End}(V^\nabla)$ and any $F \in \check{\mathscr{D}}$, we denote by $[A]_F$ the image of A under the natural map $\operatorname{End}(V^\nabla) \to T_{\check{\mathscr{D}},F}$.

Lemma 2.7. — *For each $i = 1, \ldots, p$, $[S_i + N_i]_{a(w)} \subset T^{-1,1}_{\check{\mathscr{D}},a(w)}$.*

Proof. — Since

$$\tilde{\Psi}_*(\frac{\partial}{\partial z_i})(z, w) = [S_i + N_i]_{\tilde{\Psi}(z,w)} + (L_{\exp(\sum_{i=1}^{p}(S_i+N_i)z_i)})_*\Phi_*(\frac{\partial}{\partial z_i})(z, w)$$

$\Phi_*(\frac{\partial}{\partial z_i})$ is horizontal since Φ is a horizontal mapping by § 1.6. By Lemma 2.6 $(L_{\exp(\sum_{i=1}^{p}(S_i+N_i)z_i)})_*\Phi_*(\frac{\partial}{\partial z_i})(z, w)$ is horizontal. On the other hand,

$$\tilde{\Psi}_*(\frac{\partial}{\partial z_i})(z, w) = \Psi_*(\frac{\partial}{\partial t_i})(e^{z_1}, \ldots, e^{z_p}, w) \cdot e^{z_i}$$

which tends to zero if $\Re z_i \to -\infty$ and $\Re z_j \leq C$ for other j. By continuity, this implies that

$$[S_i + N_i]_{a(w)} \in T^{-1,1}_{\check{\mathscr{D}},a(w)}.$$

□

We are ready to prove Theorem B.(ii).

Lemma 2.8. — *The holomorphic mapping*

$$\vartheta : \mathbb{H}^p \times \Delta^q \to \check{\mathscr{D}}$$
$$(z, w) \mapsto \exp(-\sum_{i=1}^{p} z_i(S_i + N_i)) \circ a(w)$$

is horizontal.

Proof. — Note that $\vartheta_*(\frac{\partial}{\partial z_i}) = [S_i + N_i]_{\vartheta(z,w)}$. Since $[S_i, N_i] = 0$, one has

$$(L_{\exp(\sum_{i=1}^p (S_i+N_i)z_i)})_*\big([S_i+N_i]_{\vartheta(z,w)}\big) = \Big[\mathrm{Ad}_{\exp(\sum_{i=1}^p (S_i+N_i)z_i)}(S_i+N_i)\Big]_{a(w)} = [S_i+N_i]_{a(w)}.$$

It then follows from Lemmas 2.6 and 2.7 that $[S_i+N_i]_{\vartheta(z,w)} \in T^{-1,1}_{\check{\mathscr{D}},\vartheta(z,w)}$. We conclude that $\vartheta_*(\frac{\partial}{\partial z_i})$ is horizontal.

On the other hand, one has

$$\vartheta_*(\frac{\partial}{\partial w_i}) = (L_{\exp(-\sum_{i=1}^p (S_i+N_i)z_i)})_* a_*(\frac{\partial}{\partial w_i}),$$

and

$$\Psi_*(\frac{\partial}{\partial w_i})(e^z, w) = \tilde{\Psi}_*(\frac{\partial}{\partial w_i})(z,w) := (L_{\exp(\sum_{i=1}^p (S_i+N_i)z_i)})_* \Phi_*(\frac{\partial}{\partial w_i})(z,w).$$

Since $\Phi_*(\frac{\partial}{\partial w_i})$ is horizontal, when $\Re z_i \to -\infty$ for $i = 1, \ldots, p$, by Lemma 2.6 and the continuity we conclude that $\Psi_*(\frac{\partial}{\partial w_i})(0,w) = a_*(\frac{\partial}{\partial w_i})$ is also horizontal. It follows from Lemma 2.6 that $\vartheta_*(\frac{\partial}{\partial w_i})$ is horizontal. In conclusion, ϑ is a horizontal mapping. We proved Theorem B.(ii). □

The rest of the paper is devoted to the proof of Theorem B.(iii). We will only deal with the case of one variable. We first start a lemma in linear algebra whose proof is direct.

Lemma 2.9. — *Let $S \in \mathrm{End}(V^\nabla)$ be semisimple with real eigenvalues. Then there exists a constant $C > 0$ so that*

$$\|\mathrm{Ad}\, e^{xS}\| \leq C \exp((\lambda_{\max} - \lambda_{\min}) \cdot |x|) \quad \textit{for all} \quad x \in \mathbb{R},$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the largest and smallest eigenvalue of S. Let $N \in \mathrm{End}(V^\nabla)$ be nilpotent. Then

$$\|\mathrm{Ad}\, e^{xN}\| \leq C'|x|^m$$

for some $C', m > 0$.

Here we fix a reference polarized Hodge structure $o \in \mathscr{D}$ which induces metrics for V^∇ and $\mathrm{End}(V^\nabla)$. $\|\mathrm{Ad}\, e^{xS}\|$ is the operator norm with respect to such metric of $\mathrm{End}(V^\nabla)$. □

The following two lemmas are due to Schmid [Sch73, Lemmas 8.12 & 8.19]. They are stated for period domains of real Hodge structures. However, their proofs can be generalized to period domains of complex polarized Hodge structures verbatim, and we thus omit their proofs here.

Lemma 2.10. — *If $g \in GL(V^\nabla)$, then for some natural distance $d_{\check{\mathscr{D}}}$ of $\check{\mathscr{D}}$, we have*

$$d_{\check{\mathscr{D}}}(ga, gb) \leq \|\mathrm{Ad}\, g\| d_{\check{\mathscr{D}}}(a,b)$$

for any points $a, b \in \check{\mathscr{D}}$. □

Lemma 2.11. — *Let $\Phi : \mathbb{H} \to \mathscr{D}$ be the period map associated to a complex polarized variation of Hodge structures on Δ^*. Fix $\alpha, k > 0$ and a reference point $o \in \mathscr{D}$. Choose $g(z) \in G = U(V^\nabla, Q)$ so that $g(z) \cdot o = \Phi(z)$. Then there exist $C, \beta > 0$ so that if $|\Im z| \leq k$, one has*

$$\|\mathrm{Ad}\, g(z)^{-1}\| \leq C|\Re z|^\beta$$

for $\Re z < -\alpha$. Here $\|\mathrm{Ad}\, g(z)^{-1}\|$ is the operator norm defined in Lemma 2.9. □

We recall here that $G := U(V^\nabla, Q)$ acts transitively on the period domain $\mathscr{D}$, and $\mathscr{D}$ admits a natural G-invariant distance $d_{\mathscr{D}}$.

Proof of Theorem B.(iii). — Let $T \in GL(V^\nabla)$ be the monodromy operator associated to the counter-clockwise generator of $\pi_1(\Delta^*)$. Note that $T \in G := U(V^\nabla, Q)$. Recall that there exist commuting $S, N \in GL(V^\nabla)$ so that

— $\exp(2\pi i(S + N)) = T$;

— S is semisimple with eigenvalues lying in $(\alpha - 1, \alpha]$;

— N is nilpotent.

Denote by $a=\Psi(0)$. Then for $|t|$ small enough, one has

$$d_{\check{\mathscr{D}}}(a,\Psi(t))<C|t| \quad \text{for some} \quad C>0,$$

which is equivalent to that

$$d_{\check{\mathscr{D}}}(a,\Psi(e^z))<Ce^x \tag{2.3.1}$$

when $x\leq -M$ for some $M>0$. Here we write $z=x+iy$. Assume now $|y|\leq 2\pi$ and $x\leq -M$. Then

$$\begin{aligned}
d_{\check{\mathscr{D}}}(\exp(-(S+N)z)a,\Phi(z)) &\leq \|\mathrm{Ad}\exp((S+N)z)\|\cdot d_{\check{\mathscr{D}}}(a,\Psi(e^z))\\
&\leq \|\mathrm{Ad}\exp(Nx)\|\cdot\|\mathrm{Ad}\exp(i(S+N)y)\|\cdot\|\mathrm{Ad}\exp(Sx)\|\cdot d_{\check{\mathscr{D}}}(a,\Psi(e^z))\\
&\leq C_1\|\mathrm{Ad}\exp(Nx)\|\cdot\|\mathrm{Ad}\exp(Sx)\|\cdot d_{\check{\mathscr{D}}}(a,\Psi(e^z))\\
&\leq C_2|x|^m\cdot\exp((\lambda_{\max}-\lambda_{\min})\cdot|x|)\cdot d_{\check{\mathscr{D}}}(a,\Psi(e^z))\\
&\leq C_3|x|^m\cdot\exp((\lambda_{\max}-\lambda_{\min})\cdot|x|)\cdot e^x\leq C_3|x|^m e^{\delta x}.
\end{aligned}$$

The first inequality is due to Lemma 2.10, the third one holds since $|y|\leq 2\pi$, the fourth one follows from Lemma 2.9, and the fifth one follows from (2.3.1). Here $\lambda_{\max}$ and $\lambda_{\min}$ are the largest and smallest eigenvalue of S. Therefore, $\lambda_{\max}-\lambda_{\min}<1$ and thus the last inequality can be achieved for some $\delta>0$. Here $C_1,\ldots,C_3>0$ are some positive constants.

Fix a reference point $o\in\mathscr{D}$ and let $g(z)\in G$ so that $g(z)\cdot o=\Phi(z)$. By Lemmas 2.10 and 2.11 one gets

$$\begin{aligned}
d_{\check{\mathscr{D}}}(g(z)^{-1}\exp(-(S+N)z)a,o) &\leq \|\mathrm{Ad}\, g(z)^{-1}\|\cdot d_{\check{\mathscr{D}}}(\exp(-(S+N)z)a,\Phi(z))\\
&\leq C_4|x|^{m+\beta}e^{\delta x}
\end{aligned} \tag{2.3.2}$$

if $|y|\leq 2\pi$ and $x<-M_2$ for some $M_2\geq M$ and $C_4,\beta>0$. Pick a small neighborhood U of o in $\mathscr{D}$ so that the distance functions $d_{\mathscr{D}}$ and $d_{\check{\mathscr{D}}}$ are mutually bounded over U. By (2.3.2) when $|y|\leq 2\pi$, $x\leq -M_3$ for some $M_3\geq M_2$, $g(z)^{-1}\exp(-(S+N)z)a$ will be entirely contained in U. Note that $g(z)\in G$, it follows that $\exp(-(S+N)z)a\in\mathscr{D}$ if $|y|\leq 2\pi$ and $x\leq -M_3$. When $|y|>2\pi$ and $x\leq -M_3$, we find some integer ℓ so that $|y-2\pi\ell|\leq 2\pi$. Then $\exp(-(S+N)(z-2\pi i\ell))a\in\mathscr{D}$. Since $\exp(-(S+N)z)a=T^{-\ell}\exp(-(S+N)(z-2\pi i\ell))a$ and $T\in G$, it follows that $\exp(-(S+N)z)a\in\mathscr{D}$. In conclusion, $\exp(-(S+N)z)a\in\mathscr{D}$ if $x\leq -M_3$. We prove the first claim in Theorem B.(iii).

Recall that the distance functions $d_{\mathscr{D}}$ and $d_{\check{\mathscr{D}}}$ are mutually bounded over U. By (2.3.2) again for some $C_5>0$ we have

$$d_{\mathscr{D}}(g(z)^{-1}\exp(-(S+N)z)a,o)\leq C_5|x|^{m+\beta}e^{\delta x}.$$

for $|y|\leq 2\pi$, $x\leq -M_3$. Since the action of $g(z)$ is $d_{\mathscr{D}}$-distance invariant, we obtain the distance estimate

$$d_{\mathscr{D}}(\exp(-(S+N)z)a,\Phi(z))\leq C_5|x|^{m+\beta}e^{\delta x}$$

for $|y|\leq 2\pi$, $x\leq -M_3$. When $|y|>2\pi$ and $x\leq -M_3$, one picks some integer ℓ so that $|y-2\pi\ell|\leq 2\pi$. Then

$$d_{\mathscr{D}}(\exp(-(S+N)(z-2\pi i\ell))a,\Phi(z-2\pi i\ell))\leq C_5|x|^{m+\beta}e^{\delta x}.$$

In other words,

$$d_{\mathscr{D}}(T^{\ell}\exp(-(S+N)z)a,T^{\ell}\Phi(z))\leq C_5|x|^{m+\beta}e^{\delta x}.$$

As T is also $d_{\mathscr{D}}$-distance invariant, it follows that

$$d_{\mathscr{D}}(\exp(-(S+N)z)a,\Phi(z))\leq C_5|x|^{m+\beta}e^{\delta x}.$$

for $x\leq -M_3$. The distance estimate is obtained. □

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