
REPRESENTATIONS OF FUNDAMENTAL GROUPS AND LOGARITHMIC SYMMETRIC DIFFERENTIAL FORMS

by

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In memory of Jean-Pierre Demailly

Abstract. — In this paper we develop some non-abelian Hodge techniques over complex quasi-projective manifolds X in both archimedean and non-archimedean contexts. In the non-archimedean case, we first generalize a theorem by Gromov-Schoen: for any Zariski dense representation $\rho : \pi_1(X) \rightarrow G(K)$, where G is a semisimple algebraic group defined over some non-archimedean local field K , we construct a ρ -equivariant harmonic map from X into the Bruhat-Tits building $\Delta(G)$ of G with some suitable asymptotic behavior. We then construct logarithmic symmetric differential forms over X when the image of such ρ is unbounded. Our main result in the archimedean case is that any semisimple representation $\sigma : \pi_1(X) \rightarrow \mathrm{GL}_N(\mathbb{C})$ is rigid provided that X does not admit logarithmic symmetric differential forms. Furthermore, such representation σ is conjugate to $\sigma' : \pi_1(X) \rightarrow \mathrm{GL}_N(\mathcal{O}_L)$ where \mathcal{O}_L is the ring of integer of some number field L , so that σ' is a complex direct factor of a \mathbb{Z} -variation of Hodge structures.

As an application we prove that a complex quasi-projective manifold X has nonzero global logarithmic symmetric differential forms if there is linear representation $\pi_1(X) \rightarrow \mathrm{GL}_N(\mathbb{C})$ with infinite images.

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0. Introduction

0.1. Motivation. — In this paper we prove a logarithmic version of a conjecture by Esnault in the linear case.

Key words and phrases. — Logarithmic symmetric differential forms, (pluri-)harmonic mapping, Bruhat-Tits buildings, tame pure imaginary harmonic bundle, rigid representation, Simpson's integrality conjecture, variation of Hodge structures.

Theorem A. — *Let X be a quasi-projective manifold and let \bar{X} be a smooth projective compactification of X such that $D := \bar{X} - X$ is a simple normal crossing divisor. Assume that there is a linear representation $\tau : \pi_1(X) \rightarrow \mathrm{GL}_N(\mathbb{C})$ so that the image is infinite. Then there exists $k \in \mathbb{N}^*$ such that*

$$H^0(\bar{X}, \mathrm{Sym}^k \Omega_{\bar{X}}(\log D)) \neq 0.$$

When X is compact Kähler, Theorem A was proven by Brunebarbe-Klingler-Totaro [BKT13]. In the present paper, we follow the general strategy in [BKT13] but the proof is more involved: to prove Theorem A, we have to establish several non-abelian Hodge results in both non-archimedean and archimedean contexts, which we believe are of independent interest.

0.2. Main results in non-abelian Hodge theories. — Our first result is the existence of harmonic maps to Euclidean buildings associated to representations of fundamental groups into algebraic groups over non-archimedean local field.

Theorem B (=Theorems 2.1 and 2.15). — *Let X be a complex quasi-projective manifold, and let K be a non-archimedean local field. For any Zariski dense representation of $\rho : \pi_1(X) \rightarrow G(K)$ where G is a semisimple algebraic group defined over K , there exists a ρ -equivariant, locally Lipschitz harmonic mapping from the universal cover \tilde{X} of X to the Bruhat-Tits building $\Delta(G)$ of G . This harmonic mapping is moreover pluriharmonic and of logarithmic energy. Furthermore, the local energy around points at infinity are finite provided that the corresponding local monodromies are quasi-unipotent.*

Based on Theorem B and ideas in previous works [Kat97, Zuo96, Eys04, Kli13], we construct logarithmic symmetric differential forms for unbounded linear representations.

Theorem C (=Theorem 3.4). — *Let X be a complex quasi-projective manifold, and let K be a non-archimedean local field. If there is a reductive representation $\pi_1(X) \rightarrow \mathrm{GL}_N(K)$ whose image is not contained in any compact subgroup of $\mathrm{GL}_N(K)$, then X has non-zero global logarithmic symmetric forms.*

A crucial step in the proof of Theorem A is the following archimedean non-abelian Hodge result.

Theorem D (=Theorem 4.3). — *Let X be a complex quasi-projective manifold, which does not admit any global logarithmic symmetric differential forms. Then any semisimple representation $\rho : \pi_1(X) \rightarrow \mathrm{GL}_N(\mathbb{C})$ is rigid.*

We can reinforce Theorem D based on Theorem C.

Corollary E (=Corollary 5.1). — *Let X be a complex quasi-projective manifold, which does not admit any global logarithmic symmetric differential forms. Then any semisimple representation $\rho : \pi_1(X) \rightarrow \mathrm{GL}_N(\mathbb{C})$ is conjugate to an integral one, i.e. a semi-simple representation $\rho' : \pi_1(X) \rightarrow \mathrm{GL}_N(\mathcal{O}_K)$ where \mathcal{O}_K is the ring of integer of a number field K . Moreover, ρ' is a complex direct factor of a \mathbb{Z} -variation of Hodge structures.*

Corollary E is the key ingredient in the proof of Theorem A.

0.3. Previous related results. — In the case when X is projective, Theorems B to D are already known to us: Theorem B was established by Gromov-Schoen in their pioneering work [GS92] of harmonic mappings into non-positively curved metric spaces. Based on Gromov-Schoen's theorem, Katzarkov [Kat97] and Zuo [Zuo96] independently proved Theorem C when X is compact. Theorem D was proved by

Arapura [Ara02] using Simpson's theorems on the properness of Hitchin fibration and the homeomorphism between Betti and Dolbeault moduli spaces.

Corollary E is related to Simpson's integrality conjecture: *a rigid linear representation of the fundamental group of a smooth projective manifold is integral*. For projective manifolds without symmetric differential forms, Simpson's conjecture was proved by Klingler [Kli13]. Recently, Esnault and Groechenig [EG18] proved that a cohomology rigid local system over a quasi-projective manifold with finite determinant and quasi-unipotent local monodromies at infinity is integral.

In the case when X is quasi-projective, it is worthwhile to mention a remarkable work by Corlette-Simpson [CS08]. For a non-archimedean local field K and a Zariski dense representation $\rho : \pi_1(X) \rightarrow \mathrm{PSL}_2(K)$ with quasi-unipotent monodromies at infinity, they proved that there exists a ρ -equivariant pluriharmonic mapping from the universal cover \tilde{X} of X to the Serre tree. With respect to the complete Poincaré-type metric on X , such pluriharmonic map has finite energy. Based on this result, they proved that rigid rank 2 local systems over quasi-projective manifolds with quasi-unipotent monodromies at infinity are of geometric origin.

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1. Preliminaries

1.1. NPC space and Euclidean buildings. — For the definitions in this subsection, we refer the readers to [BH99, Rou09].

Definition 1.1 (Geodesic space). — Let (X, d_X) be a metric space. A curve $\gamma : [0, \ell] \rightarrow X$ into X is called a geodesic if the length $d_X(\gamma(a), \gamma(b)) = b - a$ for any subinterval $[a, b] \subset [0, \ell]$. A metric space (X, d_X) is a *geodesic space* if there exists a geodesic connecting every pair of points in X .

Definition 1.2 (NPC space). — An NPC (non-positively curved) space (X, d_X) is a complete geodesic space that satisfies the following condition: for any three points $P, Q, R \in X$ and a geodesic $\gamma : [0, \ell] \rightarrow X$ with $\gamma(0) = Q$ and $\gamma(\ell) = R$

$$d^2(P, Q_t) \leq (1 - t)d^2(P, Q) + td^2(P, R) - t(1 - t)d^2(Q, R)$$

where $Q_t = \gamma(t\ell)$.

Smooth Riemannian manifold of nonpositive sectional curvature is an NPC space. Euclidean buildings are special types of NPC spaces.

Definition 1.3 (Euclidean building). — A Euclidean building of dimension n is a piecewise Euclidean simplicial complex Δ such that:

- Δ is the union of a collection \mathcal{A} of subcomplexes A , called apartments, such that the intrinsic metric d_A on A makes (A, d_A) isometric to the Euclidean space \mathbb{R}^n and induces the given Euclidean metric on each simplex.
- Given two apartments A and A' containing both simplices B and B' , there is a simplicial isometry from (A, d_A) to $(A', d_{A'})$ which leaves both B and B' pointwise fixed.
- Δ is locally finite.

Let G be a reductive group over a non-archimedean local field K . The Bruhat-Tits building $\Delta(G)$ associated to G is a Euclidean building, and $G(K)$ acts by isometries on $\Delta(G)$ and transitively on the set of apartments of $\Delta(G)$. Here $G(K)$ denotes the group of K -points of G . The dimension of $\Delta(G)$ is equal to the K -rank of the algebraic group G , namely the dimension of a maximal split torus in G .

1.2. Harmonic mapping to NPC spaces. — We mainly follow the definitions in [DM21a, §1.2] and [KS93]. Consider a map $f : \Omega \rightarrow X$ from an n -dimensional Riemannian manifold (Ω, g) to an NPC space (X, d_X) . When the target space X is a smooth Riemannian manifold of nonpositive sectional curvature, the energy of a smooth map $f : \Omega \rightarrow X$ is

$$E^f = \int_{\Omega} |df|^2 d\text{vol}_g$$

where (Ω, g) is a Riemannian domain and $d\text{vol}_g$ is the volume form of Ω . We say $f : \Omega \rightarrow X$ is harmonic if it is locally energy minimizing; i.e. for any $p \in \Omega$, there exists $r > 0$ such that the restriction $u|_{B_r(p)}$ minimizes energy amongst all maps $v : B_r(p) \rightarrow \tilde{X}$ with the same boundary values as $u|_{B_r(p)}$.

In this paper, we mainly consider the target X to be NPC spaces, not necessarily smooth. Let us recall the definition of harmonic maps in this context (cf. [KS93] for more details).

Let (Ω, g) be a bounded Lipschitz Riemannian domain. Let Ω_ϵ be the set of points in Ω at a distance least ϵ from $\partial\Omega$. Let $B_\epsilon(x)$ be a geodesic ball centered at x and $S_\epsilon(x) = \partial B_\epsilon(x)$. We say $f : \Omega \rightarrow X$ is an L^2 -map (or that $f \in L^2(\Omega, X)$) if

$$\int_{\Omega} d^2(f, P) d\text{vol}_g < \infty$$

For $f \in L^2(\Omega, X)$, define

$$e_\epsilon : \Omega \rightarrow \mathbb{R}, \quad e_\epsilon(x) = \begin{cases} \int_{y \in S_\epsilon(x)} \frac{d^2(f(x), f(y))}{\epsilon^2} \frac{d\sigma_{x,\epsilon}}{\epsilon^{n-1}} & x \in \Omega_\epsilon \\ 0 & \text{otherwise} \end{cases}$$

where $\sigma_{x,\epsilon}$ is the induced measure on $S_\epsilon(x)$. We define a family of functionals

$$E_\epsilon^f : C_c(\Omega) \rightarrow \mathbb{R}, \quad E_\epsilon^f(\varphi) = \int_{\Omega} \varphi e_\epsilon d\text{vol}_g.$$

We say f has finite energy (or that $f \in W^{1,2}(\Omega, X)$) if

$$E^f := \sup_{\varphi \in C_c(\Omega), 0 \leq \varphi \leq 1} \limsup_{\epsilon \rightarrow 0} E_\epsilon^f(\varphi) < \infty.$$

weakly to a measure which is absolutely continuous with respect to the Lebesgue measure. Therefore, there exists a function $e(x)$, which we call the energy density, so that $e_\epsilon(x) d\text{vol}_g \rightarrow e(x) d\text{vol}_g$. In analogy to the case of smooth targets, we write $|\nabla f|^2(x)$ in place of $e(x)$. Hence $|\nabla f|^2(x) \in L^1_{\text{loc}}(\Omega)$. In particular, the (Korevaar-Schoen) energy of f in Ω is

$$E^f[\Omega] = \int_{\Omega} |\nabla f|^2 d\text{vol}_g.$$

Definition 1.4 (Harmonic mapping). — We say a continuous map $f : \Omega \rightarrow X$ from a Lipschitz domain Ω is harmonic if it is locally energy minimizing; more precisely, at each $p \in \Omega$, there exists a neighborhood Ω_p of p so that all continuous comparison maps which agree with u outside of this neighborhood have no less energy.

Definition 1.5 (Locally Lipschitz). — A continuous map $f : \Omega \rightarrow X$ is called *locally Lipschitz* if for any $p \in \Omega$, there exists $B_\varepsilon(p) \subset \Omega$ and a constant $C > 0$ so that $d(f(x), f(y)) \leq Cd(x, y)$ for $x, y \in B_\varepsilon(p)$.

Remark 1.6. — It follows from the definition of $|\nabla f|^2$ that if f is locally Lipschitz, then for any $p \in \Omega$, there exists $B_\varepsilon(p) \subset \Omega$ and a constant $C > 0$ so that over $B_\varepsilon(p)$ one has $|\nabla f|^2 \leq C$.

1.3. Admissible coordinate. — The following definition of *admissible coordinate* will be used throughout the paper.

Definition 1.7. — (Admissible coordinate) Let \bar{X} be a complex manifold and let D be a simple normal crossing divisor. Let x be a point of \bar{X} , and assume that $\{D_j\}_{j=1, \dots, \ell}$ be components of D containing x . An *admissible coordinate* around x is the tuple $(U; z_1, \dots, z_n; \varphi)$ (or simply $(U; z_1, \dots, z_n)$ if no confusion arises) where

- U is an open subset of X containing x .
- there is a holomorphic isomorphism $\varphi : U \rightarrow \Delta^n$ so that $\varphi(D_j) = (z_j = 0)$ for any $j = 1, \dots, \ell$.

We define a (incomplete) Poincaré-type metric ω_P on $(\Delta^*)^\ell \times \Delta^{n-\ell}$ by

$$(1.1) \quad \omega_P = \sum_{j=1}^{\ell} \frac{\sqrt{-1} dz_j \wedge d\bar{z}_j}{|z_j|^2 (\log |z_j|^2)^2} + \sum_{k=\ell+1}^n \sqrt{-1} dz_k \wedge d\bar{z}_k.$$

Let us point out that with the notation of the definition, one can construct a complete metric on X which is of Poincaré type at any point of $D = D_1 + \dots + D_k$. Indeed, take any metric $\bar{\omega}$ on \bar{X} . For any $j = 1, \dots, k$, take a hermitian metric $\|\cdot\|_j$ on $\mathcal{O}_{\bar{X}}(D_j)$ and take a section $\sigma_j \in H^0(X, \mathcal{O}_{\bar{X}}(D_j))$ such that $D_j = (\sigma_j = 0)$ and such that $\|\sigma_j\|_j < 1$ over \bar{X} . Then it suffices to set, for some $C \in \mathbb{R}$ large enough

$$\omega := C\bar{\omega} + \sum_{j=1}^k \frac{d\|\sigma_j\| \wedge d^c\|\sigma_j\|}{\|\sigma_j\|^2 (\log \|\sigma_j\|)^2}.$$

To say that it is of Poincaré type around D , means that for any $x \in D$ and for any admissible coordinates at x as above and $C_1, C_2 > 0$ such that

$$C_1\omega_P \leq \omega \leq C_2\omega_P.$$

1.4. Tame and pure imaginary harmonic bundles. — Let \bar{X} be a compact complex manifold, $D = \sum_{i=1}^{\ell} D_i$ be a simple normal crossing divisor, $X = \bar{X} - D$ be the complement of D and $j : X \rightarrow \bar{X}$ be the inclusion.

Definition 1.8 (Higgs bundle). — A *Higgs bundle* on X is a pair (E, θ) where E is a holomorphic vector bundle with $\bar{\partial}_E$ its complex structure, and $\theta : E \rightarrow E \otimes \Omega_X^1$ is a holomorphic one form with value in $\text{End}(E)$, called the *Higgs field*, satisfying $\theta \wedge \theta = 0$.

Let (E, θ) be a Higgs bundle over a complex manifold X . Suppose that h is a smooth hermitian metric of E . Denote by D_h the Chern connection with respect to h , and by θ_h^\dagger the adjoint of θ with respect to h . We write θ^\dagger for θ_h^\dagger for short. The metric h is *harmonic* if the connection $D := D_h + \theta + \theta^\dagger$ is flat, that is, if $D^2 = 0$.

Definition 1.9 (Harmonic bundle). — A harmonic bundle on X is a Higgs bundle (E, θ) endowed with a harmonic metric h .

Let (E, θ, h) be a harmonic bundle on X . Let p be any point of D , and $(U; z_1, \dots, z_n)$ be an admissible coordinate around p . On U , we have the description:

$$(1.2) \quad \theta = \sum_{j=1}^{\ell} f_j d \log z_j + \sum_{k=\ell+1}^n f_k dz_k.$$

Definition 1.10 (Tameness). — Let t be a formal variable. For any $j = 1, \dots, \ell$, the characteristic polynomial $\det(f_j - t) \in \mathcal{O}(U^*)[t]$, is a polynomial in t whose coefficients are holomorphic functions. If those functions can be extended to the holomorphic functions over U for all j , then the harmonic bundle is called *tame* at p . A harmonic bundle is *tame* if it is tame at each point.

For a tame harmonic bundle (E, θ, h) over $\overline{X} - D$, we prolong E over \overline{X} by a sheaf of $\mathcal{O}_{\overline{X}}$ -module ${}^\circ E_h$ as follows:

$${}^\circ E_h(U) = \{\sigma \in \Gamma(U - D, E|_{U-D}) \mid |\sigma|_h \lesssim \prod_{i=1}^{\ell} |z_i|^{-\varepsilon} \text{ for all } \varepsilon > 0\}.$$

In [Moc07a] Mochizuki proved that ${}^\circ E_h$ is locally free and that θ extends to a morphism

$${}^\circ E_h \rightarrow {}^\circ E_h \otimes \Omega_{\overline{X}}(\log D),$$

which we still denote by θ .

Definition 1.11 (Pure imaginary, nilpotent residue)

Let (E, h, θ) be a tame harmonic bundle on $\overline{X} - D$. The residue $\text{Res}_{D_i} \theta$ induces an endomorphism of ${}^\circ E_h|_{D_i}$. Its characteristic polynomial has constant coefficients, and thus the eigenvalues are constant. We say that (E, θ, h) is *pure imaginary* (resp. has *nilpotent residue*) if for each component D_j of D , the eigenvalues of $\text{Res}_{D_i} \theta$ are all pure imaginary (resp. all zero).

One can verify that Definition 1.11 does not depend on the compactification \overline{X} of $\overline{X} - D$.

Theorem 1.12 (Mochizuki). — Let \overline{X} be a projective manifold and let D be a simple normal crossing divisor on \overline{X} . Let (E, θ, h) be a tame pure imaginary harmonic bundle on $X := \overline{X} - D$. Then the flat bundle $(E, D_h + \theta + \theta^\dagger)$ is semi-simple. Conversely, if (V, ∇) is a semisimple flat bundle on X , then there is a tame pure imaginary harmonic bundle (E, θ, h) on X so that $(E, D_h + \theta + \theta^\dagger) \simeq (V, \nabla)$.

2. Harmonic mapping into Bruhat-Tits buildings

Let X be a quasi-projective manifold and let $\rho : \pi_1(X) \rightarrow G(K)$ be a Zariski dense representation where G is a semi-simple algebraic group defined over a non-archimedean local field K . When X is projective, in [GS92] Gromov-Schoen proved a celebrated theorem on the existence of a ρ -equivariant pluriharmonic map from the universal cover \tilde{X} of X into the Bruhat-Tits building $\Delta(G)$. In this section, we will generalize Gromov-Schoen's theorem to the quasi-projective context. We also prove a logarithmic energy growth estimate for this map (cf. Theorem 2.15), which plays an important role in the proof of construction of logarithmic symmetric differential forms via harmonic maps (cf. Theorem 3.4).

Theorem 2.1 (Existence of harmonic maps). — Let X be a quasi-projective manifold, G be a semi-simple linear algebraic group defined on a non-archimedean local field K and $\Delta(G)$ be the Bruhat-Tits building of G . If $\rho : \pi_1(X) \rightarrow G(K)$

is a Zariski dense representation, then there is a ρ -equivariant harmonic map $\tilde{u} : \tilde{X} \rightarrow \Delta(G)$ of logarithmic energy growth (cf. Definition 2.6). Moreover, this map is locally Lipschitz and pluriharmonic.

2.1. Summary of the proof of Theorem 2.1. — We give a summary of the proof of Theorem 2.1. Note that Theorem B was proved in [DM21a] *under the additional assumption that ρ does not fix non-empty strict subsets of X* . The point of the proof of Theorem B is to remove this assumption. To do so, consider a non-empty closed minimal convex $\rho(\pi_1(X))$ -invariant subset \mathcal{C} of X . The existence of such \mathcal{C} is guaranteed by [CM09, Theorem 4.3, (Aii)]. As a convex subset of an NPC space, \mathcal{C} is itself an NPC space. Thus, for domain dimensions 1 and 2, we can apply the existence theorems of an equivariant harmonic map into an NPC space stated in Lemma 2.9 and Lemma 2.12 (and proved in [DM21a, Theorem 7.1] and [DM21a, Theorem 11.1] respectively).

The uniqueness assertion in Lemma 2.9 for harmonic maps from punctured Riemann surfaces is proved by a slight variation of the argument in [DM21b, Section 3]. The argument relies on the minimality of \mathcal{C} as well as the assumption that ρ is Zariski dense (which implies that ρ does not fix a point at infinity, cf. Lemma 2.2 (i)). We give an outline of this argument in Remark 2.10.

The harmonic map u from a punctured Riemann surface $\mathcal{R} = \bar{\mathcal{R}} \setminus \{p_1, \dots, p_n\}$ satisfies a logarithmic growth estimate towards a puncture in the form of

$$(2.1) \quad \frac{L^2}{2\pi} \log \frac{1/2}{r} \leq E^u[\Delta_{r,1/2}] \leq \frac{L^2}{2\pi} \log \frac{1/2}{r} + c, \quad 0 < r \leq \frac{1}{2}$$

where Δ is a holomorphic disk of $\bar{\mathcal{R}}$ at a puncture, $\Delta_{r,1}$ is an annulus with inner and outer radius r and 1 respectively, $E^u[\Delta_{r,1}]$ is the energy contained in $\Delta_{r,1}$ and L is the translation length of the isometry $\rho(\gamma)$ where $\gamma \in \pi_1(X)$ is the element corresponding the loop $\partial\Delta$ around the puncture. One can interpret L as the length of the minimum geodesic homotopic to the image loop $u(\partial\Delta)$ in the quotient metric space $X/\rho(\pi_1(X))$, and the lower bound of (2.1) is a consequence of the fact that the energy of the (parameterized) geodesic loop is at least $\frac{L^2}{2\pi}$. Indeed,

$$\frac{L^2}{2\pi} \log \frac{1}{r} \leq \int_r^1 \frac{L^2}{2\pi r} dr = \int_0^{2\pi} \int_r^1 \frac{1}{r^2} \left| \frac{\partial u}{\partial \theta} \right|^2 r dr d\theta \leq E^u[\Delta_{r,1}].$$

(Note that $L = 0$ is the case when the loop $u(\partial\Delta)$ is homotopically trivial.)

The upper bound in (2.1) comes from the construction of the harmonic map which we review now. The idea is to first construct a Lipschitz map v with controlled energy growth towards any puncture. This can be accomplished by defining $\theta \mapsto v(r, \theta)$ to be a parameterized geodesic loop corresponding to $\rho(\gamma)$ for each r where (r, θ) is the polar coordinates of the disk Δ . We call such a map a *prototype map*. Next, let \mathcal{R}_r be a Riemann surface with Δ_r (disk of radius r) removed around each puncture. Let u_r be the Dirichlet solution with boundary values $v|_{\partial\mathcal{R}_r}$. We thus construct a family of harmonic maps u_r and prove that the sequence of harmonic maps u_{r_i} converges uniformly to a harmonic map u on every compact subset of \mathcal{R} as $r_i \rightarrow 0$. The upper bound on the energy growth towards a puncture of u can be deduced from the energy growth of the prototype map v . The details of the above argument can be found in [DM21a, Chapter 3].

In the case of quasi-projective surfaces, the existence also relies on the construction of a prototype map but is technically more complicated because the points at infinity are the normal crossing divisors and hence much more complicated than

isolated points. The details of this construction can be found in [DM21a, Chapter 4]. By the Bochner method described in [DM21a, Section 17], the harmonic map is actually pluriharmonic; in other words, its restriction to any complex submanifold is harmonic.

The proof of the existence in higher dimensions follows the argument by Mochizuki in [Moc07b] and relies on the existence results in dimensions 1 and 2. The key is to view X as a union of hyperplanes. First, assume that $\dim_{\mathbb{C}} X = 3$. The existence result in dimension 2 guarantees that there is a pluriharmonic map defined on each hyperplane. The uniqueness result implies that the pluriharmonic maps defined on the two different hyperplanes agrees along the intersection. Thus, we can define a pluriharmonic map on X . To define a pluriharmonic map in any dimensions, we proceed by induction on the dimension. This details of this argument is given in § 2.5.

2.2. Preliminary lemmas. — Throughout the rest of this section, we fix a Zariski dense representation $\rho : \pi_1(X) \rightarrow G(K)$ as in Theorem 2.1. Below, we summarize some results regarding the action of ρ .

Lemma 2.2. — *If $H = \rho(\pi_1(X))$, then the following holds:*

- (i) *The action of H on $\Delta(G)$ is without fixed points at infinity.*
- (ii) *$\Delta(G)$ contains a non-empty closed minimal convex H -invariant set \mathcal{C} , and*
- (iii) *The action of H on \mathcal{C} is proper.*

Here, \mathcal{C} is minimal means that there does not exist a non-empty convex strict subset of \mathcal{C} invariant under H . The notion of proper is defined in [KS97, Section 2] and is used to prove the existence of equivariant harmonic maps.

Proof. — If H fixes a point at infinity, then H is contained a $P(K)$ where P is a parabolic subgroup of G . This contradicts the fact that H is Zariski dense and proves Item (i). Item (ii) follows from [CM09, Theorem 4.3, (Aii)]. We can argue as follows: suppose $\Delta(G)$ has no minimal closed convex H -invariant set. Then it contains a decreasing sequence X_n of closed convex H -invariant sets whose intersection is empty. Choose now a base point x in $\Delta(G)$ and consider the projection x_n of x to X_n . This sequence is unbounded (otherwise the intersection was not empty). Since the space is proper, it converges to some point at infinity. This point at infinity is fixed by any h in H because the distance $d(h.x_n, x_n)$ is bounded by $d(h.x, x)$ by Lemma 2.3 below. This proves Item (ii). Item (iii) is contained in [KS97, Theorem 2.2.1]. \square

Lemma 2.3. — *The closest point projection map $\Pi : \Delta(G) \rightarrow \mathcal{C}$ is a distance decreasing G -equivariant map; i.e. $\Pi(gP) = g\Pi(P)$.*

Proof. — Given $P \in \Delta(G)$, $\Pi(P)$ is the unique point that minimizes the distance from P amongst points in \mathcal{C} . Thus, for $g \in G$,

$$d(g\Pi(P), gP) = d(\Pi(P), P) \leq d(g^{-1}Q, P) = d(Q, gP), \quad \forall Q \in \mathcal{C}$$

which implies $\Pi(gP) = g\Pi(P)$. The map Π is distance decreasing by [BH99, II.2.2]. \square

As a closed convex subset of an NPC space, \mathcal{C} is itself is a NPC space. Define

$$(2.2) \quad \hat{\rho} : \pi_1(X) \rightarrow \text{Isom}(\mathcal{C})$$

by setting $\hat{\rho}(\gamma)$ is be the restriction of $\rho(\gamma)$ to \mathcal{C} .

Lemma 2.4. — *$\hat{\rho}(\pi_1(X)) \subset \text{Isom}(\mathcal{C})$ consists of only semi-simple elements.*

Proof. — Since G is semi-simple, $\Delta(G)$ is a Euclidean building without a Euclidean factor. Let $\hat{g} \in \hat{\rho}(\pi_1(X))$ such that $\hat{g} = g|_{\mathcal{C}}$ for some $g \in G$. By [Par00, Theorem 4.1] and the assumption that $\Delta(G)$ is without a Euclidean factor, g is either elliptic or hyperbolic. Thus, there exists $P_0 \in \Delta(G)$ such that $\min_{P \in \Delta(G)} d(P, gP) = d(P_0, gP_0)$. By Lemma 2.3, Π is distance decreasing and G -invariant, and thus

$$\begin{aligned} \min_{P \in \Delta(G)} d(P, gP) &= d(P_0, gP_0) \geq d(\Pi(P_0), \Pi(gP_0)) = d(\Pi(P_0), g\Pi(P_0)) \\ &\geq \min_{P \in \mathcal{C}} d(P, gP) \geq \min_{P \in \Delta(G)} d(P, gP). \end{aligned}$$

In particular,

$$d(\Pi(P_0), \hat{g}\Pi(P_0)) = d(\Pi(P_0), g\Pi(P_0)) = \min_{P \in \mathcal{C}} d(P, gP) = \min_{P \in \mathcal{C}} d(P, \hat{g}P),$$

and hence \hat{g} is a semi-simple isometry of \mathcal{C} . \square

Remark 2.5. — Semisimple isometry (resp. commuting pair of semisimple isometries) has exponential decay in the sense of [DM21a, Definition 2.1] (resp. [DM21a, Definition 3.1]).

2.3. Equivariant maps and sections. — Let \mathcal{C} be as in Lemma 2.2 and $\hat{\rho} : \pi_1(X) \rightarrow \text{Isom}(\mathcal{C})$ be as in (2.2). The set of all $\hat{\rho}$ -equivariant maps into \mathcal{C} are in one-to-one correspondence with the set of all sections of the fiber bundle $\Pi : \tilde{X} \times_{\hat{\rho}} \mathcal{C} \rightarrow X$. More precisely,

$$\tilde{f} : \tilde{X} \rightarrow \mathcal{C} \quad \longleftrightarrow \quad f : X \rightarrow \tilde{X} \times_{\hat{\rho}} \mathcal{C}$$

is defined by the relationship $f(\Pi(\tilde{p})) = [(\tilde{p}, \tilde{f}(\tilde{p}))]$.

Let \bar{X} be a projective compactification of X so that $D = \bar{X} - X$ is a simple normal crossing divisor with irreducible components D_1, \dots, D_j . Let $\bar{Y} \subset \bar{X}$ be a sufficiently ample smooth divisor such that $\bar{Y} \cap D$ is a normal crossing divisor and $\iota_Y : Y \rightarrow \bar{X}$ be the inclusion map. Then, by a version of the Lefschetz hyperplane theorem (cf. [Moc07b, Lemma 21.8] or [DM21a, Lemma 19.1]), $(\iota_Y)_* : \pi_1(Y) \rightarrow \pi_1(X)$ is onto. As discussed in [DM21a, Section 19], this fact leads us to view a section

$$u : Y \rightarrow \tilde{Y} \times_{\rho_Y} \mathcal{C}$$

as a map

$$u : Y \rightarrow \tilde{X} \times_{\hat{\rho}} \mathcal{C}.$$

Conversely, given a section $u : X \rightarrow \tilde{X} \times_{\hat{\rho}} \mathcal{C}$ and $Y \subset X$, its restriction

$$(2.3) \quad u|_Y : Y \rightarrow \tilde{X} \times_{\hat{\rho}} \mathcal{C}$$

defines a section of $\tilde{Y} \times_{\rho_Y} \mathcal{C} \rightarrow Y$ which we denote by

$$u_Y : Y \rightarrow \tilde{Y} \times_{\rho_Y} \mathcal{C}.$$

We denote the corresponding ρ_Y -equivariant map by

$$(2.4) \quad \tilde{u}_Y : \tilde{Y} \rightarrow \mathcal{C}.$$

2.4. Logarithmic energy growth. — Fix a very ample line bundle $L \rightarrow \bar{X}$ and let $H^0(\bar{X}, L)$ and $\mathbb{P}(H^0(\bar{X}, L)^*)$ be the space of global holomorphic sections and its projectified space respectively. For any element $s \in \mathbb{P}(H^0(\bar{X}, L)^*)$, we let

$$(2.5) \quad \bar{Y}_s := s^{-1}(0), \quad Y_s := \bar{Y}_s \setminus D \quad \text{and} \quad \iota_{Y_s} : Y_s \rightarrow X \quad \text{be the inclusion map.}$$

For a $\hat{\rho}$ -equivariant map $\tilde{u} : \tilde{X} \rightarrow \mathcal{C}$, let

$$(2.6) \quad u_{Y_s} : \tilde{Y}_s \rightarrow \mathcal{C}$$

be the ρ_{Y_s} -equivariant map (cf. (2.4)) where

$$(2.7) \quad \rho_{Y_s} = \hat{\rho} \circ \iota_{Y_s}.$$

If $\dim_{\mathbb{C}} X = 1$, then let $\mathfrak{R} = \{X\}$. If $\dim_{\mathbb{C}} X = n \geq 2$, then let \mathfrak{R} be the set of all \mathcal{R} where $\bar{\mathcal{R}} := Y_{s_1} \cap \dots \cap Y_{s_{n-1}}$ such that the hyperplanes $Y_{s_1} = s_1^{-1}(0), \dots, Y_{s_{n-1}} = s_{n-1}^{-1}(0)$ intersects transversely and $Y_{s_i} \cap D$ is a normal crossing divisor.

Definition 2.6. — We say a ρ -equivariant harmonic map $\tilde{u} : \tilde{X} \rightarrow \mathcal{C}$ has *logarithmic energy growth* if, for every $\mathcal{R} \in \mathfrak{R}$ and $p \in \bar{\mathcal{R}} \cap D$,

$$(2.8) \quad \frac{L_\gamma^2}{2\pi} \log \frac{1}{r} \leq E^{\tilde{u}_{\mathcal{R}}}[\Delta_{r,1}] \leq \frac{L_\gamma^2}{2\pi} \log \frac{1}{r} + c$$

where Δ is a conformal disk in $\bar{\mathcal{R}}$ centered at p and $\Delta_{r,1} = \{r < |z| < 1\}$. The constant L_γ is the translation length of $\rho([\gamma])$ where $\gamma \in \pi_1(X)$ is the element corresponding to the loop around the smooth component D_i of the divisor D ; i.e.

$$(2.9) \quad L_\gamma := \inf_{P \in \mathcal{C}} d(\rho([\gamma])P, P).$$

The constant c depends only on L_γ and the Lipschitz constant of $\tilde{u}_{\mathcal{R}}|_{\partial\Delta}$.

Remark 2.7. — The notion of logarithmic energy growth given in Definition 2.6 is slightly stronger than that of [DM21a, Definition 7.5] because of the dependence required of c .

Definition 2.8. — We say a ρ -equivariant map $\tilde{u} : \tilde{X} \rightarrow \mathcal{C}$ has *sub-logarithmic energy growth* if, for every $\mathcal{R} \in \mathcal{C}$, $p \in \bar{\mathcal{R}} \cap D$, $\epsilon > 0$, fundamental domain $F \subset \bar{\mathcal{R}}$ and $P_0 \in \tilde{X}$,

$$\lim_{|z| \rightarrow 0} d(\tilde{u}_{\mathcal{R}}(\tilde{z}), P_0) + \epsilon \log |z| = -\infty, \quad \text{for } z \in \Delta^*, \tilde{z} \in \Pi^{-1}(z) \cap F$$

where Δ is a conformal disk in $\bar{\mathcal{R}}$ centered at p and $\Delta^* = \{0 < |z| < 1\}$.

Lemma 2.9. — Let $\mathcal{C} \subset \Delta(G)$ be as in Lemma 2.2, $Y = \bar{Y} \setminus \{p_1, \dots, p_n\}$ where \bar{Y} is a compact Riemann surface, and assume that $\rho_Y : \pi_1(Y) \rightarrow \text{Isom}(\mathcal{C})$ does not fix a point at infinity. Then there exists a unique ρ_Y -equivariant harmonic map $f : \tilde{Y} \rightarrow \mathcal{C}$ with sub-logarithmic growth.

Remark 2.10. — Before we prove Lemma 2.9, we give a brief outline of the uniqueness assertion in the above lemma. The argument relies on the minimality of \mathcal{C} as well as the fact that ρ does not fix a point at infinity (since it is Zariski dense, cf. Lemma 2.2 (i)). Assume on the contrary that there exists two distinct equivariant maps u_0 and u_1 into \mathcal{C} . The minimality of \mathcal{C} as a $\rho(\pi_1(M))$ -invariant set and the equivariance of u_0 and u_1 imply that \mathcal{C} is the convex hull of $u_0(\tilde{M})$ and of $u_1(\tilde{M})$. Thus, the geodesic segment $\bar{\sigma}_P$ connecting $P = u_0(x)$ to $u_1(x)$ is contained in \mathcal{C} .

Additionally, we have the following:

- (1) The harmonicity of u_0 and u_1 implies that geodesic segments $\bar{\sigma}_P$ and $\bar{\sigma}_Q$ are parallel for any two points $P = u_0(x)$ and $Q = u_0(y)$.
- (2) The ρ -equivariance of u_0 and u_1 implies that we have a $\rho(\pi_1(M))$ -invariant field of parallel geodesic segments defined on $u_0(\tilde{M})$.

The field of $\rho(\pi_1(M))$ -invariant parallel geodesic segments can be naturally extended to the convex hull \mathcal{C} .

Next, for any point $P_0 \in \mathcal{C}$, we can extend the geodesic segment $\bar{\sigma}_{P_0}$ by joining $\bar{\sigma}_{P_0}$ and $\bar{\sigma}_{P_1}$ where P_1 is the other endpoint of $\bar{\sigma}_{P_0}$. (Note that the join is again a geodesic since the two segments are parallel.) Let P_2 be the other endpoint of $\bar{\sigma}_{P_1}$. We can repeat this process the joining $\bar{\sigma}_{P_0} \cup \bar{\sigma}_{P_1}$ and $\bar{\sigma}_{P_2}$. This process can be continued indefinitely to produce a geodesic ray $\sigma_{P_0} = \bar{\sigma}_{P_0} \cup \bar{\sigma}_{P_1} \cup \bar{\sigma}_{P_2} \cup \dots$. In this way, we construct a $\rho(\pi_1(M))$ -invariant field of parallel geodesic ray on \mathcal{C} . The parallel geodesic rays define a single point at infinity fixed by ρ , contradicting the assumption on ρ .

Proof of Lemma 2.9. — To prove existence, we use the fact that \mathcal{C} is an NPC space and apply [DM21a, Theorem 7.1] for which the assumptions are:

- (A) the action of $\rho(\pi_1(Y))$ is proper, and
- (B) $\rho(\lambda^j)$ has exponential decay for $j = 1, \dots, n$ where $\lambda^j \in \pi_1(Y)$ is the element associated to the loop around the puncture p_j .

Lemma 2.2 (iii) implies assumption (A) and Lemma 2.4 and Remark 2.5 imply assumption (B).

To prove uniqueness, we use the minimality of \mathcal{C} and a slight variation of the argument in [DM21b, Section 3] where the target space is a building. We shall assume on the contrary that $\tilde{u}_0, \tilde{u}_1 : \tilde{Y} \rightarrow \mathcal{C}$ are distinct ρ_Y -invariant harmonic maps. The following three steps lead to a contradiction to the assumption that ρ_Y does not fix a point at infinity.

STEP 1. Define the sets

$$(2.10) \quad C_0, C_1, \dots, C_n$$

inductively as follows: First, let $C_0 = \tilde{u}_0(\tilde{Y})$, and then let C_n be the union of the images of all geodesic segments connecting points of C_{n-1} . The $\rho_Y(\pi_1(Y))$ -invariance of C_0 implies the $\rho_Y(\pi_1(Y))$ -invariance of C_n . The set $\bigcup_{n=0}^{\infty} C_n$ is the convex hull of the image of \tilde{u}_0 , and the minimality of \mathcal{C} implies

$$\mathcal{C} = \bigcup_{n=0}^{\infty} C_n.$$

STEP 2. To each $Q \in \mathcal{C}$, we assign a geodesic segment $\bar{\sigma}^Q$ in \mathcal{C} as follows: First, for $Q = \tilde{u}_0(q) \in C_0$, let

$$(2.11) \quad \bar{\sigma}^Q : [0, 1] \rightarrow \mathcal{C}, \quad \bar{\sigma}^Q(t) = (1-t)\tilde{u}_0(q) + t\tilde{u}_1(q).$$

In the above, the weighted sum $(1-t)P + tQ$ is used to denote the points on the geodesic between a pair of point P and Q . Since \mathcal{C} is a convex subset of $\Delta(G)$, \tilde{u}_0 and \tilde{u}_1 are harmonic as maps into $\Delta(G)$. Thus, we can apply [DM21b, (3.16)] to conclude that $\{\bar{\sigma}^Q\}_{Q \in C_0}$ is a family of pairwise parallel of geodesic segments of uniform length. (We can assume they are all unit length by normalizing the target space.) Since \tilde{u}_0 and \tilde{u}_1 are both ρ_Y -equivariant, the assignment $Q \mapsto \bar{\sigma}^Q$ is $\rho_Y(\pi_1(Y))$ -equivariant; i.e. $\rho(\gamma)\bar{\sigma}^Q = \bar{\sigma}^{\rho(\gamma)Q}$ for any $Q \in C_0$ and $\gamma \in \rho_Y(\pi_1(Y))$.

For $n \in \mathbb{N}$, we inductively define a $\rho_Y(\pi_1(Y))$ -equivariant map from C_n to a family of pairwise parallel geodesic segments as follows: For any pair of points $Q_0, Q_1 \in C_{n-1}$, apply the Flat Quadrilateral Theorem (cf. [BH99, 2.11]) with vertices $Q_0, Q_1, P_1 := \bar{\sigma}^{Q_1}(1), P_0 := \bar{\sigma}^{Q_0}(1)$ to define a one-parameter family of parallel geodesic segments $\bar{\sigma}^{Q_t} : [0, 1] \rightarrow \mathcal{C}$ with initial point $Q_t = (1-t)Q_0 + tQ_1$ and terminal point $P_t = (1-t)P_0 + tP_1$. The inductive hypothesis implies that the

map $Q \mapsto \bar{\sigma}^Q$ defined on C_n is also $\rho_Y(\pi_1(Y))$ -equivariant. The above construction defines a $\rho_Y(\pi_1(Y))$ -equivariant map

$$Q \mapsto \bar{\sigma}^Q$$

from \mathcal{C} to a family of pairwise parallel geodesic segments contained in \mathcal{C} .

STEP 3. We extend these geodesic segments into a geodesic ray as follows: For $Q \in \mathcal{C}$, we inductively construct a sequence $\{Q_i\}$ of points in \mathcal{C} by first setting $Q_0 = Q$ and then defining $Q_i = \bar{\sigma}^{Q_{i-1}}(\frac{3}{4})$. Next, let

$$L^Q = \bigcup_{i=0}^{\infty} I^{Q_i}$$

where $I^{Q_i} = \bar{\sigma}^{Q_i}([0, 1])$. Therefore, L^Q is a union of pairwise parallel geodesic segments. Thus, $\{L^Q\}_{Q \in \mathcal{C}}$ is a family of pairwise parallel geodesic rays. Moreover, the $\rho_Y(\pi_1(Y))$ -equivariance of the map $Q \mapsto \bar{\sigma}^Q$ implies $\rho(\gamma)\bar{\sigma}^{Q_{i-1}}(\frac{3}{4}) = \bar{\sigma}^{\rho(\gamma)Q_{i-1}}(\frac{3}{4})$. Thus, if $\{Q_i\}$ is the sequence constructed starting with $Q_0 = Q$, then $\{\rho(\gamma)Q_i\}$ is the sequence constructed starting with $\rho(\gamma)Q_0 = \rho(\gamma)Q$. We thus conclude

$$\rho(\gamma)L^Q = \bigcup_{i=-\infty}^{\infty} \rho(\gamma)I^{Q_i} = \bigcup_{i=-\infty}^{\infty} I^{\rho(\gamma)Q_i} = L^{\rho(\gamma)Q}.$$

We are done by letting the geodesic ray $\sigma^Q : [0, \infty) \rightarrow \mathcal{C}$ be the extension of the geodesic segment $\bar{\sigma}^Q : [0, 1] \rightarrow \mathcal{C}$ parameterizing L^Q . Consequently, we have constructed a $\rho_Y(\pi_1(Y))$ -equivariant map

$$Q \mapsto \sigma^Q$$

from \mathcal{C} to a family of pairwise parallel geodesic rays.

The above construction shows that $\rho_Y(\pi_1(Y))$ fixes the equivalence class $[\sigma^Q]$ of geodesic rays; i.e. $\rho_Y : \pi_1(Y) \rightarrow \text{Isom}(\mathcal{C})$ fixes a point at infinity. This contradiction proves the uniqueness assertion. \square

Lemma 2.11. — *Let $\tilde{u} : \tilde{X} \rightarrow \mathcal{C}$ be a ρ -equivariant harmonic map. Then \tilde{u} has logarithmic energy growth if and only if \tilde{u} has sub-logarithmic growth.*

Proof. — Let $\mathcal{R} \in \mathcal{C}$. It follows from [DM21a, Lemma 8.9] that if $\tilde{u}_{\mathcal{R}}$ has logarithmic energy growth, then $\tilde{u}_{\mathcal{R}}$ has sub-logarithmic growth. Conversely, if $\tilde{u}_{\mathcal{R}}$ has sub-logarithmic growth, then Lemma 2.9 implies that it is the unique $\rho_{\mathcal{R}}$ -equivariant harmonic map of sub-logarithmic growth into \mathcal{C} . Since \mathcal{C} is a convex subset of $\Delta(G)$, \tilde{u} is also an ρ -equivariant harmonic map of sub-logarithmic growth into $\Delta(G)$. By [DM21a, Theorem 7.1 and Theorem 7.2], $\tilde{u}_{\mathcal{R}}$ satisfies

$$\frac{L_{\gamma}^2}{2\pi} \log \frac{1}{r} \leq E^{\tilde{u}_{\mathcal{R}}}[\Delta_{r,1}] \leq \frac{L_{\gamma}^2}{2\pi} \log \frac{1}{r} + c$$

for some constant c . We are done by showing that this constant c depends only on L_{γ} and the Lipschitz bound of $\tilde{u}_{\mathcal{R}}$ on $\partial\Delta$. Indeed, by the uniqueness assertion for the Dirichlet problem on the punctured disk Δ^* of [DM21a, Theorem 8.1], the constant c only depends on L_{γ} and k , the locally Lipschitz section which defines the boundary value of \tilde{u} . (Note that $a = b = 0$ in [DM21a, Theorem 8.1] since $\rho(\gamma)$ is semisimple.) An inspection of the proof shows that the dependence on k is simply a dependence of the Lipschitz constant of k on the boundary $\partial\Delta$. \square

2.5. Proof of Theorem 2.1. — The following lemma is a restatement of [DM21a, Theorem 11.1] when the target is $\mathcal{C} \subset \Delta(G)$ as in Lemma 2.2 and $\hat{\rho} : \pi_1(X) \rightarrow \text{Isom}(\mathcal{C})$ as in (2.2).

Lemma 2.12. — *If $\dim_{\mathbb{C}} X = 2$ and X is endowed with a global Poincaré-type metric given in [DM21a, Section 13], then there exists a $\hat{\rho}$ -equivariant harmonic map $\tilde{u} : \tilde{X} \rightarrow \mathcal{C}$ of sub-logarithmic growth.*

To prove Theorem 2.1, it is sufficient to prove the following statement:

(*) There exists a $\hat{\rho}$ -equivariant harmonic map $u : \tilde{X} \rightarrow \mathcal{C} \subset \Delta(G)$ of logarithmic energy growth. The map u is unique amongst $\hat{\rho}$ -equivariant pluriharmonic map of sub-logarithmic growth into \mathcal{C} .

To prove the assertion above, we use a slight variation of the inductive proof of [DM21a, Theorem 4]. Let

$$U = \{s \in \mathbb{P}(H^0(\bar{X}, L)^*) : \bar{Y}_s \text{ smooth and } \bar{Y}_s \cap D \text{ is a normal crossing}\}.$$

For $q \in X$, consider the subspace

$$V(q) = \{s \in H^0(\bar{X}, L) : s(q) = 0\} \text{ and } U(q) = U \cap \mathbb{P}(V(q)^*).$$

The sets $U, U(q)$ are Zariski open subsets of $\mathbb{P}(H^0(\bar{X}, L)^*), \mathbb{P}(q)$ respectively. Note that $(\iota_Y)_* : \pi_1(Y) \rightarrow \pi_1(X)$ is onto. $\rho(\pi_1(Y_s)) = \rho(\pi_1(X))$ which implies that ρ_{Y_s} does not fix a point at infinity.

Initial Step of the Induction. Assume $\dim_{\mathbb{C}} X = 2$. Lemma 2.12 implies the existence of $\hat{\rho}$ -equivariant harmonic map $u : \tilde{X} \rightarrow \mathcal{C}$. Following the proof of [DM21a, Theorem 22.1], we conclude that u is pluriharmonic.

To prove the uniqueness assertion of (*), let $v : \tilde{X} \rightarrow \mathcal{C}$ be another ρ -equivariant pluriharmonic map into \mathcal{C} with sub-logarithmic growth. For $q \in X$ and $s \in U(q)$, let Y_s and ρ_{Y_s} as in (2.5) and (2.7). Consider the sections of the fiber bundle $\tilde{X} \times_{\rho} \mathcal{C} \rightarrow X$ defined by the pluriharmonic maps u and v , and denote their restrictions to Y_s by $u_{Y_s} : Y_s \rightarrow \tilde{Y}_s \times_{\rho_{Y_s}} \mathcal{C}$ and $v_{Y_s} : Y_s \rightarrow \tilde{Y}_s \times_{\rho_{Y_s}} \mathcal{C}$ respectively as in (2.6). Since Y_s is a Riemann surface and ρ_{Y_s} does not fix a point at infinity, we can apply the uniqueness assertion of Lemma 2.9 to conclude $u_{Y_s} = v_{Y_s}$. Since q is an arbitrary point in X , we conclude $u = v$.

Inductive Step. We assume that statement (*) whenever $\dim_{\mathbb{C}} X = 2, \dots, n-1$. Now assume that $\dim_{\mathbb{C}} X = n$. The inductive hypothesis implies that, for each $s \in \mathbb{P}(H^0(\bar{X}, L)^*)$, there exists a ρ_{Y_s} -equivariant harmonic map of sub-logarithmic growth

$$u_s : \tilde{Y}_s \rightarrow \mathcal{C}.$$

Denote the associated section by

$$u^s : Y_s \rightarrow \tilde{Y}_s \times_{\rho_{Y_s}} \mathcal{C} \hookrightarrow \tilde{X} \times_{\rho} \mathcal{C}.$$

Lemma 2.13. — *For $q \in X$ and $s_1, s_2 \in U(q)$, we have $u^{s_1}(q) = u^{s_2}(q)$.*

Proof. — For $i = 1, 2$ and $q \in X$, let

$$U(s_i, q) = \{s \in U(q) : \bar{Y}_s \text{ transversal to } \bar{Y}_{s_i} \text{ and } D \cap \bar{Y}_s \cap \bar{Y}_{s_i} \text{ normal crossing}\}.$$

The set $U(s_i, q)$ is Zariski open in $U(q)$ which implies $U(s_1, q) \cap U(s_2, q) \neq \emptyset$. Fix $s \in U(s_1, q) \cap U(s_2, q)$. Let $\iota : Y_{s_i} \cap Y_s \rightarrow X$ be the inclusion map. By the inductive hypothesis, there exists pluriharmonic sections of sub-logarithmic growth

$$u^s : Y_s \rightarrow \tilde{Y}_s \times_{\rho_{Y_s}} \mathcal{C} \text{ and } \tilde{u}^{s_i} : Y_{s_i} \rightarrow \tilde{Y}_{s_i} \times_{\rho_{Y_{s_i}}} \mathcal{C}.$$

Denote their restriction by

$$u_{Y_{s_i} \cap Y_s}^s : Y_{s_i} \cap Y_s \rightarrow \widetilde{Y_{s_i} \cap Y_s} \times_{\rho_{Y_{s_i} \cap Y_s}} \mathcal{C}$$

and

$$u_{Y_{s_i} \cap Y_s}^{s_i} : Y_{s_i} \cap Y_s \rightarrow \widetilde{Y_{s_i} \cap Y_s} \times_{\rho_{Y_{s_i} \cap Y_s}} \mathcal{C}.$$

By the uniqueness assertion of the inductive hypothesis, they are in fact the same section; i.e. the sections u^s and u^{s_i} restricted to $Y_{s_i} \cap Y_s$ is equal. Since $q \in Y_{s_i} \cap Y_s$, we conclude $u^{s_i}(q) = u^s(q)$. \square

By Lemma 2.13, we can define

$$u : X \rightarrow \widetilde{X} \times_{\rho} \mathcal{C}, \quad u(q) := u_s(q) \text{ for } s \in U(q).$$

To complete the inductive step, we are left to show that u is a pluriharmonic section with sub-logarithmic growth and moreover unique amongst pluriharmonic sections with sub-logarithmic growth. Given $q \in X$ and any complex 1-dimensional subspace $\mathcal{P} \subset T_q^{1,0}(X)$, we consider the non-empty algebraic set

$$A_{\mathcal{P}}(q) = \{s \in \mathbb{P}(q) : \mathcal{P} \subset T_q^{1,0}(Y_s)\}$$

and its Zariski open subset

$$U_{\mathcal{P}}(q) = A_{\mathcal{P}}(q) \cap U(q).$$

Let $s \in U_{\mathcal{P}}(q)$. By the construction of u , its restriction u_{Y_s} is the unique pluriharmonic section of sub-logarithmic growth. Thus, all derivatives of $u|_{Y_s}$ exist in the direction of \mathcal{P} and

$$\partial_{E'} \bar{\partial}|_{\mathcal{P}} u = \partial_{E'} \bar{\partial}|_{\mathcal{P}} u_s = 0,$$

thereby proving that u is pluriharmonic with sub-logarithmic growth.

To prove uniqueness of u , let $v : \widetilde{X} \rightarrow \mathcal{C}$ be another ρ -equivariant pluriharmonic map of logarithmic energy growth. For $q \in X$, let $s \in U(q)$. The restriction v_{Y_s} of the section defined v is a pluriharmonic section of logarithmic energy growth. By the uniqueness assertion of the inductive hypothesis, we conclude $u_{Y_s} = v_{Y_s}$. Since q is an arbitrary point in X , we conclude $u = v$.

2.6. Local logarithmic energy growth estimate. — We begin with the following lemma.

Lemma 2.14. — *Let X be a quasi-projective manifold of dimension n and let \overline{X} be a projective compactification of X so that $D := \overline{X} - X$ is a simple normal crossing divisor. Fix a very ample divisor L on X . Then any smooth point x_0 of the divisor D has an admissible coordinate neighborhood $(U; z_1, \dots, z_n)$ so that for any $\zeta = (\zeta_2, \dots, \zeta_n) \in \Delta^{n-1}$, the transverse disk $\{(z_1, \zeta_2, \dots, \zeta_n) \in \Delta^n \mid |z_1| < 1\}$ is contained in some complete intersection $H_1 \cap \dots \cap H_{n-1}$, where each H_i is smooth hypersurface in $|L|$ so that H_1, \dots, H_{n-1}, D intersect transversely.*

Proof. — Since $x_0 \in D$ is a smooth point, there are thus $s_2, \dots, s_n \in H^0(\overline{X}, L)$ so that

- the hyperplanes $\overline{Y}_{s_2}, \dots, \overline{Y}_{s_n}$ are smooth, where $\overline{Y}_{s_i} := s_i^{-1}(0)$.
- $\overline{Y}_{s_2}, \dots, \overline{Y}_{s_n}, D$ intersect transversely.
- $x_0 \in \overline{Y}_{s_2} \cap \dots \cap \overline{Y}_{s_n}$.

Pick some $s_1 \in H^0(\overline{X}, L)$ so that $x_0 \notin (s_1 = 0)$. Let $u_i := \frac{s_i}{s_1}$ which are global rational functions of \overline{X} and regular on some neighborhood U of x_0 . Fix some trivialization of D on U so that s_D is given by a holomorphic function $f \in \mathcal{O}(U)$. Then $df \wedge du_2 \wedge \dots \wedge du_n(x_0) \neq 0$. After shrinking U , we may assume that

- $df \wedge du_2 \wedge \dots \wedge du_n(x) \neq 0$ for all $x \in U$;
- The map

$$\begin{aligned} \varphi : U &\rightarrow \mathbb{C}^n \\ x &\mapsto (f(x), u_2(x), \dots, u_n(x)) \end{aligned}$$

is biholomorphic to its image $\varphi(U) = \Delta_\varepsilon^n$, where $\Delta_\varepsilon := \{z \in \mathbb{C} \mid |z| < \varepsilon\}$.

In particular, it defines a local coordinate $(z_1, \dots, z_n) = (f(x), u_2(x), \dots, u_n(x))$ of U centering at x_0 . For any $(\zeta_2, \dots, \zeta_n) \in \Delta_\varepsilon^{n-1}$, the transverse disk $\{(z, \zeta_2, \dots, \zeta_n) \in \Delta_\varepsilon^n \mid |z_1| < \varepsilon\}$ is contained in $U \cap (u_2 - \zeta_2 = 0) \cap \dots \cap (u_n - \zeta_n)$. The later is contained in $(s_2 - \zeta_2 s_1 = 0) \cap \dots \cap (s_n - \zeta_n s_1 = 0)$. Recall that $\bar{Y}_{s_2}, \dots, \bar{Y}_{s_n}, D$ intersect transversely. One thus can make $\varepsilon > 0$ small enough so that the hyperplanes $(s_2 - \zeta_2 s_1 = 0), \dots, (s_n - \zeta_n s_1 = 0)$ in $|L|$ and D intersect transversely for any $(\zeta_2, \dots, \zeta_n) \in \Delta_\varepsilon^{n-1}$. The lemma follows after we compose φ with some rescaling

$$\begin{aligned} \Delta_\varepsilon^n &\rightarrow \Delta^n \\ (z_1, \dots, z_n) &\mapsto \left(\frac{z_1}{\varepsilon}, \dots, \frac{z_n}{\varepsilon}\right). \end{aligned}$$

□

Pick any smooth point x_0 of the divisor D ; i.e. x_0 is a point on a irreducible component D_i away from the crossings. Let $(U; z_1, \dots, z_n)$ be an admissible coordinate neighborhood of x_0 as in Lemma 2.14. We write

$$U(r) := \{(z_1, \dots, z_n) \in U \mid r < |z_1| < \frac{1}{2}, |z_2| < \frac{1}{2}, \dots, |z_n| < \frac{1}{2}\}.$$

Recall that ω_P is a Poincaré-type metric on $U(0)$ defined by

$$\omega_P = \frac{\sqrt{-1} dz_1 \wedge d\bar{z}_1}{|z_1|^2 (\log |z_1|^2)^2} + \sum_{k=2}^n \sqrt{-1} dz_k \wedge d\bar{z}_k.$$

Denote by $(P_{i\bar{j}})$ and $(P^{i\bar{j}})$ the components of the metric tensor and its inverse in the coordinate basis of $U(0)$.

Theorem 2.15. — *The energy density function of the pluriharmonic map \tilde{u} of Theorem 2.1 satisfies a logarithmic decay estimate of the form*

$$(2.12) \quad -\frac{L_\gamma^2}{2\pi} \log r \leq \int_{U(r)} |\nabla \tilde{u}|_{\omega_P}^2 d\text{vol}_{\omega_P} \leq -\frac{L_\gamma^2}{2\pi} \log r + C$$

for some constant $C > 0$ which does not depend on r . Here $\gamma \in \pi_1(X)$ is the element corresponding to the loop $\theta \mapsto (\frac{1}{2}e^{\sqrt{-1}\theta}, z_*)$ around the divisor D_i , and L_γ is the translation length of $\rho(\gamma)$ defined in (2.9). Moreover, the above energy $\int_{U(0)} |\nabla \tilde{u}|_{\omega_P}^2 d\text{vol}_{\omega_P}$ is finite provided that $\rho(\gamma) \in G(K)$ is quasi-unipotent.

Proof. — Let $p : \tilde{X} \rightarrow X$ be the universal covering map. Fix $z_* := (z_{2*}, \dots, z_{n*})$, $w_* := (w_{2*}, \dots, w_{n*}) \in \Delta_{\frac{1}{2}}^{n-1} := \{|z_2| < \frac{1}{2}, \dots, |z_n| < \frac{1}{2}\}$. Let Γ be the positive x -axis, define $(z, z_*) \mapsto P_{z, z_*}$ (resp. $(z, w_*) \mapsto P_{z, w_*}$) to be lifting with respect to p of the inclusion map $\Delta \setminus \Gamma \hookrightarrow U \subset X$, $z \mapsto (z, z_*)$ (resp. $z \mapsto (z, w_*)$). The function

$$\delta_{z_*, w_*}(z) := d^2(\tilde{u}(P_{z, z_*}), \tilde{u}(P_{z, w_*}))$$

defined on $\Delta \setminus \Gamma$ extends to a continuous function

$$\delta_{z_*, w_*} : \Delta^* = \{0 < |z| < 1\} \rightarrow [0, \infty).$$

Since \tilde{u} is pluriharmonic, it is harmonic with respect to any choice of a Kähler metric on \tilde{X} . We fix a smooth Kähler metric ω on X and denote its lift by $p^*\omega$.

By [GS92, Theorem 2.4], \tilde{u} is locally Lipschitz continuous with respect to the distance function on \tilde{X} induced by the metric $p^*\omega$. Both metrics ω and ω_P of (1.1) are uniformly bounded from above and below by the standard Euclidean metric $\sum_{k=1}^n \sqrt{-1} dz_k \wedge d\bar{z}_k$ on $U(r)$ for $r > 0$. Thus, the local Lipschitz continuity of \tilde{u} with respect to $p^*\omega$ implies the Lipschitz continuity of \tilde{u} with respect to $p^*\omega_P$ in $p^{-1}(U(r))$ for $r > 0$. Let $\Lambda > 0$ be the Lipschitz constant of \tilde{u} in $p^{-1}(U(\frac{1}{2}))$. Then

$$\delta_{z_*, w_*}(z) \leq \Lambda^2 |z_* - w_*|^2 \quad \text{for } |z| = \frac{1}{2}.$$

Let Y_{z_*} and Y_{w_*} be a complete intersection from Lemma 2.14. Since \tilde{u} is pluriharmonic, the maps $\tilde{u}_{Y_{z_*}}$ and $\tilde{u}_{Y_{w_*}}$ (cf. (2.4)) are harmonic maps. Consequently, the function δ_{z_*, w_*} is a continuous subharmonic function on $\Delta_{\frac{1}{2}}^*$ (cf. [KS93, Remark 2.4.3]). Since the pluriharmonic map \tilde{u} has sub-logarithmic growth, the triangle inequality implies that for $\epsilon > 0$

$$\lim_{|z| \rightarrow 0} \delta_{z_*, w_*}(z) + \epsilon \log |z| = -\infty.$$

Thus, δ_{z_*, w_*} extends to subharmonic function on $\Delta_{\frac{1}{2}} = \{|z| < \frac{1}{2}\}$ (cf. [DM21a, Lemma 8.2]). We can apply the maximum principle to conclude that

$$\delta_{z_*, w_*}(z) \leq \sup_{\zeta \in \partial \mathbb{D}_{\frac{1}{2}}} \delta_{z_*, w_*}(\zeta) \leq \Lambda^2 |z_* - w_*|^2, \quad \forall z \in \mathbb{D}_{\frac{1}{2}}.$$

Since the constant Λ is independent of $z_*, w_* \in \Delta_{\frac{1}{2}}^{n-1}$,

$$(2.13) \quad \left| \frac{\partial \tilde{u}}{\partial z_j} \right|^2(z_1, z_*) \leq \Lambda^2, \quad \forall j = 2, \dots, n,$$

where coordinates (z_1, \dots, z_n) are lifted to define local coordinates of $p^{-1}(U(0))$ and $\left| \frac{\partial \tilde{u}}{\partial z_j} \right|^2$ is defined as in [DM21a, Section 1.2].

Let L_γ be the translation length of $\rho([\gamma])$ and $\gamma \in \pi_1(X)$ be the element corresponding to the loop $\theta \mapsto (\frac{1}{2}e^{\sqrt{-1}\theta}, z_*)$ around the divisor D_i . Since the pluriharmonic map \tilde{u} has logarithmic energy growth by Lemma 2.11,

$$(2.14) \quad -\frac{L_\gamma^2}{2\pi} \log r \leq E^{\tilde{u}_{Y_{z_*}}}[\Delta_{r, \frac{1}{2}}] \leq -\frac{L_\gamma^2}{2\pi} \log r + c$$

for any fixed $z_* \in \Delta_{\frac{1}{2}}^{n-1}$ and $\Delta_{r, \frac{1}{2}} = \{(z, z_*) \in U(r) : r < |z| < \frac{1}{2}\}$. The constant $c > 0$ of (2.14) is dependent only on L_γ and the Lipschitz estimate of $\tilde{u}_{Y_{z_*}}$ on $\partial \Delta$. Thus, c depends on L_γ and Λ . Integrating (2.14) over $z_* \in \Delta_{\frac{1}{2}}^{n-1}$ while noting

$$E^{\tilde{u}_{Y_{z_*}}}[\Delta_{r, \frac{1}{2}}] = \int_{\Delta_{r, \frac{1}{2}}} \left| \frac{\partial \tilde{u}}{\partial z_1} \right|^2(z, z_*) \frac{dz \wedge d\bar{z}}{-2i},$$

we conclude

$$(2.15) \quad -\frac{L_\gamma^2}{2\pi} \log r \leq \int_{U(r)} \left| \frac{\partial \tilde{u}}{\partial z_1} \right|^2(z, z_*) d\text{vol}_0 \leq -\frac{L_\gamma^2}{2\pi} \log r + c', \quad 0 < r \leq \frac{1}{2}$$

for constants $c' \geq 0$ independent of r where $d\text{vol}_0$ is the standard Euclidean volume form on $U(r)$. Since

$$\begin{aligned} \int_{U(r)} |\nabla u|_{\omega_P}^2 d\text{vol}_{\omega_P} &= \int_{U(r)} \left(P^{1\bar{1}} \left| \frac{\partial u}{\partial z_1} \right|^2 + \sum_{j=2}^n P^{j\bar{j}} \left| \frac{\partial u}{\partial z_j} \right|^2 \right) d\text{vol}_{\omega_P} \\ &= \int_{U(r)} \left(\left| \frac{\partial u}{\partial z_1} \right|^2 + \frac{1}{|z_j|^2 (\log |z_j|^2)^2} \sum_{j=2}^n \left| \frac{\partial u}{\partial z_j} \right|^2 \right) d\text{vol}_0, \end{aligned}$$

the assertion (2.12) follows from (2.13) and (2.15).

To prove the last claim, it then suffices to show that $L_\gamma = 0$. Since the finiteness of local energy is preserved under finite unramified covers, we can assume that $\rho(\gamma)$ is unipotent. Then there exists a Borel subgroup B of G such that $\rho(\gamma) \in U(K)$, where U is the unipotent radical of B . Note that $U(K)$ fixes a sector-germ of the standard apartment A , which means that there exists a Weyl chamber C^v of the apartment A such that if u in $U(K)$, then u fixes $x + C^v$, for some x in A . In particular, $\rho(\gamma)$ fixes a point $y \in A$. Consider the minimal closed convex $\rho(\pi_1(X))$ -invariant subset $\mathcal{C} \subset \Delta(G)$ constructed in Lemma 2.2. By Lemma 2.3, the closest point projection map $\Pi : \Delta(G) \rightarrow \mathcal{C}$ is G -equivariant map, which implies that $\rho(\gamma)\Pi(y) = \Pi(\rho(\gamma)y) = \Pi(y)$. By (2.9), this implies that $L_\gamma = 0$. \square

3. Logarithmic symmetric forms and harmonic mapping

Let X be a quasi-projective manifold and let G be a semi-simple algebraic group over a non-archimedean local field K . Assume that $\rho : \pi_1(X) \rightarrow G(K)$ is a Zariski dense representation. By Theorem 2.1, there is a locally Lipschitz ρ -equivariant harmonic mapping $u : \tilde{X} \rightarrow \Delta(G)$ which is pluriharmonic of logarithmic energy growth. In this section we will construct logarithmic symmetric forms on X using this pluriharmonic mapping u . The construction we presented here is close to that in [Kli13] (cf. [Eys04, Kat97, Zuo96] for other constructions).

Let \bar{X} be a projective compactification of X so that $D = \bar{X} - X$ is a simple normal crossing divisor. We fix a complete metric ω on X which is of Poincaré type near D . For the pluriharmonic mapping $u : \tilde{X} \rightarrow \Delta(G)$, recall that $|\nabla u|^2 \in L_{\text{loc}}^1(\tilde{X})$ is the energy density function in § 1.2. By Remark 1.6, $|\nabla u|^2$ is moreover locally bounded.

Let us recall the following definition in [GS92].

Definition 3.1. — The regular locus of u is the set of all points $x \in \tilde{X}$ so that there exists a neighborhood Ω_x of x and an apartment A in $\Delta(G)$ such that $h(\Omega_x) \subset A$.

Note that $G(K)$ acts transitively on the set of apartments of $\Delta(G)$. Since u is ρ -equivariant, it follows that if x is in the regular locus of u , then $\gamma \cdot x$ also lies in the regular locus of u for any $\gamma \in \pi_1(X)$. Hence we can define $x \in X$ to be in the regular locus of u if some (thus any) $\tilde{x} \in \pi^{-1}(x)$ lies in the regular locus of u , where $\pi : \tilde{X} \rightarrow X$ denotes the universal mapping. The complement of the regular locus of u is called the singular set of u , denoted by $\mathcal{S}(u)$.

Fix now an apartment A in $\Delta(G)$, which is isometric to \mathbb{R}^N . Here N is the K -rank of G . Let $W_{\text{aff}} \subset \text{Isom}(A)$ be the affine Weyl group of $\Delta(G)$ and $W = W_{\text{aff}} \cap \text{GL}(A)$ is the finite reflection group. For the root system $\Phi = \{\alpha_1, \dots, \alpha_m\} \subset A^* - \{0\}$ of $\Delta(G)$, one has

$$\{w^* \alpha_1, \dots, w^* \alpha_m\} = \{\alpha_1, \dots, \alpha_m\} \quad \text{for any } w \in W.$$

In other words, the action of w^* on Φ is a permutation. Note that $W_{\text{aff}} = W \ltimes \Lambda$, where Λ is a lattice acts on A by translations. It follows that

$$(3.1) \quad \{w^*d\alpha_1, \dots, w^*d\alpha_m\} = \{d\alpha_1, \dots, d\alpha_m\} \quad \text{for any } w \in W_{\text{aff}}.$$

Here we consider $d\alpha_i$ as global real one-forms on A .

For any regular point $x \in X$ of u , there exists an open simply-connected neighborhood U of x so that

- the inverse image $\pi^{-1}(U) = \sqcup_{i \in I} U_i$ is a union of disjoint open sets in \tilde{X} , each of which is mapped homeomorphically onto U by π .
- For some $i \in U$, there is an apartment A_i of $\Delta(G)$ and a connected component U_i of $\pi^{-1}(U)$, so that $u(U_i) \subset A_i$.

As u is ρ equivariant and $G(K)$ acts transitively on the set of apartments of $\Delta(G)$, for any other connected component U_j of $\pi^{-1}(U)$, $u(U_j)$ is contained in some other apartments. We fix any apartment A_j which contains $u(U_j)$.

As $G(K)$ acts transitively on the set of apartments of $\Delta(G)$, choose $g_i, g_j \in G(K)$ so that $g_i(A_i) = A$ and $g_j(A_j) = A$.

Set $u_i = g_i u \circ (\pi|_{U_i})^{-1} : U \rightarrow A$ and $u_j = g_j u \circ (\pi|_{U_j})^{-1} : U \rightarrow A$. They are pluriharmonic maps and thus $\{u_i^*(d\alpha_1)_{\mathbb{C}}^{(1,0)}, \dots, u_i^*(d\alpha_m)_{\mathbb{C}}^{(1,0)}\}$ and $\{u_j^*(d\alpha_1)_{\mathbb{C}}^{(1,0)}, \dots, u_j^*(d\alpha_m)_{\mathbb{C}}^{(1,0)}\}$ are sets of holomorphic one forms on U . We claim that these two sets coincide and thus do not depend on the choice of U_i and A_i .

Choose $\gamma \in \pi_1(X)$ so that γ maps U_i to U_j isomorphically. Since $\rho(\gamma)u \circ (\pi|_{U_i})^{-1} = u \circ (\pi|_{U_j})^{-1}$, one has

$$u_j = g_j \rho(\gamma) g_i^{-1} u_i.$$

As $g_j \rho(\gamma) g_i \in G(K)$, this implies that there is $w \in W_{\text{aff}}$ so that $u_j = w u_i$. We conclude that

$$\begin{aligned} \{u_j^*(d\alpha_1)_{\mathbb{C}}^{(1,0)}, \dots, u_j^*(d\alpha_m)_{\mathbb{C}}^{(1,0)}\} &= \{u_i^* w^*(d\alpha_1)_{\mathbb{C}}^{(1,0)}, \dots, u_i^* w^*(d\alpha_m)_{\mathbb{C}}^{(1,0)}\} \\ &= \{u_i^*(d\alpha_1)_{\mathbb{C}}^{(1,0)}, \dots, u_i^*(d\alpha_m)_{\mathbb{C}}^{(1,0)}\} \end{aligned}$$

where the last equality follows from (3.1).

Let T be a formal variable. Then

$$(3.2) \quad \prod_{k=1}^m (T - u_i^*(d\alpha_k)_{\mathbb{C}}^{(1,0)}) =: T^m + \sigma_{1,x} T^{m-1} + \dots + \sigma_{m,x}$$

is well defined, and does not depend on the choice of lift U_i and apartment A_i . Its coefficients $\sigma_{k,x} \in \Gamma(U, \text{Sym}^k \Omega_U)$. As this construction is canonical, these local holomorphic symmetric differential forms $\sigma_{k,x}$ glue together into an element in $H^0(X - \mathcal{S}(u), \text{Sym}^k \Omega_{X - \mathcal{S}(u)})$, denoted by σ_k . By our construction, there is a uniform constant $C_k > 0$ so that

$$(3.3) \quad |\sigma_k|_{\omega} \leq C_k |\nabla u|_{\omega}^k \quad \text{over } X - \mathcal{S}(u).$$

Let us now apply the energy estimate of u at infinity in Theorem 2.15 to prove that these symmetric differential forms σ_k extend to logarithmic ones over \overline{X} .

Proposition 3.2. — *For any $k \in \{0, \dots, m\}$, the symmetric form σ_k extends to a logarithmic symmetric form $H^0(\overline{X}, \text{Sym}^k \Omega_{\overline{X}}(\log D))$. Moreover, if u is not constant, there exists some k so that $\sigma_k \neq 0$.*

Proof. — By [GS92, Theorem 6.4], $\mathcal{S}(u)$ is a closed subset of X of Hausdorff codimension at least two. Since u is locally Lipschitz, for any $x \in X$, there are a neighborhood Ω_x of x and a constant C_x so that $|\nabla u|_\omega \leq C_x$ on Ω_x . Hence over $\Omega_x - \mathcal{S}(u)$ one has

$$|\sigma_k|_\omega \leq C_k |\nabla u|_\omega^k \leq C_k C_x^k.$$

By a result of removable singularity in [Shi68, Lemma 3.(ii)], σ_k extends to a holomorphic symmetric form in $H^0(X, \text{Sym}^k \Omega_X)$, which we still denote by σ_k .

Pick any point x in the smooth locus of D , namely $x \in D_i - \cup_{j \neq i} D_j$ for some i . By Theorem 2.15, there is an admissible coordinate $(U; z_1, \dots, z_n)$ centered at x so that $D = (z_1 = 0)$, and one has

$$\int_{U(r)} |\nabla u|_{\omega_P}^2 d\text{vol}_{\omega_P} < -C'_1 \log r + C'_2$$

for some constants $C'_1, C'_2 > 0$ which does not depend on r . Here we write

$$U(r) := \left\{ (z_1, \dots, z_n) \in U \mid r < |z_1| < \frac{1}{2}, |z_2| < \frac{1}{2}, \dots, |z_n| < \frac{1}{2} \right\},$$

and ω_P is the Poincaré-type metric over $\Delta^* \times \Delta^{n-1}$ defined in (1.1). Write $\sigma_k = \sum_{|\alpha|=k} \tau_\alpha(z) dz^\alpha$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ with $|\alpha| := \sum_{i=1}^n \alpha_i$, and $dz^\alpha := dz_1^{\alpha_1} \cdots dz_n^{\alpha_n}$. Then τ_α are holomorphic functions over $U - D$. One has the norm estimate

$$\left(\sum_{\alpha} |\tau_\alpha|^2 \cdot |z_1|^{2\alpha_1} \log^{2\alpha_1} |z_1| \right)^{\frac{1}{k}} = |\sigma_k|_{\omega_P}^{\frac{2}{k}} \leq C'_k |\nabla u|_{\omega_P}^2$$

for some constant C'_k depending only on k . The last inequality is due to the fact that $\omega_P \sim \omega$ over $U(0)$ and (3.3). It follows that

$$\int_{U(r)} (|\tau_\alpha|^2 \cdot |z_1|^{2\alpha_1} \log^{2\alpha_1} |z_1|)^{\frac{1}{k}} d\text{vol}_{\omega_P} < -C_k C'_1 \log r + C_k C'_2.$$

Let us first prove that $\tau_\alpha(z)$ extends to a meromorphic function over U for each α . Pick some even $m > 0$ so that

$$\begin{aligned} \int_{U(r)} |z_1|^{m-1} |\tau_\alpha|^{\frac{2}{k}} idz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge idz_n \wedge d\bar{z}_n &\leq C \int_{U(r)} (|\tau_\alpha|^2 \cdot |z_1|^{2\alpha_1} \log^{2\alpha_1} |z_1|)^{\frac{1}{k}} d\text{vol}_{\omega_P} \\ &\leq -C' \log r \end{aligned}$$

for some positive constant C and C' which do not depend on r . Write

$$F(r) := \int_{U(r)} |z_1|^{m-1} |\tau_\alpha|^{\frac{2}{k}} idz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge idz_n \wedge d\bar{z}_n.$$

Then

$$\begin{aligned} \int_{U(r)} |z_1|^m |\tau_\alpha|^{\frac{2}{k}} idz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge idz_n \wedge d\bar{z}_n &= - \int_r^{\frac{1}{2}} t F'(t) dt \\ &= r F(r) + \int_r^{\frac{1}{2}} F(t) dt \\ &\leq -C' r \log r - C' \int_r^{\frac{1}{2}} \log t dt \end{aligned}$$

This yields

$$\int_{U(0)} |z_1|^m |\tau_\alpha|^{\frac{2}{k}} idz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge idz_n \wedge d\bar{z}_n < +\infty$$

By Lemma 3.3 below we conclude that $z_1^{\frac{km}{2}} \cdot \tau_\alpha$, hence τ_α extends to a meromorphic function over Δ^n .

Write $\tau_\alpha(z) = z_1^\ell \cdot b_\alpha(z)$ so that $b_\alpha(z) \in \mathcal{O}(U)$ which is not identically equal to zero on D . Take a point $y = (0, y_2, \dots, y_n) \in D \cap U$ so that $b_\alpha(y) \neq 0$. Then for some $\varepsilon > 0$ one has $|b_\alpha(z)| \geq C'_3$ over

$$V := \{(z_1, \dots, z_n) \in U(0) \mid |z_1| < \varepsilon, |z_2 - y_2| < \varepsilon, \dots, |z_n - y_n| < \varepsilon\}$$

for some constant $C'_3 > 0$. We thus have

$$-C_k C'_1 \log r + C_k C'_2 \geq \int_{V \cap U(r)} (|\tau_\alpha|^2 \cdot |z_1|^{2\alpha_1} \log^{2\alpha_1} |z_1|)^{\frac{1}{k}} d\text{vol}_{\omega_P} \geq C'_4 \int_r^\varepsilon \frac{t^{\frac{2\ell+2\alpha_1}{k}} d \log t}{|\log t|^2}$$

for some constant $C'_4 > 0$ which does not depend on r . If $\ell < -\alpha_1$, for some $0 < \varepsilon_2 < \varepsilon$ one has

$$r^{\frac{2\ell+2\alpha_1}{k}} > |\log r|^3$$

when $0 < r < \varepsilon_2$. Hence

$$\int_r^{\varepsilon_2} \frac{t^{\frac{2\ell+2\alpha_1}{k}} d \log t}{|\log t|^2} \geq \log^2 r - \log^2(\varepsilon_2)$$

which yields

$$-C_k C'_1 \log r + C_k C'_2 \geq C'_4 (\log^2 r - \log^2(\varepsilon_2))$$

for any $0 < r < \varepsilon_2$. A contradiction is obtained. Hence $\ell + \alpha_1 \geq 0$. It follows that

$$\sigma_k \in H^0(\bar{X}^\circ, \text{Sym}^k \Omega_{\bar{X}}(\log D)|_{\bar{X}^\circ}),$$

where we denote by $\bar{X}^\circ := \bar{X} - \cup_{j \neq i} D_i \cap D_j$ whose complement has codimension at least two in \bar{X} . By the Hartogs theorem, it extends to a logarithmic symmetric form on \bar{X} . The first claim is proved.

If u is not constant, then there is some connected open set $U \subset X$ and so that the pluriharmonic mapping $u_i : U \rightarrow A$ defined above is not constant. As G is semisimple, its root system $\{\alpha_1, \dots, \alpha_m\}$ generates A^* . Hence some element in $\{u_i^*(d\alpha_1)_{\mathbb{C}}^{(1,0)}, \dots, u_i^*(d\alpha_m)_{\mathbb{C}}^{(1,0)}\}$ is non zero. By the construction (3.2), $\sigma_k \neq 0$ for some $k \in \{1, \dots, m\}$. We prove the second claim. The proposition is proved. \square

The following lemma is the criterion on the meromorphicity of functions in terms of L^p -boundedness.

Lemma 3.3. — *Let f be a holomorphic function on $(\Delta^*)^\ell \times \Delta^{n-\ell}$ so that*

$$\int_{(\Delta^*)^\ell \times \Delta^{n-\ell}} |f(z)|^p idz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge idz_n \wedge d\bar{z}_n < C$$

for some real $0 < p < \infty$ and some positive constant C . Then f extends to a meromorphic function on Δ^n .

Proof. — Since $|f(z)|^p$ is plurisubharmonic function on $(\Delta^*)^\ell \times \Delta^\ell$, by the mean value inequality, for any $z = (z_1, \dots, z_n)$ with $|z_i| < \frac{1}{2}$ for each i , one has

$$|f(z)|^p \leq \frac{4^{n-\ell}}{\pi^n \prod_{i=1}^\ell |z_i|^2} \int_{\Omega_z} |f(\zeta)|^p id\zeta_1 \wedge d\bar{\zeta}_1 \wedge \cdots \wedge id\zeta_n \wedge d\bar{\zeta}_n \leq \frac{4^{n-\ell} C}{\pi^n \prod_{i=1}^\ell |z_i|^2}$$

where

$$\Omega_z := \{w \in (\Delta^*)^\ell \times \Delta^{n-\ell} \mid |w_i - z_i| < |z_i| \text{ for } i \leq \ell; |w_i - z_i| < \frac{1}{2} \text{ for } i > \ell\}.$$

It follows that

$$|f(z)| \leq C' \prod_{i=1}^{\ell} |z_i|^{-\frac{2}{p}}$$

for any $z = (z_1, \dots, z_n)$ with $|z_i| < \frac{1}{2}$. Hence $\prod_{i=1}^{\ell} z_i^{\lceil \frac{2}{p} \rceil} f(z)$ extends to a holomorphic function over Δ^n . The lemma follows. \square

Let us prove the main result of this section based on Proposition 3.2.

Theorem 3.4. — *Let X be a quasi-projective manifold. Assume that $\rho : \pi_1(X) \rightarrow \mathrm{GL}_m(K)$ is a reductive representation defined over a non-archimedean local field K (i.e. its Zariski closure G is a reductive algebraic group), whose image is not contained in any compact subgroup of $\mathrm{GL}_m(K)$. Let \bar{X} be a projective compactification of X such that $D = \bar{X} - X$ is a simple normal crossing divisor. Then*

$$H^0(\bar{X}, \mathrm{Sym}^k \Omega_{\bar{X}}(\log D)) \neq 0$$

for some positive integer k .

Proof. — Since the Zariski closure G of ρ is assumed to be reductive, there exists a semisimple G' and an algebraic torus T so that there is an isogeny

$$G' \times T \rightarrow G.$$

Then after replacing X by a finite étale cover X' , one has a Zariski dense representation $\rho' : \pi_1(X') \rightarrow G'(K) \times T(K)$ so that

$$(3.4) \quad \begin{array}{ccc} \pi_1(X') & \xrightarrow{\rho'} & G'(K) \times T(K) \\ \downarrow & & \downarrow \\ \pi_1(X) & \xrightarrow{\rho} & G(K) \end{array}$$

Since we assume that the image $\rho(\pi_1(X))$ is not contained in any compact subgroup of $\mathrm{GL}_m(K)$, it follows that the image of ρ' is also not contained in any compact subgroup of $G'(K) \times T(K)$. Let $p_1 : G'(K) \times T(K) \rightarrow G'(K)$ and $p_2 : G'(K) \times T(K) \rightarrow T(K)$ be the projection maps. Then both $\sigma_1 := p_1 \circ \rho'$ and $\sigma_2 := p_2 \circ \rho'$ are Zariski dense.

Assume first that $\sigma_1 : \pi_1(X') \rightarrow G'(K)$ is not contained in any compact subgroup. By Theorem 2.1, there is a locally Lipschitz σ_1 -equivariant harmonic mapping $u : \bar{X} \rightarrow \Delta(G')$ which is pluriharmonic of logarithmic energy growth. Note that u is not constant; or else, its image point would be fixed by σ_1 and the subgroup of $G'(K)$ fixing a point of $\Delta(G')$ is compact, which contradicts with our assumption. By Proposition 3.2, there exists a non-zero global logarithmic symmetric form $H^0(\bar{X}', \mathrm{Sym}^k \Omega_{\bar{X}'}(\log D'))$, where \bar{X}' is a projective compactification of X' with $D' := \bar{X}' - X'$ a simple normal crossing divisor.

Assume now $\sigma_1 : \pi_1(X') \rightarrow G'(K)$ is contained in some compact subgroup. Then the image of $\sigma_2 : \pi_1(X') \rightarrow T(K)$ is not contained in any compact subgroup and thus must be infinity. Note that $T(K)$ is abelian. It follows that σ_2 induces a morphism $H_1(X', \mathbb{Z}) \rightarrow T(K)$ with infinite image; in particular, by the universal coefficient theorem we conclude that $H^1(X', \mathbb{C})$ is infinite.

Claim 3.5. — $H^0(\bar{X}', \Omega_{\bar{X}'}(\log D')) \neq 0$.

Proof of Claim 3.5. — By the theory of mixed Hodge structures, we know that

$$H^1(X', \mathbb{C}) \simeq H^0(\overline{X}', \Omega_{\overline{X}'}(\log D')) \oplus H^{0,1}(\overline{X}').$$

Since $H^1(X', \mathbb{C})$ is infinite, either $H^0(\overline{X}', \Omega_{\overline{X}'}(\log D'))$ or $H^{0,1}(\overline{X}')$ is non-zero. In the latter case, by the Hodge duality, $H^0(\overline{X}', \Omega_{\overline{X}'})$ is non-zero, and thus $H^0(\overline{X}', \Omega_{\overline{X}'}(\log D'))$ is also non-zero. \square

In summary, $H^0(\overline{X}', \text{Sym}^k \Omega_{\overline{X}'}(\log D')) \neq 0$ for some $k > 0$. By Lemma 3.7 below, one has $H^0(\overline{X}, \text{Sym}^{k'} \Omega_{\overline{X}}(\log D)) \neq 0$ for some $k' > 0$. The theorem is proved. \square

Definition 3.6 (Galois morphism). — A covering map $\gamma : X \rightarrow Y$ of varieties is called *Galois with group G* if there exists a finite group $G \subset \text{Aut}(X)$ such that γ is isomorphic to the quotient map.

Lemma 3.7. — *Let $f^\circ : U \rightarrow V$ be a finite étale cover between quasi-projective manifolds which is Galois. If U admits a global logarithmic symmetric form, then V contains also global logarithmic symmetric forms.*

Proof. — Let Y be a smooth projective compactification of V and let $D_Y := Y - V$ be a simple normal crossing divisor. By Zariski's Main Theorem in the equivariant setting (cf. [GKP13, Theorem 3.8]), there are a normal projective variety X and a finite morphism $f : X \rightarrow Y$ that extends f° so that f is Galois with group G . Hence we have $f \circ g = f$ for any $g \in G \subset \text{Aut}(X)$. Note that X has at most quotient singularities. Denote by D_Y^{sing} the set of singular points of D_Y , namely the points lying at least two irreducible components of D_Y . Then D_Y^{sing} is a closed subvariety of X of codimension at least two. Write $Y^\circ := X - D_Y^{\text{sing}}$, and $X^\circ := f^{-1}(Y^\circ)$. Then X° is smooth, and $D_X^\circ := X^\circ - U$ is a smooth divisor in X° . Moreover, it follows from the proof of [Den22, Lemma A.12] that at any $x \in D_X^\circ$ there are an admissible coordinate $(\Omega; x_1, \dots, x_n)$ around x with $D_X^\circ \cap \Omega = (x_1 = 0)$ and an admissible coordinate $(\Omega'; y_1, \dots, y_n)$ around $f(x)$ with $D_Y \cap \Omega' = (y_1 = 0)$ so that

$$(3.5) \quad f(x_1, \dots, x_n) = (x_1^k, x_2, \dots, x_n).$$

For any $g \in G$ and any $\omega \in H^0(X^\circ, \text{Sym}^k \Omega_{X^\circ}(\log D_X^\circ))$, one has $g^* \omega \in H^0(X^\circ, \text{Sym}^k \Omega_{X^\circ}(\log D_X^\circ))$ since $g : (X^\circ, D_X^\circ) \rightarrow (X^\circ, D_X^\circ)$ is an automorphism of the log pair (X°, D_X°) .

We now take a log resolution of $(\tilde{X}, \tilde{D}) \rightarrow (X, D_X)$ which is an isomorphism over X° , where $D_X := X - U$. By the assumption, there is a logarithmic symmetric form $\tilde{\omega} \in H^0(\tilde{X}, \text{Sym}^k \Omega_{\tilde{X}}(\log \tilde{D}))$. Its restriction to X° gives rise to a logarithmic one form $\omega \in H^0(X^\circ, \text{Sym}^k \Omega_{X^\circ}(\log D_X^\circ))$. We define $\eta := \prod_{g \in G} g^* \omega$, which is a non-zero, G -invariant logarithmic symmetric form in $H^0(X^\circ, \text{Sym}^{|G|k} \Omega_{X^\circ}(\log D_X^\circ))$. By the local description of f in (3.5), η descends to a logarithmic symmetric form

$$\eta' \in H^0(Y^\circ, \text{Sym}^{|G|k} \Omega_{Y^\circ}(\log D_Y^\circ)),$$

where $D_Y^\circ = Y^\circ - V$. Since $Y - Y^\circ$ is of codimension at least two, by the Hartogs theorem, η' extends to a non-zero logarithmic symmetric form in $H^0(Y, \text{Sym}^{|G|k} \Omega_Y(\log D_Y))$. The lemma is proved. \square

4. Rigid representations

In this section we will prove Theorem D based on gauge theoretical methods and Theorem 1.12. We begin with the following observation.

Lemma 4.1. — *Let X be a quasi-projective manifold. Let \overline{X} be a projective compactification of X so that $D = \overline{X} - X$ is a simple normal crossing divisor. Assume that $H^0(\overline{X}, \text{Sym}^k \Omega_{\overline{X}}(\log D)) = 0$ for all k . Then any tame harmonic bundle (E, θ, h) on X is a nilpotent harmonic bundle, i.e. for any $x \in X$ and $v \in T_{X,x}$, $\theta(v) : E_x \rightarrow E_x$ is a nilpotent endomorphism. Moreover, (E, θ, h) is pure imaginary.*
Proof. — Consider the prolongation $({}^\diamond E_h, \theta, h)$ in § 1.4. Since θ extends to a morphism

$${}^\diamond E_h \rightarrow {}^\diamond E_h \otimes \Omega_{\overline{X}}(\log D),$$

its characteristic polynomial $\det(t - \theta)$ is of the form

$$\det(t - \theta) = \sum_{k=1}^{\text{rank } E} \sigma_k t^k,$$

where $\sigma_k \in H^0(\overline{X}, \text{Sym}^{n-k} \Omega_{\overline{X}}(\log D))$ for any $0 \leq k \leq n$.

Since $H^0(\overline{X}, \text{Sym}^k \Omega_{\overline{X}}(\log D)) = 0$ for all k , one has $\det(t - \theta) = t^{\text{rank } E}$. Hence for any $x \in \overline{X}$ and $v \in T_{\overline{X}}(-\log D)$, one has

$$\det(t - \theta(v)) = \det(t - \theta)(v) = \sum_{k=1}^{\text{rank } E} \sigma_k(v) t^k = t^{\text{rank } E}.$$

Hence, $\theta(v)$ is nilpotent. From the above expression of $\det(t - \theta)$, it also follows that $\text{Res}_{D_i}(\theta)$ is zero for each i , and therefore (E, θ, h) is in particular pure imaginary. \square

The following lemma is a variant of [Sim92, Lemma 2.7].

Lemma 4.2. — *Let X be a quasi-projective manifold. Let \overline{X} be a projective compactification of X so that $D = \overline{X} - X$ is a simple normal crossing divisor. Assume that $H^0(\overline{X}, \text{Sym}^k \Omega_{\overline{X}}(\log D)) = 0$ for all k . Then for any compact set $K \subset X$, there is a constant C_K so that any tame harmonic bundle (E, θ, h) on X satisfies that*

$$|\theta|_h \leq C_K$$

over K .

Proof. — By Lemma 4.1, (E, θ, h) is nilpotent harmonic bundle. For any polydisk $(U; z_1, \dots, z_n)$ on X , writing $\theta_i := \theta(\frac{\partial}{\partial z_i})$ one has

$$\partial_i \bar{\partial}_i \log |\theta_i|_h^2 \geq -\frac{\langle R_{i\bar{i}}^{\text{End}(E)}(\theta_i), \theta_i \rangle_h}{|\theta_i|_h^2}.$$

By the flatness of $D_h + \theta + \theta_h^\dagger$, we have $R_{i\bar{i}}^E = -[\theta_i, \theta_i^\dagger]$, so $R_{i\bar{i}}^{\text{End}(E)}(\theta_i) = -[[\theta_i, \theta_i^*], \theta_i]$. This gives

$$\begin{aligned} \langle R_{i\bar{i}}^{\text{End}(E)}(\theta_i), \theta_i \rangle_h &= -\langle [[\theta_i, \theta_i^*], \theta_i], \theta_i \rangle_h \\ &= -\text{tr}([\theta_i, \theta_i^*], \theta_i \theta_i^*) \\ &= -\text{tr}([\theta_i, \theta_i^*][\theta_i, \theta_i^*]) \\ &= -|[\theta_i, \theta_i^*]|^2 \\ &\leq -\frac{|\theta_i|^4}{4^{\text{rank } E - 1}}. \end{aligned}$$

where the last inequality follows from [CD21, Proposition 2.1] since θ_i is nilpotent. Hence

$$\partial_i \bar{\partial}_i \log |\theta_i|_h^2 \geq \frac{|\theta_i|^4}{4^{\text{rank } E - 1}}$$

and thus by the Ahlfors lemma, one has

$$|\theta_i|_h^2 \leq \frac{4^{\text{rank } E - 1}}{(1 - |z_i|^2)^2}$$

on the polydisk U . If we shrink the polydisk, we can show the uniform boundedness of $|\theta|_h$. The lemma is proved. \square

Let us prove the main result of this section.

Theorem 4.3. — *Let X be a quasi-projective manifold of dimension n . Let \bar{X} be a projective compactification of X so that $D = \bar{X} - X$ is a simple normal crossing divisor. Assume that $H^0(\bar{X}, \text{Sym}^k \Omega_{\bar{X}}(\log D)) = 0$ for all $k \in \mathbb{N}$. Then the Betti moduli space $M_B(X, \text{GL}_N(\mathbb{C})) = \text{Hom}(\pi_1(X), \text{GL}_N(\mathbb{C})) // \text{GL}(N, \mathbb{C})$ is zero dimensional. In particular, any semi-simple representation $\rho : \pi_1(X) \rightarrow \text{GL}_N(\mathbb{C})$ is rigid.*

Recall that a semi-simple representation $\rho : \pi_1(X) \rightarrow \text{GL}_N(\mathbb{C})$ is called *rigid* if $[\rho] \in M_B(X, \text{GL}_N(\mathbb{C}))$ is an isolated point.

Proof of Theorem 4.3. — Since $H^0(\bar{X}, \text{Sym}^k \Omega_{\bar{X}}(\log D)) = 0$ for all k , by Lemma 4.1 any tame harmonic bundle (E, θ, h) on X is pure imaginary.

We shall use an alternative way to study tame harmonic bundle. Let us fix a complex vector bundle V equipped with a smooth metric h . Let $\mathcal{A}(X, V)$ be the space of smooth unitary connections of (V, h) . Denote by $\mathcal{A}_{loc}^{1,p}(X, V)$ the Sobolev space $W_{loc}^{1,p}$ of unitary connections (cf. [Weh04, Appendix B] for more details). We also denote by $\mathcal{C}^\infty(X, \Omega_X^{1,0} \otimes \text{End}(V))$ (resp. $W_{loc}^{1,p}(X, \Omega_X^{1,0} \otimes \text{End}(V))$) the space of smooth (resp. Sobolev- $W_{loc}^{1,p}$) $(1, 0)$ -forms with values in $\text{End}(V)$. The space of harmonic bundles modeled over V can thus be identified with the set \mathcal{H} of pairs $(A, \theta) \in \mathcal{A}(X, V) \times \mathcal{C}^\infty(X, \Omega_X^{1,0} \otimes \text{End}(V))$ satisfying

$$\begin{aligned} & - \bar{\partial}_A \theta = 0 \\ & - \theta \wedge \theta = 0 \\ & - (d_A + \theta + \theta^\dagger)^2 = 0, \text{ where } \theta^\dagger \text{ is the adjoint of } \theta \text{ with respect to } h. \end{aligned}$$

Let $\mathcal{G}_{loc}^{2,p}$ be the gauge group of (V, h) , which is the set of $W_{loc}^{2,p}$ -isometries of (V, h) (see also [Weh04, Appendix A]). We define the action of $g \in \mathcal{G}_{loc}^{2,p}$ on $(A, \theta) \in \mathcal{A}_{loc}^{1,p}(X, V) \times W_{loc}^{1,p}(X, \Omega_X^{1,0} \otimes \text{End}(V))$ by setting

$$d_{g^*A} := g^{-1} d_A g, \quad g^* \theta := g^{-1} \theta g.$$

By [Weh04, Lemmas A.5 and A.6], when $p > 2n$, $g^{-1} \in \mathcal{G}_{loc}^{2,p}$ and $(g^*A, g^*\theta) \in \mathcal{A}_{loc}^{1,p}(X, V) \times W_{loc}^{1,p}(X, \Omega_X^{1,0} \otimes \text{End}(V))$.

Since $M_B(X, \text{GL}_N(\mathbb{C}))$ is an affine variety, to prove that it is zero dimensional, it suffices to show that it is compact. We take an arbitrary sequence of points $\varphi_n \in M_B(X, \text{GL}_N(\mathbb{C}))$ in any connected component. We shall prove that $\{\varphi_n\}_{n \geq 1}$ has a subsequence which converges to a point $\varphi \in M_B(X, \text{GL}_N(\mathbb{C}))$.

By Theorem 1.12, φ_n correspond to tame pure imaginary harmonic bundles. Since they are in the same component of $M_B(X, \text{GL}_N(\mathbb{C}))$, they can be modeled over the same smooth vector bundle V . Hence there are $(A_n, \theta_n) \in \mathcal{H}$ so that (V, h, A_n, θ_n) is tame, and the monodromy representation of $(V, d_{A_n} + \theta_n + \theta_n^\dagger)$ is a representative of φ_n . By Lemma 4.2, $|\theta_n|_h$ are locally uniformly bounded. Therefore, $|F_{A_n}|_h = |[\theta_n, \theta_n^\dagger]|_h$ are also locally uniformly bounded. By Uhlenbeck's compactness theorem (cf. [Weh04, Theorem A']), there exists $g_n \in \mathcal{G}_{loc}^{2,p}$ so that after taking a subsequence $B_n := g_n^* A_n$ converges weakly in $\mathcal{A}_{loc}^{1,p}(X, V)$ to a connection $A \in \mathcal{A}_{loc}^{1,p}(X, V)$. Write $a_n := d_{A_n} - d_{B_n}$. This is equivalent to that a_n converges $W_{loc}^{1,p}$ -weakly to zero. By the Sobolev embedding theorem ([Weh04, Lemma B.2]), if $p > 2n$, $\hat{\theta}_n := g_n^* \theta_n$ is in

\mathcal{C}^1 and a_n converges to zero locally in \mathcal{C}^0 -topology. Moreover, a_n converges strongly in L^p_{loc} to zero. Since $F_{B_n} = F_A - d_A(a_n) + a_n \wedge a_n$, it follows that F_{B_n} converges L^p_{loc} -weakly to F_A according to [Weh04, Lemma B.3].

Note that

$$d_{B_n} + g_n^* \theta_n + (g_n^* \theta_n)^\dagger = g_n^*(d_{A_n} + \theta_n + \theta_n^\dagger).$$

Let us stress here that while the connection $d_{B_n} + g_n^* \theta_n + (g_n^* \theta_n)^\dagger$ is not smooth, it is \mathcal{C}^0 and satisfies $(d_{B_n} + g_n^* \theta_n + (g_n^* \theta_n)^\dagger)^2 = 0$. Its monodromy representation $\rho_n : \pi_1(X) \rightarrow \mathrm{GL}_N(\mathbb{C})$ can also be defined and is conjugate to that of $d_{A_n} + \theta_n + \theta_n^\dagger$. Therefore, $\varphi_n = [\rho_n]$.

Note that

$$(\bar{\partial}_{A_1} + \bar{\partial}_{A_1}^*)(\tilde{\theta}_n) = \bar{\partial}_{A_1}(\tilde{\theta}_n) = \bar{\partial}_{B_n}(\tilde{\theta}_n) + [A_1^{0,1} - B_n^{0,1}, \tilde{\theta}_n] = [A_1^{0,1} - B_n^{0,1}, \tilde{\theta}_n].$$

We cover X by countably many relatively compact unit balls U_m so that the smaller balls $U'_m \Subset U_m$ are also a covering of X . Since $\bar{\partial}_{A_1} + \bar{\partial}_{A_1}^*$ is an elliptic operator of order 1, by the Schauder estimate (cf. [Nic20, Exercise 10.3.14]), one has

$$\begin{aligned} \|\tilde{\theta}_n\|_{L^p, U'_m} &\leq C_m \left(\|\tilde{\theta}_n\|_{L^p, U_m} + \|[A_1^{0,1} - B_n^{0,1}, \tilde{\theta}_n]\|_{L^p, U_m} \right) \\ &\leq C'_m \left(\|\tilde{\theta}_n\|_{L^p, U_m} + \|[A_1^{0,1} - B_n^{0,1}]\|_{L^\infty, U_m} \|\tilde{\theta}_n\|_{L^p, U_m} \right) \end{aligned}$$

where C_m and C'_m are uniform constants depending only on U_m and U'_m . Since $|\theta_n|_h = |\tilde{\theta}_n|_h$, $|\tilde{\theta}_n|_h$ is uniformly bounded on each U_m by Lemma 4.2. Since $A_1 - B_n$ converges to $A_1 - A$ locally in \mathcal{C}^0 -topology, it follows that $\|[A_1^{0,1} - B_n^{0,1}]\|_{L^\infty, U_m}$ is uniformly bounded over U_m . By the Banach-Alaoglu theorem (cf. [Weh04, Theorem B.4]), over each U'_m , $\tilde{\theta}_n$ has a subsequence which converges weakly in $W^{1,p}$. Since U_m is countably many, one can work by taking a diagonal subsequence to conclude that $\tilde{\theta}_n$ has a subsequence which converges weakly in $W^{1,p}_{loc}$ to some θ . By the Sobolev embedding theorem again, $\tilde{\theta}_n$ converges to θ locally in \mathcal{C}^0 -topology. Hence $\tilde{\theta}_n^\dagger$ also converges to θ^\dagger locally in \mathcal{C}^0 -topology. This implies that $[\tilde{\theta}_n, (\tilde{\theta}_n)^\dagger]$ (resp. $\tilde{\theta}_n \wedge \tilde{\theta}_n^\dagger$) converges to $[\theta, \theta^\dagger]$ (resp. $\theta \wedge \theta$) locally in \mathcal{C}^0 -topology. Recall that $\tilde{\theta}_n \wedge \tilde{\theta}_n^\dagger = 0$, and that F_{B_n} converges L^p_{loc} -weakly to F_A . It follows that $\theta \wedge \theta = 0$, and that $F_{B_n} + [\tilde{\theta}_n, (\tilde{\theta}_n)^\dagger] = 0$ converges L^p_{loc} -weakly to $F_A + [\theta, \theta^\dagger]$. Hence $F_A + [\theta, \theta^\dagger] = 0$.

Note that $d_A(\tilde{\theta}_n) = d_{B_n}(\tilde{\theta}_n) + [a_n, \tilde{\theta}_n] = [a_n, \tilde{\theta}_n]$. Hence $d_A(\tilde{\theta}_n)$ converges to zero locally in \mathcal{C}^0 -topology. On the other hand,

$$d_A(\tilde{\theta}_n) - d_A \theta = d_{A_1}(\tilde{\theta}_n - \theta) + [A - A_1, \tilde{\theta}_n - \theta]$$

converges to zero weakly in L^p_{loc} . It then follows that $d_A \theta = 0$. Since $d_A \theta = 0$ and $\theta \wedge \theta = 0$, it follows that

$$(d_A + \theta + \theta^\dagger)^2 = F_A + [\theta, \theta^\dagger] = 0.$$

We stress here that $(B_n, \tilde{\theta}_n)$ converges to $(A, \theta) \in \mathcal{A}_{loc}^{1,p}(X, V) \times W_{loc}^{1,p}(X, \Omega_X^{1,0} \otimes \mathrm{End}(V))$ locally in \mathcal{C}^0 -topology.

By Proposition 4.4 below, there is a gauge transformation $g \in \mathcal{G}_{loc}^{2,p}$ so that $(g^* A, g^* \theta)$ is smooth. One can thus replace the gauge transformations g_n by $g_n g \in \mathcal{G}_{loc}^{2,p}$ so that $g_n^*(A_n, \theta_n)$ converges to a smooth $(A, \theta) \in \mathcal{H} \subset \mathcal{A}(X, V) \times \mathcal{C}^\infty(X, \Omega_X^{1,0} \otimes \mathrm{End}(V))$ locally in \mathcal{C}^0 -topology. Hence (V, h, A, θ) is a harmonic bundle on X .

We will prove that (V, h, A, θ) is tame pure imaginary. Since (V, h, A_n, θ_n) is tame, by Lemma 4.1 one has $\det(t - \theta_n) = t^N$. Since $\tilde{\theta}_n$ converge to θ locally in \mathcal{C}^0 -topology, its characteristic polynomial $\det(t - \tilde{\theta}_n) = \det(t - \theta_n) = t^N$ converges locally uniformly to $\det(t - \theta)$ on X , and thus $\det(t - \theta) = t^N$. By Lemma 4.1 (V, h, A, θ)

is tame and pure imaginary. In virtue of Theorem 1.12, for the monodromy representation $\rho : \pi_1(X) \rightarrow \mathrm{GL}_N(\mathbb{C})$ of $d_A + \theta + \theta^\dagger$, its conjugate class $[\rho]$ is a point in $M_B(X, \mathrm{GL}_n(\mathbb{C}))$, denoted by φ .

Let us prove that φ_n converges to φ . Since $M_B(X, \mathrm{GL}_n(\mathbb{C})) = \mathrm{Hom}(\pi_1(X), \mathrm{GL}_N(\mathbb{C})) // \mathrm{GL}_N(\mathbb{C})$, it suffices to show that the monodromy representation ρ_n of $D_n := d_{B_n} + \theta_n + \theta_n^\dagger$ converges to that of $D := d_A + \theta + \theta^\dagger$. Since $\pi_1(X)$ is finitely generated, we choose generators $\gamma_1, \dots, \gamma_m$. Then $\mathrm{Hom}(\pi_1(X), \mathrm{GL}_N(\mathbb{C}))$ is an affine subvariety of

$$\underbrace{\mathrm{GL}_N(\mathbb{C}) \times \dots \times \mathrm{GL}_N(\mathbb{C})}_{m \text{ times}}$$

whose defining equations arise from the relations of $\gamma_1, \dots, \gamma_m$. The image of a representation $\tau : \pi_1(X) \rightarrow \mathrm{GL}_N(\mathbb{C})$ is $(\tau(\gamma_1), \dots, \tau(\gamma_m))$. We fix smooth loops ℓ_1, \dots, ℓ_m on X representing $\gamma_1, \dots, \gamma_m$. The holonomies $\mathrm{Hol}(\ell_i, D_n) = \rho_n(\gamma_i)$ and $\mathrm{Hol}(\ell_i, D) = \rho(\gamma_i)$. Note that $D - D_n := \phi_n$ converges locally uniformly to zero. Hence by some standard arguments in ordinary differential equations one can show that for any ℓ_i , $\mathrm{Hol}(\ell_i, D_n)$ converges to $\mathrm{Hol}(\ell_i, D)$ in $\mathrm{GL}_N(\mathbb{C})$. It follows that $\rho_n(\gamma_i)$ converges to $\rho(\gamma_i)$ for any i . In conclusion, we prove that after taking a subsequence, φ_n converges to some $\varphi \in M_B$. This implies that M_B is compact, which must be zero dimensional since it is an affine variety. \square

In [Uhl82] Uhlenbeck proved that a weakly Yang-Mills connection on a compact Riemannian manifold is gauge equivalent to a smooth one. This result was extended by Wehrheim in [Weh04] to some non-compact manifolds. As required in the proof of Theorem 4.3, we have to prove a similar result for “weakly harmonic bundle” on quasi-projective manifolds.

Proposition 4.4. — *Let X be a quasi-projective manifold of dimension n . Let V be a complex vector bundle equipped with a smooth metric h . Denote by $\mathcal{A}_{loc}^{1,p}(X, V)$ the Sobolev space $W_{loc}^{1,p}$ of unitary connections as above. If $(A, \theta) \in \mathcal{A}_{loc}^{1,p}(X, V) \times W_{loc}^{1,p}(X, \Omega_X^{1,0} \otimes \mathrm{End}(V))$ with $p > 2n$ satisfies*

$$\begin{aligned} & - \bar{\partial}_A \theta = 0 \\ & - \theta \wedge \theta = 0 \\ & - (d_A + \theta + \theta^\dagger)^2 = 0, \text{ where } \theta^\dagger \text{ is the adjoint of } \theta \text{ with respect to } h, \end{aligned}$$

then there is a gauge transformation $g \in \mathcal{G}_{loc}^{2,p}$ so that $g^*(A, \theta)$ is smooth.

Proof. — We first find an increasing exhausting sequence of compact submanifolds X_k of X with smooth boundary ∂X_k that are deformation retracts of X . Write $A_k := A|_{X_k}$ and $\theta_k := \theta|_{X_k}$. We apply Theorem 4.5 to find smooth connections $\tilde{A}_k \in \mathcal{A}(X_k, V)$ and gauge transformations $g_k \in \mathcal{G}^{2,p}(X_k, V)$ such that

$$d_{A_k}^*(g_k^* \tilde{A}_k - A_k) = 0, \quad *(g_k^* \tilde{A}_k - A_k)|_{\partial X_k} = 0.$$

By [Weh04, Lemma 8.4], we have

$$d_{\tilde{A}_k}^*(\tilde{A}_k - u_k^* A_k) = 0, \quad *(\tilde{A}_k - u_k^* A_k)|_{\partial X_k} = 0.$$

where $u_k := g_k^{-1} \in \mathcal{G}^{2,p}$. Denote $B_k := u_k^* A_k \in \mathcal{A}^{1,p}(X_k, V)$ and $\varphi_k := u_k^* \theta_k \in W^{1,p}(X_k, T_{X_k}^* \otimes \mathrm{End}(V))$. Write $\alpha := \tilde{A}_k - u_k^* A_k \in \mathcal{A}^{1,p}(X_k, V)$. Note that

$$d_{\tilde{A}_k}(\alpha) = F_{\tilde{A}_k} - F_{B_k} + \alpha \wedge \alpha = F_{\tilde{A}_k} + [\varphi_k, \varphi_k^\dagger] + \alpha \wedge \alpha$$

Then

$$(d_{\tilde{A}_k}^* d_{\tilde{A}_k} + d_{\tilde{A}_k}^* d_{\tilde{A}_k})(\alpha) = d_{\tilde{A}_k}^*(F_{\tilde{A}_k} + [\varphi_k, \varphi_k^\dagger] + \alpha \wedge \alpha)$$

By the Sobolev multiplication theorem (cf. [Weh04, Lemma B.3]), $[\varphi_k, \varphi_k^\dagger]$ and $\alpha \wedge \alpha$ are in $W^{1,p}(X_k, \Lambda^2 T_{X_k}^* \otimes \text{End}(V))$. Hence the term on the right in the above equation is in L^p . By the elliptic regularity, $\alpha \in W^{2,p}(X_k, T_{X_k}^* \otimes \text{End}(V))$. Hence $B_k = \tilde{A}_k - \alpha \in \mathcal{A}^{2,p}(X_k, V)$. Since $\bar{\partial}_{B_k}(\varphi_k) = 0$, it follows that

$$(4.1) \quad (\bar{\partial}_{\tilde{A}_k} + \bar{\partial}_{\tilde{A}_k}^*)(\varphi_k) = \bar{\partial}_{\tilde{A}_k}(\varphi_k) = [\alpha^{0,1}, \varphi_k].$$

By the Sobolev multiplication theorem again, $[\alpha^{0,1}, \varphi_k] \in W^{1,p}(X_k, \Lambda^2 T_{X_k}^* \otimes \text{End}(V))$. By the elliptic regularity, $\varphi_k \in W^{2,p}(X_k, T_{X_k}^* \otimes \text{End}(V))$. One can iterate this procedure to show the smoothness of $u_k^* A|_{X_k}$ and $u_k^* \theta|_{X_k}$. By Proposition 4.6, there is a gauge transformation $u \in \mathcal{G}_{loc}^{2,p}(X, V)$ so that $u^* A$ is smooth. The same ‘‘bootstrapping’’ method as (4.1) implies the smoothness of $u^* \theta$. The proposition is proved. \square

The following two results are extracted from [Weh04].

Theorem 4.5 ([Weh04, Theorem F]). — *Let M be a compact Riemannian n -manifold with smooth boundary (that might be empty). Let V be a complex vector bundle on M equipped with a smooth metric h . Let $p > \max\{1, \frac{n}{2}\}$. Let $\hat{A} \in \mathcal{A}^{1,p}(M, V)$ be a unitary Sobolev $W^{1,p}$ -connection of (V, h) . Then there exist a smooth unitary connection $A \in \mathcal{A}(M, V)$ and a gauge transformation $u \in \mathcal{G}^{2,p}$ such that*

$$\begin{aligned} d_{\hat{A}}^*(u^* A - \hat{A}) &= 0 \\ *(u^* A - \hat{A})|_{\partial M} &= 0. \end{aligned}$$

\square

Proposition 4.6 ([Weh04, Proposition 9.8]). — *Let $M = \bigcup_{k \in \mathbb{N}} M_k$ be a Riemannian n -manifold exhausted by an increasing sequence of compact submanifolds M_k that are deformation retracts of M . Let V be a smooth vector bundle on M equipped with a metric h . Let $A \in \mathcal{A}_{loc}^{1,p}(M, V)$ and suppose that for each $k \in \mathbb{N}$ there is a gauge transformation $u_k \in \mathcal{G}^{2,p}(M_k)$ such that $u_k^* A|_{M_k}$ is smooth. Then there exists a gauge transformation $u \in \mathcal{G}_{loc}^{2,p}$ such that $u^* A$ is smooth. \square*

Remark 4.7. — Theorem 4.3 is proved by Arapura [Ara02] when X is projective. His proof is based on the properness of Hitchin map of the moduli of Higgs bundles, together with Simpson’s work on the analytic isomorphism between moduli of Higgs bundles and Betti moduli spaces. These results are still unknown in the quasi-projective cases.

Proposition 4.8. — *Let X be a quasi-projective manifold. Assume that $\rho : \pi_1(X) \rightarrow \text{GL}_N(\mathbb{C})$ is a semi-simple representation which is locally rigid. Assume moreover that the corresponding tame harmonic bundle (E, θ, h) of ρ has nilpotent residue in the sense of Definition 1.11. Then ρ underlies a complex variation of Hodge structures.*

Recall that $\rho : \pi_1(X) \rightarrow \text{GL}_N(\mathbb{C})$ is called *locally rigid* if $[\rho] = [\rho'] \in M_B(X, \text{GL}_N(\mathbb{C}))$ for any deformation ρ' of ρ . It is straightforward that a rigid representation is locally rigid. Proposition 4.8 was proved by Simpson [Sim92] when X is compact, and by Corlette-Simpson [CS08] for semisimple $\rho : \pi_1(X) \rightarrow \text{GL}_N(\mathbb{C})$ with quasi-unipotent local monodromies at infinity.

Proof of Proposition 4.8. — It suffices to assume ρ is simple. By Theorem 1.12, there is a tame pure imaginary harmonic bundle (E, θ, h) so that the monodromy representation of $D := \nabla_h + \theta + \theta_h^\dagger$ is ρ . Here ∇_h for the Chern connection of (E, h) . One can see that $(E, t\theta, h)$ is also a harmonic bundle for any $t \in U(1)$. It is tame by Definition 1.10. Take a projective compactification \bar{X} of X with $D := \bar{X} - X$ a

simple normal crossing divisor. For every irreducible component D_i of D , since the eigenvalues of $\text{Res}_{D_i}\theta$ are all zero by our assumption, so is $\text{Res}_{D_i}(t\theta) = t\text{Res}_{D_i}(\theta)$. Hence $(E, t\theta, h)$ is a tame pure imaginary. It follows from Theorem 1.12 that the flat connection $D_t := \nabla_h + t\theta + \bar{t}\theta_h^\dagger$ is semi-simple.

Note that for any $t \in U(1)$ the flat connection $D_t := \nabla_h + t\theta + \bar{t}\theta_h^\dagger$ is a deformation of D . Fix some $t \in U(1)$ which is not root of unity. By the assumption that ρ is local rigid, the *semi-simple* flat connection $D_t := \nabla_h + t\theta + \bar{t}\theta_h^\dagger$ is thus gauge equivalent to D . In other words, letting V be the underlying smooth vector bundle of E , there is a smooth automorphism $\varphi : V \rightarrow V$ so that $D_t = \varphi^*D := \varphi^{-1}D\varphi$. Hence φ^*h defined by $\varphi^*h(u, v) := h(\varphi(u), \varphi(v))$ is the harmonic metric for the flat bundle $(V, \varphi^*D) = (V, D_t)$. By the unicity of harmonic metric of simple flat bundle in [Moc07b, Theorem 25.21], there is a constant $c > 0$ so that $c\varphi^*h = h$. Let us replace the automorphism φ of V by $\sqrt{c}\varphi$ so that we will have $\varphi^*h = h$. Since the decomposition of D_t with respect to the metric h into a sum of unitary connection and self-adjoint operator of V is unique, it implies that

$$\begin{aligned}\nabla_h &= \varphi^*\nabla_h \\ t\theta + \bar{t}\theta_h^\dagger &= \varphi^*(\theta + \theta_h^\dagger).\end{aligned}$$

Since $E = (V, \nabla_h^{0,1})$, the first equality means that φ is a holomorphic isomorphism of E . The second one implies that

$$t\theta = \varphi^*\theta := \varphi^{-1}\theta\varphi.$$

Consider the prolongation ${}^\circ E_h$ over \bar{X} via norm growth defined in § 1.4. Since $\varphi : (E, h) \rightarrow (E, h)$ is moreover an isometry, it thus extends to a holomorphic isomorphism ${}^\circ E_h \rightarrow {}^\circ E_h$, which we still denote by φ . Recall that (E, θ, h) and $(E, t\theta, h)$ is tame, we thus have the following diagram

$$\begin{array}{ccc} {}^\circ E_h & \xrightarrow{t\theta} & {}^\circ E_h \otimes \Omega_{\bar{X}}(\log D) \\ \downarrow \varphi & & \downarrow \varphi \otimes \text{id} \\ {}^\circ E_h & \xrightarrow{\theta} & {}^\circ E_h \otimes \Omega_{\bar{X}}(\log D). \end{array}$$

This diagram is commutative. Indeed, since $t\theta = \varphi^*\theta$ over X , it follows that $t\theta = \varphi^*\theta$ for the extended morphisms over \bar{X} by the continuity. Now we apply the same arguments in [Sim92, Lemma 4.1] to proceed as follows. The eigenvalues of $\varphi : {}^\circ E_h \rightarrow {}^\circ E_h$ are holomorphic functions, which are thus constant since \bar{X} is compact. Consider the generalized eigenspace ${}^\circ E_{h,\lambda}$ defined by $\ker(\varphi - \lambda)^\ell = 0$ for some sufficiently big ℓ . Since

$$(\varphi - t^{-1}\lambda)^k \theta = t^{-k} \theta (\varphi - \lambda)^k,$$

it follows that $\theta : {}^\circ E_{h,\lambda} \rightarrow {}^\circ E_{h,t^{-1}\lambda} \otimes \Omega_{\bar{X}}(\log D)$. Since t is not root of unity, the eigenvalues of φ break up into a finite number of chains of the form $t^i\lambda, \dots, t^{-j}\lambda$ so that $t^{i+1}\lambda$ and $t^{-j-1}\lambda$ are not eigenvalues of φ . Therefore, there is decomposition ${}^\circ E_h = \bigoplus_{i=0}^m E_i$ so that

$$\theta : E_i \rightarrow E_{i+1} \otimes \Omega_{\bar{X}}(\log D).$$

In the language of [Den22, Definition 2.10] $({}^\circ E_h = \bigoplus_{i=0}^m E_i, \theta)$ is a *system of log Hodge bundles*. One can apply [Den22, Proposition 2.12] to show that the harmonic metric h is moreover a *Hodge metric* in the sense of [Den22, Definition 2.11], *i.e.* $h(u, v) = 0$ if $u \in E_i|_X$ and $v \in E_j|_X$ with $i \neq j$. Hence by [Sim88, Section 8]

(E, θ, h) corresponds to a complex variation of Hodge structures in X . This proves the proposition. \square

5. Simpsons's integrality conjecture and proof of Theorem A

In [Sim92], Simpson conjectured that for any projective manifold X a rigid representation $\rho : \pi_1(X) \rightarrow \mathrm{GL}_N(\mathbb{C})$ is conjugate to an *integral* one, *i.e.* a representation $\pi_1(X) \rightarrow \mathrm{GL}_N(\mathcal{O}_K)$ where K is a number field and \mathcal{O}_K is the ring of integer of K . This is the so-called *Simpson's integrality conjecture*. Let us first prove the Simpson conjecture for quasi-projective manifolds which do not admit logarithmic symmetric forms.

Corollary 5.1. — *Let X be a quasi-projective manifold. Let \bar{X} be a projective compactification of X so that $D = \bar{X} - X$ is a simple normal crossing divisor. Assume that $H^0(\bar{X}, \mathrm{Sym}^m \Omega_{\bar{X}}(\log D)) = 0$ for all positive integer m . Then any semi-simple representation $\rho : \pi_1(X) \rightarrow \mathrm{GL}_N(\mathbb{C})$ is integral. Moreover, ρ is a complex direct factor of a \mathbb{Z} -variation of Hodge structures.*

Proof. — It suffices to assume that ρ is irreducible. By Theorem 4.3, ρ is a rigid representation, and thus for semi-simple representations in the affine variety $\mathrm{Hom}(\pi_1(X), \mathrm{GL}_N(\mathbb{C}))$ near ρ , it is conjugate to ρ . Since $\mathrm{Hom}(\pi_1(X), \mathrm{GL}_N(\bar{\mathbb{Q}}))$ is dense in $\mathrm{Hom}(\pi_1(X), \mathrm{GL}_N(\mathbb{C}))$, and $\pi_1(X)$ is finitely generated, it follows that after taking some conjugation, there is a number field K so that $\rho : \pi_1(X) \rightarrow \mathrm{GL}_N(K)$, which is an irreducible representation over $GL_N(\bar{K})$ (so called *absolutely irreducible*). For each finite place v of K , let $\rho_v : \pi_1(X) \rightarrow \mathrm{GL}_N(K_v)$ be the extended one of ρ through $K \hookrightarrow K_v$, where K_v is the non-archimedean completion of K with respect to v . ρ_v is also semisimple since the semi-simplicity is preserved under field extension in characteristic zero. Moreover, the Zariski closure is reductive of ρ_v is reductive.

If for some prime v , ρ_v is unbounded, namely its image is not contained in any compact subgroup of $\mathrm{GL}_N(K_v)$, by Theorem 3.4 we have

$$H^0(\bar{X}, \mathrm{Sym}^k \Omega_{\bar{X}}(\log D)) \neq 0$$

for some positive integer k . A contradiction is obtained. Hence ρ_v is bounded for every finite place v of K . It follows that $\rho(\pi_1(X))$ lies in $\mathrm{GL}_N(\mathcal{O}_K)$, where \mathcal{O}_K denotes the ring of integer of K .

For every embedding $\sigma : K \rightarrow \mathbb{C}$, the composition $\sigma\rho : \pi_1(X) \rightarrow \mathrm{GL}_N(\mathbb{C})$ is also semi-simple and hence rigid by Theorem 4.3 again. By Proposition 4.8, $\sigma\rho$ underlies a complex variation of Hodge structures for each embedding $\sigma : K \rightarrow \mathbb{C}$. Conditions in [LS18, Proposition 7.1 and Lemma 7.2] are fulfilled, so we conclude that ρ is a complex direct factor of a \mathbb{Z} -variation of Hodge structures. \square

Remark 5.2. — Corollary 5.1 was proved by Klingler in [Kli13, Corollary 1.8] when X is compact. Recently, Esnault and Groechenig [EG18] proved that a cohomology rigid local system over a quasi-projective variety with finite determinant and quasi-unipotent local monodromies at infinity is integral.

Proof of Theorem A. — Let σ be the semisimplification of τ . If σ has finite image, then we can replace X be a finite étale cover X' so that the image of $\pi_1(X')$ under σ is zero. Hence $\tau(\pi_1(X'))$ is contained in a unipotent subgroup U of $GL_N(\mathbb{C})$. Since U is unipotent and τ has infinite image, the abelianization $H_1(X', \mathbb{Z})$ of $\pi_1(X')$ must be infinite. The universal coefficient theorem implies that $H^1(X', \mathbb{C})$ is infinite. Take a smooth projective compactification of \bar{X}' so that $D' := \bar{X}' - X'$ is a simple normal

crossing divisor. It follows from Claim 3.5 that $H^0(\overline{X'}, \Omega_{\overline{X'}}(\log D'))$ is non-zero. By Lemma 3.7, any element in $H^0(\overline{X'}, \Omega_{\overline{X'}}(\log D'))$ gives rise to a non-zero logarithmic symmetric forms in $H^0(\overline{X}, \text{Sym}^{|G|} \Omega_{\overline{X}}(\log D))$. We proved the theorem if σ has finite image.

Assume now σ has infinite image. Assume by contradiction that

$$H^0(\overline{X}, \text{Sym}^k \Omega_{\overline{X}}(\log D)) = 0$$

for all $k > 0$. It follows from Corollary 5.1 that σ is a direct factor of a representation $\rho : \pi_1(X) \rightarrow \text{GL}_m(\mathbb{Z})$ underlying a \mathbb{Z} -variation of Hodge structures. Let $\Phi : X \rightarrow \mathcal{D}/\Gamma$ be the corresponding period mapping, where \mathcal{D} is the period domain and $\Gamma = \rho(\pi_1(X))$ is the monodromy group which acts discretely on \mathcal{D} . Since ρ has infinite image, Φ has positive dimensional image. By the work of Griffiths, we know that there is a Zariski open set $X_1 \subset \overline{X}$ containing X so that Φ extends to a proper holomorphic map $X_1 \rightarrow \mathcal{D}/\Gamma$. Its image Z is thus a proper subvariety of \mathcal{D}/Γ . By Sommese [Som78, Proposition IV], we know that there is a desingularization $Y \rightarrow Z$ so that Y is a quasi-projective manifold. Take a projective compactification \overline{Y} of Y so that $D_Y = \overline{Y} - Y$ is simple normal crossing. By [Den20], the meromorphic map $X_1 \dashrightarrow Y$ is rational. One can take a resolution of indeterminacy $X'_1 \rightarrow X_1$ so that $f : X'_1 \rightarrow Y$ is an algebraic morphism. By [Bru18, BC20], we know that the log cotangent bundle $\Omega_{\overline{Y}}(\log D_Y)$ is big. Therefore, for some k big enough,

$$H^0(\overline{Y}, \text{Sym}^k \Omega_{\overline{Y}}(\log D_Y)) \neq 0.$$

Take a projective compactification \overline{X}'_1 of X_1 so that $D_1 = \overline{X}'_1 - X'_1$ is simple normal crossing and f extends to a dominant log morphism $\bar{f} : (\overline{X}'_1, D_1) \rightarrow (\overline{Y}, D_Y)$. One thus can pull back the log symmetric forms in $H^0(\overline{Y}, \text{Sym}^k \Omega_{\overline{Y}}(\log D_Y))$ via \bar{f} to obtain non-zero log symmetric forms in $H^0(\overline{X}'_1, \text{Sym}^k \Omega_{\overline{X}'_1}(\log D_1))$. In the same manner, these log symmetric forms give rise to non-zero log symmetric forms in $H^0(\overline{X}, \text{Sym}^k \Omega_{\overline{X}}(\log D))$. The theorem is proved. \square

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