DEFORMATION OPENNESS OF BIG FUNDAMENTAL GROUPS AND APPLICATIONS

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ABSTRACT. In 2001, de Oliveira, Katzarkov, and Ramachandran conjectured that the property of smooth projective varieties having big fundamental groups is stable under small deformations. This conjecture was proven by Benoît Claudon in 2010 for surfaces and for threefolds under suitable assumptions. In this paper, we prove this conjecture for smooth projective varieties admitting a big complex local system. Moreover, we address a more general conjecture by Campana and Claudon concerning the deformation invariance of the Γ-dimension of projective varieties.

As an application, we establish the deformation openness of pseudo-Brody hyperbolicity for projective varieties endowed with a big and semisimple complex local system.

To achieve these results, we develop the deformation regularity of equivariant pluriharmonic maps into Euclidean buildings and Riemannian symmetric spaces in families, along with techniques from the reductive and linear Shafarevich conjectures.

CONTENTS

1. Introduction

In [Sha77], Shafarevich conjectured that the universal cover of a smooth complex projective variety X is holomorphically convex. If true, this conjecture implies the existence of a proper holomorphic fibration $\mathrm{sh}_X : X \to \mathrm{Sh}(X)$ (referred to as the Shafarevich morphism), which contracts to a point precisely those closed subvarieties Z for which $\mathrm{Im}[\pi_1(Z) \to \pi_1(X)]$ is finite. In [Kol93, Cam94], Campana and Kollár independently established the existence of such a map, up to birational equivalence (so-called Γ-*reduction* or *Shafarevich map*, cf. Theorem 1.2 below). In [Eys04, EKPR12], Eyssidieux, Katzarkov, Pantev and Ramachandran proved the Shafarevich conjecture for smooth projective varieties with *linear fundamental groups*. Recall that a finitely generated group Γ is called *linear* (resp. *reductive*) if there exists an almost faithful (i.e. with finite kernel) linear (resp. *reductive*) representation $\Gamma \to \mathrm{GL}_N(\mathbb{C})$. For more recent progress on the Shafarevich conjecture, see [DYK23].

In [Kol95, §18.4], Kollár asked the question of how Shafarevich maps for complex projective varieties vary in families. Precise conjectures were later proposed by de Oliveira, Katzarkov, and Ramachandran [dOKR02], as well as by Campana and Claudon [Cla10]. This paper aims to explore these conjectures, offering some applications to the hyperbolicity of algebraic varieties.

1.1. **Deformation of varieties with big fundamental groups.** Let X be a compact Kähler manifold X. We say X has *big* (or *generically large* in [Kol95]) fundamental group if for any positive-dimensional irreducible subvariety Z of X passing through a very general point, the image $\mathrm{Im}[\pi_1(Z) \to \pi_1(X)]$ is an infinite group. Similarly, a representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ is called *big* if for any positive-dimensional irreducible subvariety Z of X passing through a very general point, the image $\varrho(\mathrm{Im}[\pi_1(Z) \to \pi_1(X)])$ is an infinite group.

As seen in the above definition, we emphasize that throughout this paper, when discussing properties of the fundamental group $\pi_1(X)$ of a compact Kähler manifold X, or representations $\pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$, we are not solely referring to the abstract group structure of $\pi_1(X)$. Instead, we also take into account the structure inherently tied to the complex structure of X itself.

In [dOKR02, Conjecture 1.1], de Oliveira, Katzarkov, and Ramachandran proposed the following intriguing conjecture.

Conjecture 1.1 ([dOKR02])**.** *Let $f : \mathscr{X} \to \mathbb{D}$ be a smooth projective family of varieties over the unit disk $\mathbb{D}$. For each $t \in \mathbb{D}$, let $X_t := f^{-1}(t)$ denote the fiber over t. If X_0 has big fundamental group, then X_t also has big fundamental group for sufficiently small $t \in \mathbb{D}$.*

Conjecture 1.1 as stated here is more general than in [dOKR02, Conjecture 1.1], as we do not assume the existence of a linear representation of $\pi_1(X_0)$ with infinite image. This conjecture was proved by Claudon for the case $\dim X_0 = 2$ (cf. [Cla10, Corollaire 1.1]). Furthermore, he established Conjecture 1.1 for $\dim X_0 = 3$ (cf. [Cla10, Théorème 0.4]), contingent upon a conjecture regarding the virtual abelianity of the fundamental groups of special Kähler orbifold surfaces (cf. [Cla10, Conjecture 3]). However, beyond these works, there has been no progress on this conjecture in higher-dimensional cases.

In this paper, we prove the following theorem.

Theorem A (=Theorem 7.16.(ii))**.** *Conjecture 1.1 holds if there exists a big representation $\varrho : \pi_1(X_0) \to \mathrm{GL}_N(\mathbb{C})$. In particular, it holds if $\pi_1(X_0)$ is linear.*

1.2. Γ-dimension under deformation. In [Cam94], Campana introduced the notion of the Γ-*dimension* (distinct from the Von Neumann dimension in [Ati76]!), based on his construction of the Γ-*reduction*. This construction was also obtained independently by Kollár for algebraic varieties [Kol93] under the name *Shafarevich map*.

Theorem 1.2 ([Cam94, Kol93])**.** *Let X be a compact Kähler manifold and let $H \lhd \pi_1(X)$ be a normal subgroup. Then there exists proper surjective,* almost holomorphic *map $\gamma_{(X,H)} : X \dashrightarrow \Gamma_H(X)$ with connected fibers such that for any closed positive-dimensional subvariety $Z \subset X$ passing through a very general point of X, $\gamma_{(X,H)}(Z)$ is a point if and only if $\mathrm{Im}[\pi_1(Z) \to \pi_1(X)/H]$ is finite. Such a map is unique up to bimeromorphic equivalence.* □

Here, an almost holomorphic map refers to a meromorphic map whose indeterminacy locus does not dominate its image. When H is a finite subgroup, one can verify that $\gamma_{(X,H)}$ is bimeromorphic to $\gamma_{(X,\{e\})}$, and in this case we simply write $\gamma_X : X \dashrightarrow \Gamma(X)$ for $\gamma_{(X,H)} : X \dashrightarrow \Gamma_H(X)$, and call it the Γ-reduction of X.

The following definition is introduced in [Cam94, Définition 4.1].

Definition 1.3 (Γ-reduction, Γ-dimension)**.** Let X be a compact Kähler manifold and let $H \lhd \pi_1(X)$ be a normal subgroup. The almost holomorphic map $\gamma_{(X,H)} : X \dashrightarrow \Gamma_H(X)$ in Theorem 1.2 is called the Γ-*reduction* or *Shafarevich map* of (X, H). The Γ-*dimension* of (X, H), denoted by $\gamma d(X, H)$, is $\dim \Gamma_H(X)$.

Remark 1.4. The Γ-dimension, roughly speaking, measures the positivity of the fundamental group of a compact Kähler manifold. By Theorem 1.2, it can be viewed as an analogue of the Kodaira dimension of canonical bundles for the fundamental groups of compact Kähler manifolds, with the Γ-reduction playing a role similar to the Iitaka fibration. In two extreme cases, X has big (resp. finite) fundamental group if and only if $\gamma d(X) = \dim X$ (resp. $\gamma d(X) = 0$). In [Cla10], varieties with big fundamental groups are also referred to as being of π_1-*general type*, analogous to the concept of maximal Kodaira dimension. In [Cam95], Campana established an upper bound for $\gamma d(X)$ in terms of an algebraic-geometric invariant $\kappa^+(X)$, which is determined by the positivity properties of Ω_X^p for any $p > 0$.

As an analogue to the invariance of plurigenera proved by Siu [Siu98, Siu02] (cf. also [Pău07]), Campana and Claudon proposed the following conjecture, which is considerably more general than Conjecture 1.1.

Conjecture 1.5 ([Cla10, Conjecture 2]). *Let $f : \mathscr{X} \to \mathbb{D}$ be a smooth proper Kähler family. Then $t \mapsto \gamma d(X_t)$ is a continuous (hence constant) function on $\mathbb{D}$.*

Here we say $f : \mathscr{X} \to \mathbb{D}$ is a *smooth proper Kähler family* if $\mathscr{X}$ is a Kähler manifold and f is a holomorphic proper submersion with connected fibers.

In [Cla10], Claudon proved Conjecture 1.5 when $\dim X_0 = 2$, and in the case where $\dim X_0 = 3$ and $\kappa(X_0) \leq 2$.

We prove Conjecture 1.5 for Kähler varieties with rigid and integral fundamental groups.

Theorem B (=Theorem 3.7). *Let $f : \mathscr{X} \to \mathbb{D}$ be a proper smooth Kähler family. If there exists an almost faithful reductive representation $\varrho : \pi_1(X_0) \to \mathrm{GL}_N(\mathbb{C})$, which is rigid and integral, then $t \mapsto \gamma d(X_t)$ is constant on $\mathbb{D}$.*

Remark 1.6. By a deep theorem of Esnault and Groechenig [EG18], any *cohomologically rigid* local system on a smooth projective variety is integral. Consequently, Theorem B also holds if X_0 is a smooth projective variety and $\varrho : \pi_1(X_0) \to \mathrm{GL}_N(\mathbb{C})$ is assumed to be only almost faithful and cohomologically rigid.

We also prove the lower semi-continuity for Γ-dimension of varieties with linear fundamental groups, which includes Theorem A as a special case.

Theorem C (=Theorems 7.16.(i) and 5.6). *Let $f : \mathscr{X} \to \mathbb{D}$ be a smooth projective family.*

(i) *Let $M_t := M_{\mathrm{B}}(\pi_1(X_t), \mathrm{GL}_N)(\mathbb{C})$ denote the Betti moduli space of $\pi_1(X_t)$. Define $H_t \triangleleft \pi_1(X_t)$ as the intersection of the kernels of all reductive representations $\varrho : \pi_1(X_t) \to \mathrm{GL}_N(\mathbb{C})$. Then the function $t \mapsto \gamma d(X_t, H_t)$ on $\mathbb{D}$ is* lower semicontinuous.

(ii) *If $\pi_1(X_0)$ is linear, then $t \mapsto \gamma d(X_t)$ is also a lower semicontinuous function on $\mathbb{D}$.*

To prove Theorems A to C, we establish the deformation regularity of equivariant harmonic maps to Euclidean buildings or symmetric spaces in families (see Theorems 2.5, 4.29, 5.4 and 7.7). We then apply these results, together with techniques in [Eys04, EKPR12, DYK23] on the Shafarevich conjecture to prove Theorems A to C.

1.3. **Applications to hyperbolicity.** We give an application of Theorem C.(i) to the hyperbolicity of algebraic varieties under deformation. First, we first recall the following definition of hyperbolicity (see [Dem20, CDY22] for related results and other notions of hyperbolicity).

Definition 1.7 (Pseudo Brody hyperbolicity). A compact complex manifold X is called *pseudo-Brody hyperbolic* if there is a proper Zariski closed subset $Z \subsetneq X$ such that any non-constant holomorphic map $f : \mathbb{C} \to X$ has image in Z. If $Z = \varnothing$, X is simply called *Brody hyperbolic*.

Recall the following classical result on the openness of Brody hyperbolicity (see e.g. [Dem20, Proposition 1.10]).

Theorem 1.8. *Let $f : \mathscr{X} \to \mathbb{D}$ be a holomorphic proper submersion from a complex manifold $\mathscr{X}$ to the unit disk with connected fibers. If X_0 is Brody hyperbolic, then there exists $\varepsilon > 0$ such that X_t is Brody hyperbolic for $|t| < \varepsilon$.*

There has long been a folklore conjecture that such openness properties hold for pseudo-Brody hyperbolicity.

Conjecture 1.9. *Let $f : \mathcal{X} \to \mathbb{D}$ be as in Theorem 1.8. If X_0 is pseudo-Brody hyperbolic, then X_t is also pseudo Brody hyperbolic for sufficiently small t.*

This conjecture is indeed a consequence of the following generalized Green-Griffiths-Lang conjecture.

Conjecture 1.10 (Generalized GGL conjecture)**.** *Let X be a smooth projective variety. Then X is pseudo-Brody hyperbolic if and only if it is of general type.*

Let us explain how to prove Conjecture 1.9 assuming Conjecture 1.10. If X_0 is pseudo-Brody hyperbolic, then X_0 is of general type by Conjecture 1.10. By the invariance of plurigenera, X_t is of general type for any $t \in \mathbb{D}$. By Conjecture 1.10 again, X_t is pseudo-Brody hyperbolic. This proves Conjecture 1.9.

In [CDY22, Theorem C], we prove Conjecture 1.10 if there exists a big and reductive representation $\pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$. We will apply Theorem C.(i) together with [CDY22, Theorem C] and [DYK23] to prove the following result on Conjecture 1.9.

Theorem D (=Theorem 6.1)**.** *Let $f : \mathcal{X} \to \mathbb{D}$ be a smooth projective family. Assume that there is a big and reductive representation $\varrho : \pi_1(X_0) \to \mathrm{GL}_N(\mathbb{C})$. If X_0 is pseudo-Brody hyperbolic, then there exists $\varepsilon > 0$ such that X_t is also pseudo-Brody hyperbolic for sufficiently small t.*

As a consequence, we prove the following result.

Corollary E. *Let X be a smooth projective variety. Assume that either*

(a) *there is a complex variation of Hodge structure ($\mathbb{C}$-VHS for short) $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ with discrete monodromy Γ such that the period map $X \to \mathscr{D}/\Gamma$ is generically finite onto the image, or*
(b) *there is a big representation $\tau : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ such that the Zariski closure of $\tau(\pi_1(X))$ is a semisimple algebraic group.*

Then every small projective deformation of X is pseudo-Brody hyperbolic.

Remark 1.11. The hyperbolicity results stated in this subsection also hold if we replace pseudo-Brody hyperbolicity with the stronger notion, known as pseudo-Picard hyperbolicity, introduced in [Den23, CDY22]. To lighten the notion, we consider only pseudo-Brody hyperbolicity in this paper, and interested readers can refer to [CDY22] for other notions of hyperbolicity.

1.4. **Notation and Convention.**

(a) Throughout this paper, $\pi_1(X)$ always refers to the topological fundamental group of the variety X.
(b) For a complex space X, we denote by X^{reg} the regular locus of X.
(c) Unless otherwise specified, all projective varieties are assumed to be defined over the field of complex numbers.
(d) Denote by $\mathbb{D}$ the unit disk in $\mathbb{C}$, and by $\mathbb{D}^*$ the punctured unit disk. We write $\mathbb{D}_\varepsilon := \{z \in \mathbb{C} \mid |z| < \varepsilon\}$, $\mathbb{D}_\varepsilon^* := \{z \in \mathbb{C} \mid 0 < |z| < r\}$.

(e)   A *smooth projective family* $f : \mathscr{X} \to \mathbb{D}$ is a smooth projective morphism from $\mathscr{X}$ to $\mathbb{D}$ with connected fibers. A *smooth proper Kähler family* $f : \mathscr{X} \to \mathbb{D}$ is a holomorphic proper submersion from a Kähler manifold $\mathscr{X}$ to $\mathbb{D}$ with connected fibers. For any $t \in \mathbb{D}$, let $X_t := f^{-1}(t)$ denote the fiber of f over t. For notational simplicity, we often write X instead of X_0. Denote by $\pi_{\mathscr{X}} : \widetilde{\mathscr{X}} \to \mathscr{X}$ the universal covering map.

(f)   Let $f : \mathscr{X} \to \mathbb{D}$ be as above, and let $\varrho : \pi_1(X_0) \to \mathrm{GL}_N(K)$ be a representation, where K is any field. We denote by $\varrho_t : \pi_1(X_t) \to \mathrm{GL}_N(K)$ the representation induced by the composition of ϱ with the natural isomorphism $\pi_1(X_t) \to \pi_1(X_0)$ induced by f.

(g)   Let $\Omega \subset \mathbb{C}^n$ be an open domain endowed with a Riemannian metric g, and let $z \in \Omega$ be any point. We denote by $\mathbb{B}_r(z)$ the Euclidean ball of radius r in Ω centered at z, and by $B_r(z)$ the ball of radius r in Ω centered at z with respect to the Riemannian metric g.

(h)   Let G be a reductive group defined over a non-archimedean local field K. We denote by $\Delta(G)_K$ the Bruhat-Tits building of G, which is a non-positively curved (NPC) space. Denote by $d(\bullet, \bullet)$ the distance function on $\Delta(G)_K$. Let $\mathscr{D}G$ denote the derived group of G, which is semisimple.

(i)   A linear representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$, where K is a field, is called *reductive* if the Zariski closure of $\varrho(\pi_1(X))$ is a reductive algebraic group over $\overline{K}$.

2. DEFORMATION OF HARMONIC BUNDLES

The aim of this section is to prove Theorem 2.5, which was used in the proofs of Theorem B and Theorem C.

2.1. **Some preliminary on Higgs bundle and harmonic bundle.** We first recall the definition of Higgs bundles and harmonic bundles. We refer the readers to [Cor88, Sim88, Sim92] for further details.

Definition 2.1. Let X be a complex manifold. A *Higgs bundle* on X consists of $(E, \bar{\partial}_E, \theta)$ where E is a holomorphic vector bundle with $\bar{\partial}_E$ its complex structure, and $\theta : E \to E \otimes \Omega_X^1$ is a holomorphic one form with value in $\mathrm{End}(E)$, say *Higgs field*, satisfying $\theta \wedge \theta = 0$. We will usually write (E, θ) instead of $(E, \bar{\partial}_E, \theta)$ if no confusion arises.

Let (E, θ) be a Higgs bundle over a complex manifold X. Suppose h is a smooth hermitian metric of E. Denote by ∇_h the Chern connection with respect to h, and by θ_h^* the adjoint of θ with respect to h. The metric h is *harmonic* if the operator $D_h := \nabla_h + \theta + \theta_h^*$ is integrable, that is, if $D_h^2 = 0$. We write θ^* instead of θ_h^* to simply the notation if no confusion arises.

Definition 2.2 (Harmonic bundle). A harmonic bundle on a complex manifold X is a Higgs bundle (E, θ) endowed with a harmonic metric h.

Definition 2.3. For a harmonic bundle (E, θ, h) on a complex manifold X, its *canonical form* is a semi-positive closed $(1, 1)$-form on X defined by $\sqrt{-1}\mathrm{tr}(\theta \wedge \theta^*)$.

Let (X, ω) be a compact Kähler manifold. Assume that $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ is a reductive representation. Let (V_ϱ, D) be the flat bundle on X with the monodromy representation being ϱ. Let h be any smooth hermitian metric for V_ϱ. Then there exists a unique decomposition $D = \nabla_h + \Phi_h$ such that ∇_h is a unitary connection of (V_ϱ, h) and Φ_h is a self-adjoint operator, namely, we have

$$\langle \Phi_h(e), e' \rangle_h = \langle e, \Phi_h(e') \rangle_h, \quad \forall \, e, e' \in V_\varrho.$$

Say h is a *harmonic metric* if $\nabla_h^* \Phi_h = 0$. In this case, by the Siu-Sampson formula, we have the pluriharmonicity of h. In other words, if we decompose $\nabla_h = \nabla_h' + \nabla_h''$ and $\Phi_h = \Phi_h' + \Phi_h''$ into the $(1, 0)$ and $(0, 1)$-parts, then $(V_\varrho, \nabla_h'', \Phi_h', h)$ is a harmonic bundle. By the theorem of Corlette [Cor88], such harmonic metric exists and is unique up to some obvious ambiguity. We will also refer to (V_ϱ, D, h) as a *harmonic bundle* if no confusion arises. In this case, the canonical form associated with the harmonic bundle $(V_\varrho, D_h'', \Phi_h', h)$ introduced in Definition 2.3 is uniquely defined, and we denote it by $\omega_\varrho := \sqrt{-1}\mathrm{tr}(\Phi_h' \wedge \Phi_h'')$. Note that it does not depend on the choice of the Kähler metric ω on X.

Definition 2.4 (Canonical form). We call the above semi-positive closed $(1, 1)$-form ω_ϱ the *canonical form* associated with the reductive representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$.

We leave it to the reader to verify as an exercise that for any $g \in \mathrm{GL}_N(\mathbb{C})$, we have $\omega_\varrho = \omega_{g\varrho g^{-1}}$, where $g\varrho g^{-1}$ is the conjugate of ϱ by g.

2.2. **Deformation of canonical forms.** Let us prove the main result of this section on the continuity of canonical forms in families.

Theorem 2.5. *Let $f : \mathcal{X} \to \mathbb{D}$ be a proper smooth Kähler family. Let $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ be a reductive representation. Then the fiberwise defined canonical form ω_{ϱ_t} on X_t associated with the representation $\varrho_t : \pi_1(X_t) \to \mathrm{GL}_N(\mathbb{C})$ varies continuously with respect to $t \in \mathbb{D}$.*

The proof of Theorem 2.5 follows essentially from the same arguments in [Sim92, Sim94b], based on Uhlenbeck's weak compactness theorem [Uhl82]. We first recall some preliminary results.

Let g denote the Euclidean metric $\sum dz_i \cdot d\bar{z}_i$ of $\mathbb{D}^n$. Let ω be a Kahler form on $\mathbb{D}^n$ such that there exists a constant $C_0 > 0$ such that $C_0^{-1} \cdot \omega \leq g \leq C_0 \cdot \omega$. Let (E, θ, h) be a harmonic bundle on $\mathbb{D}^n$. We have the expression $\theta = \sum_{i=1}^n f_i \cdot dz_i$ for holomorphic sections $f_i \in \mathrm{End}(E)$ on $\mathbb{D}^n$. Recall the following result in [Moc06].

Lemma 2.6 ([Moc06, Lemma 2.13]). *There exist a constant C_1 depending only on $\mathrm{rank}\, E$, the upper bound of absolute values of eigenvalues of $f_1, \ldots, f_n$ and the above constant C_0, such that*

$$|\theta|_{\omega, h}^2 (x) \leq C_1, \quad \forall x \in \mathbb{D}_{\frac{1}{2}}^n.$$

Here $|\theta|_{\omega, h}$ denotes the norm of θ with respect the metric h and ω. □

Lemma 2.7. *There exists a constant C_2 depending only on $\mathrm{rank}\, E$ and C_0 such that absolute values of eigenvalues of $f_1(x), \ldots, f_n(x)$ for each $x \in \mathbb{D}_{\frac{1}{2}}^n$ are bounded from above by $C_2\|\theta\|_{\omega, h}^2$, where*

$$\|\theta\|_{\omega, h}^2 := \int_{\mathbb{D}^n} |\theta|_{\omega, h}^2 d\mathrm{Vol}_\omega.$$

Proof. Consider the characteristic polynomial $P_i := \det(t - f_i) = t^r + \sigma_{i,1}t^{r-1} + \cdots + \sigma_{i,r}$, where $r = \operatorname{rank} E$ and $\sigma_{i,j}$ are holomorphic functions on $\mathbb{D}^n$. For any $x \in \mathbb{D}^n$, we denote by $\lambda_i(x)$ the maximum of the absolute values of eigenvalues of $f_i(x)$. Then by the classical inequalities between the norms of roots and coefficients of a polynomial, one has

$$\lambda_i(x) \le 2 \max_{k=1,\dots,r} |\sigma_{i,k}(x)|^{\frac{1}{k}}.$$

Note that there exists a constant C_3 depending only on $\operatorname{rank} E$ such that

$$s(x) := \max_{k=1,\dots,r} |\sigma_{i,k}(x)|^{\frac{2}{k}} \le C_3 |f_i(x)|_h^2.$$

For each $z \in \mathbb{D}^n_{\frac{1}{2}}$, we set

$$\Omega_z := \{w \in \mathbb{D}^n \mid |w_i - z_i| \le \frac{1}{2} \text{ for each } i\}.$$

Since $s(x)$ is a psh function, we have

$$\begin{aligned}
\log |s(z)|^2 &\leqslant \frac{4^n}{\pi^n} \int_{\Omega_z} \log |s(w)|^2 d\mathrm{vol}_g \\
&\leqslant \log\Big(\frac{4^n}{\pi^n} \cdot \int_{\Omega_z} |s(w)|^2 d\mathrm{vol}_g\Big) \\
&\le \log(\frac{4^n}{\pi^n}) + \log \int_{\mathbb{D}^n} |s(w)|^2 d\mathrm{vol}_g \\
&\le \log C_3 (\frac{4^n}{\pi^n}) + \log \int_{\mathbb{D}^n} |f_i|_h^2 d\mathrm{vol}_g.
\end{aligned}$$

On the other hand, note that there exists a constant C_4 depending only on C_0 such that

$$\max_{i=1,\dots,n} \int_{\mathbb{D}^n} |f_i|_h^2 d\mathrm{vol}_g \le C_4 \int_{\mathbb{D}^n} |\theta|_{\omega,h}^2 d\mathrm{vol}_\omega = C_4 \|\theta\|_{\omega,h}^2.$$

The above two inequalities yield the lemma. □

Lemmas 2.6 and 2.7 imply the following result.

Lemma 2.8. *There exists a uniform constant C depending only on* $\operatorname{rank} E$*, the constant C_0 above, and the L^2-norm $\|\theta\|_{\omega,h}^2$ such that*

$$|\theta|_{\omega,h}^2 (x) \le C, \quad \forall x \in \mathbb{D}^n_{\frac{1}{2}}.$$

□

Proof of Theorem 2.5. It suffices to prove the continuity of ω_{ϱ_t} at $t = 0$. Write X for X_0 for notational simplicity. We fix a smooth trivialization $F : \mathcal{X} \to X \times \mathbb{D}$ and let $F_t : X_t \to X$ be the induced diffeomorphism. To prove the theorem, it is equivalent to show that $F_{t,*}\omega_{\varrho_t}$ are continuously varying forms on X at $t = 0$.

Fix a Kähler metric g on X and let $g_t := g|_{X_t}$ be the induced Kähler metric on X_t. Let (V_ϱ, D) be the flat vector bundle on $\mathcal{X}$ whose monodromy representation is ϱ. By Corlette's theorem, for each $t \in \mathbb{D}$, there exists a harmonic metric h_t for $(V_t, D_t) := (V_\varrho, D)|_{X_t}$. Let (E_t, θ_t, h_t) be the corresponding harmonic bundle on X_t. By the same arguments as we will see in (4.9),

the energy of the ϱ_t-equivariant harmonic map from $\widetilde{X}_t$ to GL_N/U_N induced (E_t, θ_t, h_t) is a continuous function on $\mathbb{D}$. Hence, there exists a constant $C_1 > 0$ such that the energy

$$\int_{X_t} |\theta_t + \theta_t^*|_{g_t, h_t}^2 d\mathrm{Vol}_{g_t} \leq C_1$$

for any $t \in \mathbb{D}_{\frac{1}{2}}$. This implies that

$$\|\theta_t\|_{g_t, h_t}^2 \leq C_1 \text{ for any } t \in \mathbb{D}_{\frac{1}{2}}. \tag{2.1}$$

We can find a constant $\varepsilon \in (0, \frac{1}{2})$ and admisisble coordinate systems $\{\Omega_1, \dots, \Omega_m\}$ centered at points in X_0 such that the following properties hold:

(a)   there are biholomorphisms $\varphi_i : \mathbb{D}^n \times \mathbb{D}_\varepsilon \to \Omega_i$ with $f \circ \varphi_i(z, t) = t$.
(b)   Set $\Omega_i(\frac{1}{2}) := \varphi_i(\mathbb{D}_{\frac{1}{2}}^n \times \mathbb{D}_\varepsilon)$. Then $\cup_{i=1}^m \Omega_i(\frac{1}{2}) = f^{-1}(\mathbb{D}_\varepsilon)$.
(c)   There exists a constant $C_0 > 0$ such that for the Euclidean metric $g = \sum dz_i \cdot d\bar{z}_i$ of $\mathbb{D}^n$, for each $t \in \mathbb{D}_\varepsilon$, we have

$$C_0^{-1} \varphi_i^* g_t \leq g \leq C_0 \varphi_i^* g_t.$$

By Lemma 2.8 and (2.1), this implies that there exists a uniform constant $C > 0$ such that for any $t \in \mathbb{D}_{\frac{1}{2}}$ and any $x \in X_t$, we have

$$|\theta_t(x)|_{g_t, h_t}^2 \leq C. \tag{2.2}$$

Let C_2 be a constant such that $C_2^{-1} F_t^* g_0 \leq g_t \leq C_2 F_t^* g_0$ for each $t \in \mathbb{D}_\varepsilon$. Then we have the uniform bound for the Chern curvature of (E_t, h_t) with respect to the metric $F_t^* g_0$ as follows:

$$|R(E_t, h_t)(x)|_{F_t^* g_0, h_t} \leq C_2 |R(E_t, h_t)(x)|_{g_t, h_t} = C_2 |[\theta_t, \theta_t^*](x)|_{g_t, h_t} \leq 2C^2 C_2. \tag{2.3}$$

Pick $\{t_i\}_{i \in \mathbb{N}}$ be an arbitrary sequence of points in $\mathbb{D}_\varepsilon$ that converge to 0. To simplify notation, let $F_i : X_i \to X$ be $F_{t_i} : X_{t_i} \to X$ and (E_i, θ_i, h_i) be $(E_{t_i}, \theta_{t_i}, h_{t_i})$. Using (2.3), we can apply Uhlenbeck's weak compactness theorem [Uhl82], following the approach outlined in [Sim94b, §7].

Fix some large positive integer p. By [Sim94b, Proposition 7.9], after subtracting a subsequence, there exists a harmonic bundle (E, θ, h) on X together with a sequence of L_2^p-isometries (i.e. gauge transform in [Uhl82]) $\eta_i : F_{i,*}(E_i, h_i) \xrightarrow{\cong} (E, h)$ such that $\eta_{i,*} F_{i,*}(\theta_i) - \theta$, $\eta_{i,*} F_{i,*}(\theta_i^*) - \theta^*$, and $\eta_{i,*} F_{i,*}(D_i) - D$ converges to zero strongly in C^0. Here D_i (resp. D) is flat connection associated with the harmonic bundle (E_i, θ_i, h_i) (resp. (E, θ, h)).

Note that the monodromy representation $\tau_i : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ of the flat bundle

$$(\eta_{i,*} F_{i,*}(E_i), \eta_{i,*} F_{i,*}(D_i))$$

is conjugate to that of D_i. Hence there exists an element $g_i \in \mathrm{GL}_N(\mathbb{C})$ such that $\tau_i = g_i \varrho g_i^{-1}$. Hence we have $[\tau_i] = [\varrho]$ in $M_B(\pi_1(X), \mathrm{GL}_N)(\mathbb{C})$. Let τ be the the monodromy representation of D. Since $\eta_{i,*} F_{i,*}(D_i) - D$ converges to zero strongly in C^0, it follows that $\lim_{i \to \infty} [\tau_i] = [\tau]$. Hence τ is conjugate to ϱ. One thus has

$$\omega_\varrho = \sqrt{-1} \mathrm{tr}(\theta_0 \wedge \theta_0^*) = \sqrt{-1} \mathrm{tr}(\theta \wedge \theta^*) = \omega_\tau. \tag{2.4}$$

Note that

$$F_{i,*}(\omega_{\varrho_{t_i}}) = F_{i,*}\left(\sqrt{-1}\mathrm{tr}(\theta_i \wedge \theta_i^*)\right) = \sqrt{-1}\mathrm{tr}\left(F_{i,*}(\theta_i) \wedge F_{i,*}(\theta_i^*)\right)$$

$$= \sqrt{-1}\mathrm{tr}\left(\eta_{i,*}F_{i,*}(\theta_i) \wedge \eta_{i,*}F_{i,*}(\theta_i^*)\right). \tag{2.5}$$

Since $\eta_{i,*}F_{i,*}(\theta_i) - \theta$ and $\eta_{i,*}F_{i,*}(\theta_i^*) - \theta^*$ converges to zero strongly in C^0,

$$\sqrt{-1}\mathrm{tr}\left(\eta_{i,*}F_{i,*}(\theta_i) \wedge \eta_{i,*}F_{i,*}(\theta_i^*)\right)$$

converges to $\sqrt{-1}\mathrm{tr}(\theta \wedge \theta^*)$ strongly in C^0. (2.4) and (2.5) imply that $F_{i,*}(\omega_{\varrho_{t_i}})$ converges to ω_ϱ strongly in C^0. As $\{t_i\}$ is any sequence in $\mathbb{D}_{\frac{1}{2}}$ converging to 0, this implies the continuity of ω_{ϱ_t} at X_0. The theorem is proved.   □

Remark 2.9. Later, in Theorem 7.7, we will establish a regularity result for equivariant harmonic maps into symmetric spaces in families. This result enables us to strengthen Theorem 2.5, showing that ω_{ϱ_t} varies *smoothly* with respect to t, although this is not required for the proofs of Theorems B and C. See Remark 7.8.

3. Γ-DIMENSION AND RIGID AND INTEGRAL FUNDAMENTAL GROUPS

In this section we will prove Theorem B, utilizing Theorem 2.5.

3.1. Some preliminary for ℂ-VHS and Griffiths bundle.
We recall some notions of complex variation of Hodge structures (ℂ-VHS for short). We refer the readers to [Gri70, Sch73, Sim88, CMSP17, SS22] for more details.

Let X be a compact Kähler manifold and let $\tau : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ be a reductive representation underlying a ℂ-VHS of weight w. By [Sim88], it corresponds to a *system of Hodge bundles* $(E = \oplus_{p+q=w}E^{p,q}, \theta)$ endowed with a Hodge metric h induced by τ. Precisely, we have

(1) (E, θ, h) is a harmonic bundle on X.
(2)   The decomposition $E = \oplus_{p+q=w}E^{p,q}$ is orthogonal with respect to h.
(3)   We have

$$\theta : E^{p,q} \to E^{p-1,q+1} \otimes \Omega_X$$

(4)   The connection $\nabla_h + \theta + \theta^*$ is flat, whose monodromy representation is τ.

In Griffiths [Gri70] (cf. also [CMSP17, Problem 13.3.3]), Griffiths introduced a line bundle on X (so-called *Griffiths line bundle*) defined by

$$L_\tau := (\det E^{w,0})^w \otimes (\det E^{w-1,1})^{w-1} \otimes \cdots \otimes (\det E^{1,w-1})$$

endowed with the metric g induced by h.

We denote by $\theta_{p,q} := \theta|_{E^{p,q}}$, and $h_{p,q} := h|_{E^{p,q}}$. Then the Chern curvature of $(E^{p,q}, h_{p,q})$ is given by

$$R_{p,q} := -\theta_{p,q}^* \wedge \theta_{p,q} - \theta_{p+1,q-1} \wedge \theta_{p+1,q-1}^*.$$

Therefore, the Chern curvature of (L_τ, g) is

$$R(L_\tau, g) = w\mathrm{tr}(-\theta_{w,0}^* \wedge \theta_{w,0}) + \sum_{p=1}^{w-1}(w-q)\mathrm{tr}(-\theta_{w-q,q}^* \wedge \theta_{w-q,q} - \theta_{w-q+1,q-1} \wedge \theta_{w-q+1,q-1}^*)$$

$$= \sum_{q=0}^{w-1} \mathrm{tr}(-\theta_{w-q,q}^* \wedge \theta_{w-q,q})$$

$$= \sum_{q=0}^{w-1} \mathrm{tr}(\theta_{w-q,q} \wedge \theta_{w-q,q}^*)$$

$$= \mathrm{tr}(\theta \wedge \theta^*).$$

This implies the following result, which is already well-known to experts:

Lemma 3.1. *We have the integrality of the cohomology class*

$$\{\frac{1}{2\pi}\omega_\tau\} = \{\frac{\sqrt{-1}}{2\pi}\mathrm{tr}(\theta \wedge \theta^*)\} = c_1(L_\tau) \in H^2(X, \mathbb{Z}).$$

Let $\mathscr{D}$ be the period domain of the $\mathbb{C}$-VHS τ and let Γ be the monodromy group $\tau(\pi_1(X))$. Assume that Γ acts discretely on $\mathscr{D}$. Then the quotient $\mathscr{D}/\Gamma$ is a complex space and the period map is a holomorphic map $p : X \to \mathscr{D}/\Gamma$, whose differential is θ.

Lemma 3.2. *Let* $\tau : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ *be a* $\mathbb{C}$-VHS *with discrete monodromy. Let* ω *be a Kähler form on a compact Kähler n-fold X. Let* k_0 *be the largest integer k such that*

$$\int_X (\sqrt{-1}\mathrm{tr}(\theta \wedge \theta^*))^k \wedge \omega^{n-k} > 0.$$

Then k_0 *is the dimension of the image* $p(X)$.

Proof. Since θ is the differential of p, it follows that $\theta|_Z$ is trivial for any fiber Z of p. Hence, $\sqrt{-1}\mathrm{tr}(\theta \wedge \theta^*)|_Z$ is also trivial. On the other hand, for any $\xi \in T_X$ such that $dp(\xi) \neq 0$, we have $\mathrm{tr}(\theta \wedge \theta^*)(\xi, \bar{\xi}) = |\theta(\xi)|_h > 0$. Let X° be a Zariski dense open subset of p such that $p|_{X^\circ}$ is a proper submersion. This implies that, at each point $x \in X^\circ$, the semi-positive $(1,1)$-form $\sqrt{-1}\mathrm{tr}(\theta \wedge \theta^*)$ has exactly $\dim p(X)$ strictly positive eigenvalues. Therefore, $(\sqrt{-1}\mathrm{tr}(\theta \wedge \theta^*))^k \wedge \omega^{n-k}$ is a positive measure on X° for $k = \dim p(X)$, and is trivial for $k > \dim p(X)$ on X°. The lemma is proved. □

3.2. Notion of Shafarevich morphism.

Let us give the definition of the Shafarevich morphism (cf. [Eys04, Eys11, DYK23]).

Definition 3.3 (Shafarevich morphism). Let X be a compact Kähler manifold, and let $H \triangleleft \pi_1(X)$ be a normal subgroup. A proper holomorphic fibration $\mathrm{sh}_H : X \to \mathrm{Sh}_H(X)$ onto a complex normal variety $\mathrm{Sh}_H(X)$ is called the *Shafarevich morphism* associated with (X, H) if, for any closed subvariety Z of X, $\mathrm{sh}_H(Z)$ is a point if and only if the image

$$\mathrm{Im}[\pi_1(Z) \to \pi_1(X)/H]$$

is finite. We shall write $\mathrm{sh}_X : X \to \mathrm{Sh}(X)$ for the Shafarevich morphism associated with $(X, \{e\})$, and refer to it as the Shafarevich morphism of X.

Let M be a subset of the *Betti moduli space* $M_{\mathrm{B}}(\pi_1(X), \mathrm{GL}_N)(\mathbb{C})$ (cf. [Sim94a] for the definition). Let $H = \bigcap_\tau \ker \tau$, where τ ranges over all reductive representations $\tau : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ with $[\tau] \in M$. If the Shafarevich morphism $\mathrm{sh}_H : X \to \mathrm{Sh}_H(X)$ exists, then we shall write

$$\mathrm{sh}_M : X \to \mathrm{Sh}_M(X)$$

in place of $\mathrm{sh}_H : X \to \mathrm{Sh}_H(X)$, and call it the Shafarevich morphism associated with M.

In [Eys04] (cf. [DYK23, Proposition 3.37] for the singular case), it is shown that when X is projective and $M := M_{\mathrm{B}}(\pi_1(X), \mathrm{GL}_N)(\mathbb{C})$, the Shafarevich morphism associated with M exists and is algebraic.

Lemma 3.4. *Let X be a smooth projective variety. Set $M := M_{\mathrm{B}}(\pi_1(X), \mathrm{GL}_N)(\mathbb{C})$. If there exists an almost faithful reductive representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$, then the Shafarevich morphism $\mathrm{sh}_M : X \to \mathrm{Sh}_M(X)$ associated with M is the Shafarevich morphism of X.*

Proof. Since $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ is almost faithful, then for any closed subvariety Z of X, $\varrho(\mathrm{Im}[\pi_1(Z) \to \pi_1(X)])$ is finite if and only if $\mathrm{Im}[\pi_1(Z) \to \pi_1(X)]$ is finite. The lemma follows from Definition 3.3. □

Remark 3.5. Let $H \triangleleft \pi_1(X)$ be a normal subgroup. Note that the Shafarevich morphism $\mathrm{sh}_H : X \to \mathrm{Sh}_H(X)$ associated with (X, H), if exists, is bimemorphic to the Γ-reduction of (X, H). Hence, we have $\dim \mathrm{Sh}_H(X) = \gamma d(X, H)$.

The next result is well-known, and we provide a proof for the sake of completeness.

Lemma 3.6. *Let τ be a $\mathbb{C}$-VHS on a compact Kähler manifold X with discrete monodromy Γ. The Stein factorization $g : X \to Y$ of its period map $p : X \to \mathscr{D}/\Gamma$ is the Shafarevich morphism associated with τ.*

Proof. Let Z be a connected component of any fiber of g. Recall that (cf. [CMSP17]), there exists a real semisimple algebraic group $G \subset \mathrm{GL}_N$ such that $G(\mathbb{R})$ acts transitively on $\mathscr{D}$. For any point $P \in \mathscr{D}$, the stabilizer of P under the action of $G(\mathbb{R})$ is a compact subgroup K of $G(\mathbb{R})$. Moreover, we have $\tau(\pi_1(X)) \subset G(\mathbb{R})$. Then $\tau(\mathrm{Im}[\pi_1(Z) \to \pi_1(X)])$ is contained in some conjugate K' of K. Since Γ is discrete, $K' \cap \Gamma$ is finite. Thus, $\tau(\mathrm{Im}[\pi_1(Z) \to \pi_1(X)]) \subset K' \cap \Gamma$ is finite.

Now, let Z be a closed subvariety of X such that $\tau(\mathrm{Im}[\pi_1(Z) \to \pi_1(X)])$ is finite. Then, for some finite étale cover Z' of Z, $\tau(\mathrm{Im}[\pi_1(Z') \to \pi_1(X)])$ is trivial. By the rigidity of VHS (cf. [Sch73, Proposition 7.24]), the pullback of the $\mathbb{C}$-VHS underlying τ on the desingularization of Z' is thus trivial. Hence, the composite map $Z' \to Z \xrightarrow{p} \mathscr{D}/\Gamma$ is trivial. This completes the proof of the lemma. □

3.3. **Proof of Theorem B.** In this subsection, we will prove Theorem B.

Theorem 3.7 (=Theorem B). *Let $f : \mathscr{X} \to \mathbb{D}$ be a proper smooth Kähler family. If there exists a faithful reductive representation $\varrho : \pi_1(X_0) \to \mathrm{GL}_N(\mathbb{C})$, which is rigid and integral, then $t \mapsto \gamma d(X_t)$ is constant on $\mathbb{D}$.*

Proof. By the integrality and rigidity of ϱ, after replacing ϱ by some conjugate, there exists a number field L such that we have

$$\varrho : \pi_1(X_0) \to \mathrm{GL}_N(O_L).$$

Hence we have

$$\varrho_t : \pi_1(X_t) \to \mathrm{GL}_N(O_L)$$

for each $t \in \mathbb{D}$ by the definition of ϱ_t.

Let $\mathrm{Ar}(L)$ be the set of archimedean places of L, consisting of all embedding $\sigma : L \to \mathbb{C}$. Then the direct sum

$$\oplus_{\sigma \in \mathrm{Ar}(L)} \sigma \varrho_t : \pi_1(X_t) \to \oplus \mathrm{GL}_N(\mathbb{C})$$

has discrete image for each $t \in \mathbb{D}$. Let $\tau_t : \pi_1(X_t) \to \mathrm{GL}_M(\mathbb{C})$ be $\oplus_{\sigma \in \mathrm{Ar}(L)} \sigma \varrho_t$.

Since the rigidity of ϱ implies the rigidity of ϱ_t, as we have the natural isomorphism between $M_{\mathrm{B}}(\pi_1(X_t), \mathrm{GL}_N)$ and $M_{\mathrm{B}}(\pi_1(X_0), \mathrm{GL}_N)$, by [Sim92], for any embedding $\sigma \in \mathrm{Ar}(L)$, $\sigma \varrho_t : \pi_1(X_t) \to \mathrm{GL}_N(\mathbb{C})$ corresponds to a $\mathbb{C}$-VHS on X_t. Hence, τ_t underlies a $\mathbb{C}$-VHS with discrete monodromy. We stress here that the period domains of these $\mathbb{C}$-VHS τ_t might vary in t.

Let ω_{τ_t} be the canonical form on X_t associated with τ_t defined in Definition 2.4. By Theorem 2.5, the cohomology classes $\{\omega_{\tau_t}\}_{t \in \mathbb{D}}$ vary continuously in $H^2(\mathscr{X}, \mathbb{R}) \simeq H^2(X_t, \mathbb{R})$ with respect to t.

On the other hand, by Lemma 3.1, $\frac{1}{2\pi} \omega_{\tau_t} \in H^2(X_t, \mathbb{Z})$ for each t. Therefore, the section $(t \mapsto \{\omega_{\tau_t}\}) \in \Gamma(\mathbb{D}, R^2 f_*(\mathbb{R}))$ is flat with respect to the Gauss-Manin connection.

Let ω be a Kähler form on $\mathscr{X}$ which gives the polarization. Denote by $\omega_t = \omega|_{X_t}$. Then the smooth section $(t \mapsto \{\omega_t\}) \in \Gamma(\mathbb{D}, R^2 f_*(\mathbb{R}))$ is also flat with respect to the Gauss-Manin connection of the local system $R^2 f_*(\mathbb{R})$. It follows that for each $k \in \mathbb{N}$,

$$h_k : \mathbb{D} \to \mathbb{N}$$

$$t \mapsto \int_{X_t} (\omega_{\tau_t})^k \wedge (\omega_t)^{n-k}$$

is a constant function on $\mathbb{D}$, where n denotes the relative dimension of f. Set

$$k_0 = \max_k \{h_k \neq 0\}.$$

Since τ_t has discrete image, its image $\Gamma_t := \tau_t(\pi_1(X_t))$ acts discretely on $\mathscr{D}_t$, and thus $\mathscr{D}_t/\Gamma_t$ is a complex space. Denote by $p_t : X_t \to \mathscr{D}_t/\Gamma_t$ the corresponding period map of τ_t. By Lemma 3.2, k_0 is the dimension of the image $p_t(X_t)$. By Lemma 3.6, we have $\dim \mathrm{Sh}_{\tau_t}(X_t) = k_0$ for each $t \in \mathbb{D}$, where $\mathrm{sh}_{\tau_t} : X_t \to \mathrm{Sh}_{\tau_t}(X_t)$ is the Shafarevich morphism of τ_t. Since ϱ_t is assumed to be almost faithful, it follows that $\ker \tau_t$ is a finite subgroup of $\pi_1(X_t)$. This implies that sh_{τ_t} is the Shafarevich morphism of X_t. Hence $\gamma d(X_t) = k_0$ for any $t \in \mathbb{D}$. The theorem is proved. □

4. DEFORMATION OF HARMONIC MAPS INTO EUCLIDEAN BUILDINGS

To prove Theorems A and C, we need to apply techniques used in studying the reductive Shafarevich conjecture (see [Eys04, DYK23]). This involves considering local systems over non-archimedean local fields on projective varieties. In this context, we apply tools developed by Gromov and Schoen [GS92] regarding the existence of equivariant harmonic maps to Bruhat-Tits buildings. In this section, we will explore the deformation of these harmonic maps in families (see Theorem 2.5). Analogous to the canonical form introduced in Definition 2.4, there is a notion of *canonical currents* (see Definition 5.1) associated with representations of fundamental groups into algebraic groups over non-archimedean local fields. Our ultimate

goal is to apply Theorem 4.29 to establish a result similar to Theorem 2.5 concerning the deformation of canonical currents (see Theorem 5.4).

4.1. **NPC spaces.** For the definitions in this subsection, we refer the readers to [BH99, Rou09, KP23].

Definition 4.1 (Geodesic space). Let (X, d_X) be a metric space. A curve $\gamma : [0, \ell] \to X$ into X is called a geodesic if the length $d_X(\gamma(a), \gamma(b)) = b - a$ for any subinterval $[a, b] \subset [0, \ell]$. A metric space (X, d_X) is a *geodesic space* if there exists a geodesic connecting every pair of points in X.

Definition 4.2 (NPC space). An NPC (non-positively curved) space $(\mathcal{N}, d_{\mathcal{N}})$ is a complete geodesic space that satisfies the following condition: for any three points $P, Q, R \in \mathcal{N}$ and a geodesic $\gamma : [0, \ell] \to \mathcal{N}$ with $\gamma(0) = Q$ and $\gamma(\ell) = R$, we have

$$d^2(P, Q_t) \le (1 - t)d^2(P, Q) + t d^2(P, R) - t(1 - t)d^2(Q, R)$$

for any $t \in [0, 1]$, where $Q_t := \gamma(t\ell)$.

An NPC space generalizes a Hadamard manifold, which is a complete, simply connected Riemannian manifold with non-positive sectional curvature. A relevant example of an NPC space that is not a Riemannian manifold, pertinent to this paper, is the Bruhat-Tits building $\Delta(G)_K$ associated with a reductive algebraic group G defined over a non-archimedean local field K. While we do not provide the full definition of Bruhat-Tits buildings here, readers can refer to sources such as [Rou09] and [KP23] for precise details. Notably, a Bruhat-Tits building consists of a collection of apartments, each of which is an isometric and totally geodesic embedding of Euclidean space $\mathbb{R}^N$, where $N = \dim(\Delta(G)_K)$. Additionally, the group $G(K)$, which denotes the K-points of G, acts isometrically on $\Delta(G)_K$ and transitively on its set of apartments. The dimension of $\Delta(G)_K$ equals the K-rank of the algebraic group G, which is the dimension of a maximal K-split torus in G.

In this paper, we mainly work on two types of NPC spaces:

(1) The Riemannian symmetric spaces $\mathrm{GL}_N(\mathbb{C})/\mathrm{U}_N$ and $\mathrm{SL}_N(\mathbb{C})/\mathrm{SU}_N$.
(2) The Bruhat-Tits building $\Delta(G)_K$ of a reductive algebraic group G over a non-archimedean local field K.

4.2. **Some preliminary of harmonic maps to NPC spaces.** For further details of definitions and results in this subsection, we refer the readers to [GS92, KS93, KS97], [BDDM22, §2.2] and [DM24, §4].

Consider a map $f : \Omega \to \mathcal{N}$ from an n-dimensional Riemannian manifold (Ω, g) to an NPC space $(\mathcal{N}, d_{\mathcal{N}})$. When the target space $\mathcal{N}$ is a smooth Riemannian manifold of nonpositive sectional curvature, the energy of a smooth map $f : \Omega \to \mathcal{N}$ is

$$E^f = \int_\Omega |df|^2 \mathrm{dvol}_g$$

where (Ω, g) is a Riemannian domain and dvol_g is the volume form of Ω. We say $f : \Omega \to \mathcal{N}$ is harmonic if it is locally energy minimizing; i.e. for any $x \in \Omega$, there exists $r > 0$ such that the restriction $u|_{B_r(x)}$ minimizes energy amongst all finite energy maps $v : B_r(x) \to \mathcal{N}$ with the same boundary values as $u|_{B_r(x)}$. Here $B_r(x)$ denotes the geodesic ball of radius r centered at x.

In this paper, we also consider the target $\mathcal{N}$ to be NPC spaces, not necessarily smooth. Let us recall the definition of harmonic maps in this context (cf. [KS93] for more details).

Let (Ω, g) be a bounded Lipschitz Riemannian domain. Let Ω_ε be the set of points in Ω at a distance least ε from $\partial\Omega$. Denote by $S_\varepsilon(x) := \partial B_\varepsilon(x)$. We say $f : \Omega \to \mathcal{N}$ is an L^2-map (or that $f \in L^2(\Omega, \mathcal{N})$) if for some point $P \in \Omega$, we have

$$\int_\Omega d^2(f(x), P) d\mathrm{vol}_g < \infty.$$

For $f \in L^2(\Omega, \mathcal{N})$, define

$$e_\varepsilon^f : \Omega \to \mathbb{R}, \quad e_\varepsilon^f(x) = \begin{cases} \int_{y \in S_\varepsilon(x)} \frac{d^2(f(x), f(y))}{\varepsilon^2} \frac{d\sigma_{x,\varepsilon}}{\varepsilon^{n-1}} & x \in \Omega_\varepsilon \\ 0 & \text{otherwise} \end{cases}$$

where $\sigma_{x,\varepsilon}$ is the induced measure on $S_\varepsilon(x)$. We define a family of functionals

$$E_\varepsilon^f : C_c(\Omega) \to \mathbb{R}, \quad E_\varepsilon^f(\varphi) = \int_\Omega \varphi e_\varepsilon^f d\mathrm{Vol}_g.$$

We say f has *finite energy*, denoted by $f \in W^{1,2}(\Omega, \mathcal{N})$, if

$$E^f[\Omega] := \sup_{\varphi \in C_c(\Omega), 0 \le \varphi \le 1} \limsup_{\varepsilon \to 0} E_\varepsilon^f(\varphi) < \infty.$$

In this case, it was proven in [KS93, Theorem 1.10] that there exists an absolutely continuous function $e^f(x)$ with respect to Lebesgue measure, which we call the *energy density*, such that $e_\varepsilon^f(x) d\mathrm{vol}_g$ converges weakly to $e^f(x) d\mathrm{vol}_g$ as ε tends to 0. In analogy to the case of smooth targets, we write $|\nabla f|^2(x)$ in place of $e^f(x)$. Hence $|\nabla f|^2(x) \in L^1_{\mathrm{loc}}(\Omega)$. In particular, the (Korevaar-Schoen) energy of f in Ω is

$$E^f[\Omega] = \int_\Omega |\nabla f|^2 d\mathrm{vol}_g. \tag{4.1}$$

Furthermore, we say $f \in L^2(\Omega, \mathcal{N})$ has *locally finite energy* if for any $x_0 \in \Omega$, there exists a bounded Lipschitz Riemannian subdomain Ω_{x_0} containing x_0 such that $f \in W^{1,2}(\Omega_{x_0}, \mathcal{N})$.

Remark 4.3. If (M, g) is a closed Riemannian manifold and $\varrho : \pi_1(M) \to \mathrm{Isom}(\mathcal{N})$ is a homomorphism, where $\mathrm{Isom}(\mathcal{N})$ is the isometry group of the NPC space $(\mathcal{N}, d_\mathcal{N})$, then for any ϱ-equivariant map $u : \widetilde{M} \to \mathcal{N}$ with locally finite energy, the energy density $|\nabla u|^2$ is a locally L^1-function on $\widetilde{M}$ that is invariant under $\pi_1(M)$-action. It then descends to a function on M, which we abusively denote by $|\nabla u|^2$. Denote

$$^g E^u := \int_M |\nabla u|^2 d\mathrm{Vol}_g$$

if this integral exists (as a finite number). In this case, we say u is a finite energy map and define $^g E^u$ as the energy of u. We write

$$u \in W_\varrho^{1,2}(\widetilde{M}, \mathcal{N}).$$

If the dependence of energy on the metric g is clear from context, we will omit the superscript g and write E^u instead.

Definition 4.4 (Harmonic maps). We say a continuous map $f : \Omega \to \mathcal{N}$ from a Lipschitz Riemannian domain (Ω, g) is harmonic if it is locally energy minimizing; more precisely, at each $x \in \Omega$, there exists $r > 0$ such that $u|_{B_r(x)} \in W^{1,2}(B_r(x), \mathcal{N})$ and $E^{u|_{B_r(x)}} \le E^v$ for any $v \in W^{1,2}(B_r(x), \mathcal{N})$ with $\text{tr}(u|_{B_r(x)}) = \text{tr}(v)$. Here, $\text{tr}(\cdot)$ is the trace of a finite energy map defined in [KS93, §1.12].

For $V \in \Gamma\Omega$ where $\Gamma\Omega$ is the set of Lipschitz vector fields on Ω, in [KS93, §2.3], the *directional energy* $|f_*(V)|^2$ is similarly defined. The real valued L^1_{loc} function $|f_*(V)|^2$ generalizes the norm squared on the directional derivative of f. The generalization of the pull-back metric is the continuous, symmetric, bilinear, non-negative and tensorial operator

$$\pi_f(V, W) = \Gamma\Omega \times \Gamma\Omega \to L^1(\Omega, \mathbb{R})$$

where

$$\pi_f(V, W) = \frac{1}{2}|f_*(V + W)|^2 - \frac{1}{2}|f_*(V - W)|^2.$$

We refer to [KS93, §2.3] for more details.

Let $(x_1, \dots, x_n)$ be local coordinates of (Ω, g), and $g = (g_{ij})$, $g^{-1} = (g^{ij})$ be the local metric expressions. Then energy density function of f can be written (cf. [KS93, (2.3vi)])

$$|\nabla f|^2 = g^{ij}\pi_f\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$$

Next assume (Ω, g) is a Hermitian domain and let $(z_1 = x_1 + ix_2, \dots, z_n = x_{2n-1} + ix_{2n})$ be local complex coordinates. If we extend π_f linearly over $\mathbb{C}$, then we have

$$\frac{1}{4}|\nabla f|^2 = g^{i\bar{j}}\pi_f\left(\frac{\partial f}{\partial z_i}, \frac{\partial f}{\partial \bar{z}_j}\right).$$

Definition 4.5 (Proper action). Let $\Gamma \subset \text{Isom}(\mathcal{N})$ be a finitely generated group with $\gamma_1, \dots, \gamma_p$ being the finite set of generators, where $\mathcal{N}$ be an NPC space. Define

$$\delta : \mathcal{N} \to [0, \infty), \quad \delta(P) = \max d(\gamma_\alpha.P, P), \alpha = 1, \dots, p. \tag{4.2}$$

We will call the action *proper* if the sublevel sets of the function δ are bounded in $\mathcal{N}$; i.e. if $P_0 \in \mathcal{N}$, then for any $L > 0$, there is a number $R > 0$ (depending on L, P_0) so that $\{P \in \mathcal{N} : \delta(P) \le L\} \subset B_R(P_0)$.

The following result is implicitly addressed in [KS97]. For the sake of completeness, we provide a detailed proof here.

Lemma 4.6. *Let Γ and $\mathcal{N}$ be as in Definition 4.5. If Γ does not fix a point at the visual boundary $\partial\mathcal{N}$ of $\mathcal{N}$ (see [BH99, Chapter II.8] for the definition), then the action of Γ on $\mathcal{N}$ is proper.*

Proof. Let $\overline{\mathcal{N}} := \mathcal{N} \cup \partial\mathcal{N}$. We equip $\overline{\mathcal{N}}$ with the *cone topology* (see [BH99, Chapter II, Definition 8.6]) Since $\mathcal{N}$ is locally compact, $\overline{\mathcal{N}}$ is compact.

Assume that Γ is not proper. Then there exists some $L > 0$ such that $C_L := \{P \in \mathcal{N} \mid \delta(P) \le L\}$ is an unbounded subset in $\mathcal{N}$. Then we can choose a sequence of points $\{P_n\}_{n\in\mathbb{N}} \subset \mathcal{N}$ such that $\lim_{n\to\infty} P_n = Q$ for some $Q \in \partial\mathcal{N}$. Note that for any P_n, we have

$$d(\gamma_\alpha.P_n, P_n) \le L,$$

where $\{\gamma_1, \ldots, \gamma_p\}$ is a finite set of generators of Γ. This implies that, for any $\gamma \in \Gamma$, one has

$$d(\gamma.P_n, P_n) \le \ell(\gamma)L,$$

where $\ell(\gamma)$ is the word length of γ with respect to $\{\gamma_1, \ldots, \gamma_p\}$. Thus, we have $\gamma.Q = Q$ for each $\gamma \in \Gamma$. This contradicts with the assumption that Γ does not fix a point at ∂N. The lemma is proved. $\qquad\square$

Definition 4.7. Let M be a closed Riemannian manifold without boundary, N be a locally compact NPC space, and $\varrho : \pi_1(M) \to N$ be a representation. The metric space $(L^2_\varrho(\tilde{M}, N), d_2)$ is defined as follows: Let $L^2_\varrho(\tilde{M}, N)$ to be the space of ϱ-equivariant, locally L^2-maps. The distance function d_2 is given by

$$d_2(u, v) = \left(\int_M d^2(u(x), v(x))\, d\mu \right)^{\frac{1}{2}}.$$

Here, since $x \to d^2(u(x), v(x))$ is $\pi_1(M)$-invariant on $\tilde{M}$, it descends to a function on M. By [KS97, Lemma 2.1.2], $(L^2_\varrho(\tilde{M}, N), d_2)$ is an NPC space. Note that $W^{1,2}_\varrho(\tilde{M}, N)$ is a subspace of $L^2_\varrho(\tilde{M}, N)$ (cf. Remark 4.3).

Definition 4.8. Let M, N, and ϱ be as in Definition 4.7. We say $\{u_k\}$ is a energy minimizing sequence if $u_k \in W^{1,2}_\varrho(\tilde{M}, N)$ and

$$\lim_{k \to \infty} E^{u_k} = \inf_{v \in W^{1,2}_\varrho(\tilde{M}, N)} E^v.$$

Proposition 4.9. *Assume M is compact, N is locally compact and ϱ is proper. If $\{u_k\}$ is a energy minimizing sequence that are uniformly locally Lipschitz continuous, then a subsequence of $\{u_k\}$ converges uniformly to a harmonic map v. Moreover, the energy density measures and the directional energy density measures of the u_k converge weakly to those of v.*

Proof. Let $\mathcal{F} \subset \tilde{M}$ be a fundamental domain for the action of $\pi_1(M)$ on $\tilde{M}$. We fix a point $P_0 \in N$. The compactness of $\overline{\mathcal{F}}$ and the uniform local Lipschitz continuity of $\{u_k\}$ implies that there exists $c > 0$ such that

$$d(u_k(\gamma_\alpha x), u_k(x)) \le c d_{\tilde{M}}(\gamma_\alpha x, x), \quad \forall x \in \mathcal{F}, \quad \alpha = 1, \ldots, p$$

where $\gamma_1, \ldots, \gamma_p$ are the generators of $\pi_1(M)$. The equivariance of u_k implies that for any $x \in \mathcal{F}$ and any $\alpha \in \{1, \ldots, p\}$, we have

$$d(\varrho(\gamma_\alpha)(u_k(x)), u_k(x)) = d(u_k(\gamma_\alpha.x), u_k(x)) \le c_1 := c \sup_{\beta=1,\ldots,p; x \in \mathcal{F}} d_{\tilde{M}}(\gamma_\beta(x), x).$$

Since ϱ is proper, it follows that $\delta(u_k(x)) \le c_1$ for $x \in \mathcal{F}$ and for all k, where δ is the function defined in (4.2).

Claim 4.10. *There exists some constant $R > 0$ such that $d(u_k(\gamma.\overline{\mathcal{F}}), \varrho(\gamma).P_0) \le R$ for any $k \in \mathbb{N}$ and any $x \in \overline{\mathcal{F}}$.*

Proof. By Definition 4.5, there exists some $R > 0$ such that

$$\{P \in N \mid \delta(P) \le c_1\} \subset B_R(P_0).$$

It follows that $u_k(\mathcal{F}) \subset B_R(P_0)$ for any $k \in \mathbb{N}$. Since u_k is ϱ-equivariant and continuous, it follows that for any $x \in \overline{\mathcal{F}}$, and any $k \in \mathbb{N}$, we have

$$d(u_k(\gamma.x), \varrho(\gamma).P_0) = d(\varrho(\gamma).u_k(x), \varrho(\gamma).P_0) = d(u_k(x), P_0) \leq R.$$

□

Since $\mathcal{N}$ is locally compact, u_k is uniformly locally Lipschitz continuous, and $\widetilde{M} = \cup_{\gamma \in \pi_1(M)} \overline{\mathcal{F}}$, by the Arzela-Ascoli theorem, the above claim yields that there exists a subsequence of $\{u_k\}$ that converges to a map $v \in W_\varrho^{1,2}(\tilde{M}, \mathcal{N})$. The lower semicontinuity of energy implies that v is energy minimizing, and thus v is harmonic. Last statement follows from [KS97, Theorem 3.9]. □

4.3. Regular and singular points of harmonic maps.
In [GS92], Gromov and Schoen investigated the regularity properties of harmonic maps into a class of NPC spaces they defined as F-connected. It is straightforward to verify that Bruhat-Tits buildings satisfy the conditions for being F-connected.

Definition 4.11 (F-connected). An N-dimensional complex $\mathfrak{F}$ is called *F-connected* if it is an NPC space, each of its simplices is isometric to a linear image of the standard simplex and any two adjacent simplices are contained in an apartment A, an isometric and totally geodesic subcomplex isometric to $\mathbb{R}^N$.

Definition 4.12 (Regular points and singular points). Let $u : \Omega \to \mathfrak{F}$ be a harmonic map from a Riemannian domain into an F-connected complex. A point $x \in \Omega$ is said to be a *regular point* of u if there exists a neighborhood $\mathcal{U}$ of x in X and an apartment $A \subset \mathfrak{F}$ such that $u(\mathcal{U}) \subset A$. The set of regular points of u is denoted by $\mathcal{R}(u)$.

Theorem 4.13 ([GS92]). *If $u : \Omega \to \mathfrak{F}$ is a harmonic map from an n-dimensional Riemannian domain Ω into an F-connected complex, then the Hausdorff dimension of $\mathcal{S}(u)$ is at most $n-2$.*

Definition 4.14. Let u be as in Theorem 4.13. For $x \in \Omega$, set

$$\text{Ord}^u(x) = \lim_{\sigma \to 0} \frac{e^{c\sigma^2}\sigma \displaystyle\int_{B_\sigma(x)} |\nabla u|^2 d\mu}{\min_{Q \in \Delta(G)} \displaystyle\int_{\partial B_\sigma(x)} d^2(u, Q) d\Sigma}.$$

As a limit of non-decreasing sequence of functions, $x \mapsto \text{Ord}^u(x)$ is a upper semicontinuous function. Thus, we have the following:

(a)   By [GS92, Lemma 1.3], $\text{Ord}^u(x) \geq 1$ for all $x \in \Omega$.
(b)   By [GS92, Theorem 6.3.(i)], if $x_i \to x$ and $\text{Ord}^u(x_i) > 1$, then $\text{Ord}^u(x) > 1$.

Lemma 4.15 ([GS92], proof of Theorem 6.4). *Let u be as in Theorem 4.13 and $\tilde{S}_0(u)$ to be the set of points $x \in \Omega$ such that $\text{Ord}^u(x) > 1$. Then $\tilde{S}_0(u)$ is a closed set such that $\dim_\mathcal{H}(\tilde{S}_0(u)) \leq n-2$.* □

Lemma 4.16 ([GS92], proof of Proposition 2.2, Theorem 2.3). *Let u be as in Theorem 4.13. For $x \in \Omega$, let $\alpha := \text{Ord}^u(x)$. There exists a constant $c > 0$ and $\sigma_0 > 0$ such that*

$$\sigma \mapsto \frac{e^{c\sigma^2}}{\sigma^{n-1+2\alpha}} \int_{\partial B_\sigma(x)} d^2(u, u(x)) d\Sigma$$

and

$$\sigma \mapsto \frac{e^{c\sigma^2}}{\sigma^{n-2+2\alpha}} \int_{B_\sigma(x)} |\nabla u|^2 d\mu$$

are non-decreasing functions in the interval $(0, \sigma_0)$. □

Remark 4.17. For a finite energy map $u : \Omega \to \mathfrak{F}$ into an F-connected complex, $|\nabla u|^2 \in L^1_{\text{loc}}$ is not necessarily defined at all points of Ω. On the other hand, it follows from Lemma 4.16 that for a harmonic map u, we can define $|\nabla u|^2$ at every point of $x \in \Omega$ by setting

$$|\nabla u|^2(x) = \lim_{\sigma \to 0} \frac{1}{c_n \sigma^n} \int_{B_\sigma(x)} |\nabla u|^2 \, d\mu$$

where $c_n \sigma^n$ is the volume of a ball or radius σ in Euclidean space.

To further study the local behavior of harmonic maps, we consider the following notion (cf. Remark 4.19 below).

Definition 4.18 (F-connected cone). An F-connected complex T is called an F-*connected cone* if there exists an isometric embedding of T in Euclidean space as a geometric cone; that is, after identifying T as a subset of Euclidean space via the isometric embedding, we have

$$Q \in T \Rightarrow \lambda Q \in T, \forall \lambda \in [0, \infty). \tag{4.3}$$

Henceforth, we will always assume that T is a subset of Euclidean space. Note that the origin $\vec{0}$ of the Euclidean space is the cone point, i.e. the vertex of the cone.

Remark 4.19. A neighborhood of a point $P_0 \in \Delta(G)_K$ is isometric to a neighborhood of the origin in the tangent cone $T_{P_0}\Delta(G)$ (cf. [BH99, p. 190] for the definition). Two simplices (which are actually simplicial cones) in $T_{P_0}\Delta(G)$ are contained in a totally geodesic subcomplex $T_{P_0}A$ where A is an apartment of $\Delta(G)_K$. In other words, $T_{P_0}\Delta(G)_K$ is an N-dimensional F-connected complex. Thus, when we study the local behavior of harmonic maps $u : \Omega \to \Delta(G)_K$ at a point $x_0 \in \Omega$, we can assume that u maps into the N-dimensional, F-connected cone $T := T_{P_0}\Delta(G)_K$ where $P_0 = u(x_0)$.

The following theorem follows from [GS92, Sections 5 and 6]

Theorem 4.20 ([GS92]). *Let T be an F-connected cone and $l : \mathbb{R}^n \to T$ be a homogeneous degree 1 harmonic map; i.e. $l(\lambda x) = \lambda l(x)$.*

(i)   *The map l can be decomposed as $l = J \circ v$ where $J : \mathbb{R}^m \to T$ is an isometric and totally geodesic map and $v : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map of rank m.*

(ii) *Let $S \subset T$ be the union of all apartments containing the m-flat $\mathbb{F} = l(\mathbb{R}^n)$. Then S is an F-connected cone isometric to $\mathbb{R}^m \times T_2$ where T_2 is a lower dimensional F-connected cone. Furthermore, S is essentially regular and l is effectively contained in S in the sense of* [GS92, Section 5].

(iii) *Let $\mathcal{M}$ be a family of Riemannian metrics defined on a Euclidean unit ball $\mathbb{B}_1(0) \subset \mathbb{R}^n$ and, for $g \in \mathcal{M}$, define*

$$||g|| = \sum_{i,j=1}^n \sup_{\mathbb{B}_1(0)} |g_{ij} - \delta_{ij}| + \sum_{i,j,k=1}^n \sup_{\mathbb{B}_1(0)} \left|\frac{\partial g_{ij}}{\partial x_k}\right| + \sum_{i,j,k,l=1}^n \sup_{\mathbb{B}_1(0)} \left|\frac{\partial^2 g_{ij}}{\partial x_k \partial x_l}\right| \tag{4.4}$$

where $g = (g_{ij})$ is the matrix expression of g with respect to the Euclidean coordinates $(x_1, \ldots, x_n)$. Then there exists $\kappa_0 > 0$, $\delta_0 > 0$ and $\sigma_0 \in (0, 1)$ such that if $g \in \mathcal{M}$ with $||g|| < \kappa_0$ and $u : (\mathbb{B}_1(0), g) \to T$ is a harmonic map with

$$\sup_{\mathbb{B}_1(0)} d(u(x), l(x)) \le \delta_0,$$

then $u(\mathbb{B}_{\sigma_0}(0)) \subset S$. □

Remark 4.21. Theorem 4.20.(iii) explicitly addresses the dependence on the domain metric g in [GS92, Theorem 5.1]. This dependence can be deduced by following the proof in either [GS92] or [DM21, Lemma 5.5]. Specifically, κ_0 must be chosen sufficiently small such that, if $||g|| \le \kappa_0$, then g meets the criteria stated prior to [DM21, Lemma 5.5]; namely:

- For any smooth submanifold S of $\mathbb{B}_1(0)$, $\frac{15}{16}\text{Vol}(S) \le \text{Vol}_g(S) \le \frac{17}{16}\text{Vol}(S)$ where $\text{Vol}(S)$ is the Euclidean volume.
- The error term $e^{c\sigma^2}$ that appears in the monotonicity formulae of Lemma 4.16 is ≤ 2 for all $\sigma \in (0, 1]$.
- There exists a constant c_0 such that for any subharmonic function $f : \mathbb{B}_1(0) \to \mathbb{R}$ with respect to the metric g, we have

$$\sup_{\mathbb{B}_{\frac{15r}{16}}(0)} f \le \frac{c_0}{r^{n-1}} \int_{\partial \mathbb{B}_r(0)} f \, d\Sigma.$$

Let $u : \Omega \to T$ be a harmonic map into an F-connected cone with cone point $u(x_0) = 0$. Use normal coordinates centered at x_0 to identify $x_0 = 0$ and let $\mathbb{B}_r(0) = \{x = (x^1, \ldots, x^n) \in \mathbb{R}^n : |x| < r\}$. Let

$$\mu(\sigma) = \left(\sigma^{1-n} \int_{\partial \mathbb{B}_\sigma(0)} d^2(u, u(0)) d\Sigma \right)^{-\frac{1}{2}}.$$

Definition 4.22. The *blow up map* is defined by

$$u_\sigma : \mathbb{B}_1(0) \to T, \quad u_\sigma(x) = \mu(\sigma) u(\sigma x).$$

Here, we are using the property eq. (4.3) of the geometric cone T in Euclidean space.

By [GS92, Proposition 3.3] and the paragraph proceeding it, there exists a sequence $\sigma_i \to 0$ such that u_{σ_i} converges locally uniformly to a non-constant homogeneous harmonic map u_* of degree $\alpha := \text{Ord}^u(x_0)$. Consequently, Theorem 4.20 (see also [GS92, Theorem 6.3]) implies the following.

Lemma 4.23 ([GS92]). *If $\text{Ord}^u(x_0) = 1$, then there exists $\sigma_0 > 0$ such that $u(\mathbb{B}_{\sigma_0}(x_0)) \subset \mathbb{R}^m \times T_2$ where T_2 is a lower dimensional F-connected cone. If we write*

$$u = (u^1, u^2) : \mathbb{B}_{\sigma_0}(x_0) \to \mathbb{R}^m \times T_2, \tag{4.5}$$

then $u^1 : \mathbb{B}_{\sigma_0}(x_0) \to \mathbb{R}^m$ is a smooth harmonic map of rank m and $u^2 : \mathbb{B}_{\sigma_0}(x_0) \to T_2$ is a harmonic map with $\text{Ord}^{u^2}(x_0) > 1$.

Remark 4.24. Let X be a smooth projective variety, $\widetilde{X}$ its universal cover, $\pi_X : \widetilde{X} \to X$ the universal covering map, $\varrho : \pi_1(X) \to G(K)$ a representation, and $u : \widetilde{X} \to \Delta(G)_K$ a ϱ-equivariant harmonic map.

Since $G(K)$ acts transitively on the apartments of $\Delta(G)_K$, it follows that if $x \in \widetilde{X}$ is a regular point of u, then every point of $\pi_X^{-1}(\pi_X(x))$ is also a regular point of u. We abusively denote by $\mathcal{R}(u) \subset X$ the image $\pi_X(\mathcal{R}(u))$.

Definition 4.25. Let $u : \Omega \to \Delta(G)_K$ be as in Remark 4.24. For $x_0 \in \tilde{\mathcal{S}}_0(u)$ (cf. Lemma 4.15), define $m_{x_0} = 0$. For $x_0 \in \Omega \backslash \tilde{\mathcal{S}}_0(u)$, let m_{x_0} be the integer m in eq. (4.5). Let $\mathrm{rank}(u) :=$ $\sup_{x_0 \in \Omega} m_{x_0}$. We say the point $x_0 \in \Omega$ is a *critical point* if $m_{x_0} < \mathrm{rank}(u)$. We denote the set of critical points by $\tilde{\mathcal{S}}(u)$. Define $\tilde{\mathcal{R}}(u) = \Omega \backslash \tilde{\mathcal{S}}(u)$.

4.4. **Equivariant harmonic maps to Bruhat-Tits buildings in families.** Throughout this subsection, let $f : \mathcal{X} \to \mathbb{D}$ be a smooth projective family. Let $\pi_{\mathcal{X}} : \widetilde{\mathcal{X}} \to \mathcal{X}$ be the universal covering map. Then the fiber $\widetilde{X}_t$ of $\widetilde{\mathcal{X}} \to \mathbb{D}$ at $t \in \mathbb{D}$ is the universal cover of $X_t := f^{-1}(t)$.

Definition 4.26. For $x_0 \in \widetilde{X}_0$ (resp. $x_0 \in X_0$), we may take an open neighborhood Ω of x_0 in $\widetilde{\mathcal{X}}$ (resp. $\mathcal{X}$) together with a biholomorphism $\varphi : \mathbb{D}^n \times \mathbb{D}_\varepsilon \to \Omega$ such that x_0 is identified with $(0, \ldots, 0, 0)$ via φ and $f \circ \pi_{\mathcal{X}} \circ \varphi(z_1, \ldots, z_n, t) = t$ (resp. $f \circ \varphi(z_1, \ldots, z_n, t) = t$). For simplicity of notation, we write $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{D}^n$. We call such coordinate system an *admissible coordinate system* centered at x_0.

We first prove the following regularity result in the absolute case.

Lemma 4.27. *If $u : \Omega \to \Delta(G)$ is a harmonic map from a Lipschitz Riemannian domain, then the energy density $|\nabla u|^2$ is a continuous function.*

Proof. By the absolute continuity of Lebesgue integrals,

$$e_\sigma : \Omega \to \mathbb{R}$$

$$x \mapsto \frac{e^{c\sigma^2}}{\sigma^{n-2+2\alpha}} \int_{\mathbb{B}_\sigma(x)} |\nabla u|^2 d\mu$$

is continuous. By Lemma 4.16, e_σ are non-decreasing functions in the interval $(0, \sigma_0)$ for some $\sigma_0 > 0$, whose limit is $|\nabla u|^2(x)$. As a decreasing limit of continuous functions, the energy density function $|\nabla u|^2(x)$ is upper semicontinuous.

Pick any $x_0 \in \Omega$. If $\mathrm{Ord}^u(x_0) > 1$, then by [DM24, Lemma 4.13], $|\nabla u|^2(x_0) = 0$. The upper semi-continuity of $|\nabla u|^2(x)$ implies that $|\nabla u|^2(x)$ is continuous at x_0.

If $\mathrm{Ord}^u(x_0) = 1$, by Lemma 4.23, there exists $\sigma_0 > 0$ such that $u(\mathbb{B}_{\sigma_0}(x_0)) \subset \mathbb{R}^m \times \mathcal{F}_2$, where $\mathcal{F}_2$ is another F-complex. By (4.5), we write

$$u = (u^1, u^2) : \mathbb{B}_{\sigma_0}(x_0) \to \mathbb{R}^m \times \mathcal{F}_2,$$

and $u^1 : \mathbb{B}_{\sigma_0}(x_0) \to \mathbb{R}^m$ is a smooth harmonic map of rank m, and $u^2 : \mathbb{B}_{\sigma_0}(x_0) \to \mathcal{F}_2$ is a harmonic map with $\mathrm{Ord}^{u^2}(x_0) > 1$. Hence $|\nabla u^2|(x)$ is continuous at x_0 by the above arguments. Since $|\nabla u|^2(x) = |\nabla u^1|^2(x) + |\nabla u^2|(x)$, and u^1 is smooth, it follows that $|\nabla u|(x)$ is continuous at x_0. The lemma is proved. $\square$

In the relative case, we have the following result.

Lemma 4.28. *Let T be an F-connected cone, and let g_t be a family of Riemannian metrics on $\mathbb{D}^n$. Suppose $u_t : \mathbb{D}^n \to T$, $t \in \mathbb{D}_\varepsilon$, is a family of uniformly Lipschitz harmonic maps with respect to g_t, such that*

- $u_t(\mathbf{z})$ *varies uniformly in* t *in the following sense:* $\forall \varepsilon_0 > 0, \exists \delta_0 > 0$ *such that* $\mathbf{z} \in \mathbb{D}^n, t_1, t_2 \in \mathbb{D}_\varepsilon$ *with* $|t_1 - t_2| < \delta_0 \Rightarrow |u_{t_1}(\mathbf{z}) - u_{t_2}(\mathbf{z})| < \varepsilon_0$.
- *The component functions of* $g_t(\mathbf{z})$ *with respect to the coordinates* $(z_1, \ldots, z_n)$ *of* $\mathbb{D}^n$ *varies smoothly in* t.
- *For any* $\mathbb{B}_r(\mathbf{z}) \subset \mathbb{D}^n$, *and any* $t_0 \in \mathbb{D}_\varepsilon$

$$\lim_{t \to t_0} \int_{\mathbb{B}_r(\mathbf{z})} |\nabla u_t|^2 dVol_{g_t} = \int_{\mathbb{B}_r(\mathbf{z})} |\nabla u_{t_0}|^2 dVol_{g_{t_0}}. \tag{4.6}$$

Then the map $(\mathbf{z}, t) = (z_1, \ldots, z_n, t) \mapsto |\nabla u_t|^2(\mathbf{z})$ *is a continuous function in* $\mathbb{D}^n \times \mathbb{D}_\varepsilon$.

Proof. We will show that $(\mathbf{z}, t) = (z_1, \ldots, z_n, t) \mapsto |\nabla u_t|^2(z_1, \ldots, z_n)$ is continuous at $(\mathbf{0}, 0) \in \mathbb{D} \times \mathbb{D}_\varepsilon$ where $\mathbf{0} = (0, \ldots, 0) \in \mathbb{D}^n$. The same argument will show that this function is continuous at any other point of $\mathbb{D}^n \times \mathbb{D}_\varepsilon$. We consider the following two cases:

CASE 1: $\text{Ord}^{u_0}(\mathbf{0}) > 1$. Thus, $|\nabla u_0|^2(\mathbf{0}) = 0$. Assume on the contrary that $(\mathbf{z}, t) \mapsto |\nabla u_t|^2(\mathbf{z})$ is not continuous at $(\mathbf{0}, 0)$; i.e. there exists $\varepsilon > 0$ and $\mathbf{z}_i = (z_{i1}, \ldots, z_{in}) \to \mathbf{0}$ such that $\varepsilon \leq |\nabla u_{t_i}|^2(\mathbf{z}_i)$.

To derive a contradiction, we first apply the monotonicity property of energy in Lemma 4.16 and note that the order is ≥ 1, to obtain

$$\varepsilon \leq \frac{e^{cr}}{\omega_{2n} r^{2n}} \int_{\mathbb{B}_r(\mathbf{z}_i)} |\nabla u_{t_i}|^2 d\text{Vol}_{g_{t_i}} \tag{4.7}$$

where ω_{2n} is the volume of the unit ball in $\mathbb{R}^{2n}$. Second,

$$\left| \int_{\mathbb{B}_r(\mathbf{z}_i)} |\nabla u_{t_i}|^2 d\text{Vol}_{g_{t_i}} - \int_{\mathbb{B}_r(\mathbf{0})} |\nabla u_{t_i}|^2 d\text{Vol}_{g_{t_i}} \right| \leq \int_{\mathbb{B}_r(\mathbf{z}_i) \triangle \mathbb{B}_r(\mathbf{0})} |\nabla u_{t_i}|^2 d\text{Vol}_{g_{t_i}}$$
$$\leq C\text{Vol}_{g_{t_i}}(\mathbb{B}_r(\mathbf{z}_i) \triangle \mathbb{B}_r(\mathbf{0}))$$

where C is the Lipschitz bound. Combining the above two inequalities with the fact that $\text{Vol}(\mathbb{B}_r(\mathbf{z}_i) \triangle \mathbb{B}_r(\mathbf{0})) \to 0$ as $i \to \infty$, we obtain

$$\varepsilon \leq \lim_{i \to \infty} \frac{1}{\omega_{2n} r^{2n}} \int_{\mathbb{B}_r(\mathbf{0})} |\nabla u_{t_i}|^2 d\text{Vol}_{g_{t_i}}.$$

Combining this inequality with assumption eq. (4.6), we get

$$\varepsilon \leq \frac{1}{\omega_{2n} r^{2n}} \int_{\mathbb{B}_r(\mathbf{0})} |\nabla u_0|^2 d\text{Vol}_{g_0}.$$

By Remark 4.17, we have

$$\lim_{r \to 0} \frac{1}{\omega_{2n} r^{2n}} \int_{\mathbb{B}_r(\mathbf{0})} |\nabla u_0|^2 d\text{Vol}_{g_0} = |\nabla u_0|^2(\mathbf{0}),$$

we conclude $\varepsilon \leq |\nabla u_0|^2(\mathbf{0})$, a contradiction.

CASE 2: $\text{Ord}^{u_0}(\mathbf{0}) = 1$. Choose normal coordinates centered at $\mathbf{0}$. Let $(g_{t\sigma, ij})$ be the matrix expression of the metric $g_{t\sigma}(x) := g_t(\sigma x)$ in $\mathbb{B}_1(0)$. We will let $\mathcal{M}$ be the collection of metrics $g_{t\sigma}$. Note that $||g_{t\sigma}|| \to 0$ (cf. Theorem 4.20) as $|t| \to 0$ and $\sigma \to 0$.

Denote the distance function on T by d_T. Define a scaling factor

$$\mu(\sigma) = \left(\sigma^{1-n} \int_{\partial \mathbb{B}_\sigma(0)} d^2(u_0, u_0(0)) d\Sigma \right)^{-\frac{1}{2}}.$$

and a new distance function on T by setting $d_\sigma := \mu(\sigma) d_T$. The blow up maps of u_0 at x_0 is defined by

$$u_{0\sigma} : \mathbb{B}_1(0) \to (T, d_\sigma), \quad u_{0\sigma}(x) = u_0(\sigma x).$$

Choose a subsequence $\sigma_i \to 0$ such that $u_{0\sigma_i}$ converges to a tangent map u_{0*}. Also define

$$u_{t\sigma} : \mathbb{B}_1(0) \to (T, d_\sigma), \quad u_{t\sigma}(x) = u_t(\sigma x).$$

Here, we note that $u_{t\sigma}$ is not the usual blow up map of u_t since the choice of the scaling $\mu(\sigma)$ to define the target distance function is not consistent with the scaling used for blow up maps of u_t. For the blow up maps of u_t, the correct scaling is $\left(\sigma^{1-n} \int_{\partial \mathbb{B}_\sigma(0)} d^2(u_t, u_t(0)) d\Sigma \right)^{-\frac{1}{2}}$.

As in Remark 4.19, we assume u_0 maps into the tangent cone $T := T_{P_0}(\Delta(G))$. Since $\mathrm{Ord}^{u_0}(\mathbf{0}) = 1$, $u_{0*} : \mathbb{B}_1(0) \to T$ is a homogeneous degree 1 harmonic map. By Theorem 4.20, u_{*0} is effectively contained in a essentially regular subcomplex $\mathbb{R}^m \times T_2$ where T_2 is a lower dimensional F-connected cone.

Let $\kappa_0 > 0$, $\delta_0 > 0$ and $\sigma_0 > 0$ be from Theorem 4.20. Fix $\tau_1 > 0$ and i sufficiently large such that $\|g_{t\sigma_i}\| < \kappa_0$ for $|t| < \tau_1$ and $\sup_{\mathbb{B}_1(0)} d(u_{0\sigma_i}, u_{0*}) < \delta_0$. Since $u_{t\sigma_i}$ converges uniformly to $u_{0\sigma_i}$, there exists $\tau_0 \in (0, \tau_1)$ such that $|t| < \tau_0$ implies $\sup_{\mathbb{B}_1(0)} d(u_{t\sigma_i}, u_{0*}) < \delta_0$. Hence, Theorem 4.20 implies that there exists $\sigma_0 > 0$ such that $u_{t\sigma_i}(\mathbb{B}_{\sigma_0}(0)) \subset \mathbb{R}^m \times B$, or equivalently, $u_t(\mathbb{B}_{\sigma_0 \sigma_i}(0)) \subset \mathbb{R}^m \times B$. We write

$$u_t = (u_t^1, u_t^2), \quad u_t^1 : \mathbb{B}_{\sigma_0 \sigma_i}(0) \to \mathbb{R}^m, \ u_t^2 : \mathbb{B}_{\sigma_0 \sigma_i}(0) \to T_2 \tag{4.8}$$

in terms of the product structure. Since $u_t^1 : \mathbb{B}_{\sigma_0 \sigma_i}(0) \to \mathbb{R}^m$ is a smooth harmonic map for each t, the uniform convergence of u_t^1 to u_0^1 implies the local C^∞-convergence of u_t^1 to u_0^1 by elliptic regularity. Thus,

$$(\mathbf{z}, t) \mapsto \frac{\partial u_i^1}{\partial z_j}(z_1, \ldots, z_n, t) \text{ is continuous in } \mathbb{B}_{\sigma_0 \sigma_i}(0), \ \forall i = 1, \ldots, N \text{ and } j = 1, \ldots, m.$$

Since $\mathrm{Ord}^{u_0^2}(\mathbf{0}) > 1$, CASE 1 implies that $(\mathbf{z}, t) \to |\nabla u_t^2|^2(\mathbf{z})$ is continuous at $(\mathbf{0}, 0)$. Combining the continuity of $|\nabla u_t^1|^2$ and $|\nabla u_t^2|^2$ at $(\mathbf{0}, 0)$, we conclude that $(\mathbf{z}, t) \to |\nabla u_t|^2(\mathbf{z})$ is continuous at $(\mathbf{0}, 0)$. The lemma is proved. □

Let us now state and prove the main result of this section.

Theorem 4.29. *Let $f : \mathcal{X} \to \mathbb{D}$ be the smooth projective family. Let $\varrho : \pi_1(\mathcal{X}) \to G(K)$ be a Zariski dense representation, where G is a semisimple algebraic group over a non-archimedean local field K. Then there is a ϱ-equivariant map $u : \widetilde{\mathcal{X}} \to \Delta(G)_K$ such that*

(1) For each $t \in \mathbb{D}$, $u_t := u|_{\widetilde{X}_t} : \widetilde{X}_t \to \Delta(G)_K$ is ϱ_t-equivariant and pluriharmonic.

(2) Let $m = \mathrm{rank}(u_0)$ and let $x_0 \in \tilde{\mathcal{R}}(u_0)$; i.e. x_0 is a non-critical point of u_0 defined in Definition 4.25. Assume that u_0 maps some neighborhood of x_0 into the tangent cone $T := T_{u(x_0)}(\Delta(G)_K)$ (cf. Remark 4.19). Then there exists an admissible coordinate system

$\Omega \simeq \mathbb{D}^n \times \mathbb{D}_\varepsilon$ centered at x_0 (cf. Definition 4.26), an apartment A of T, and a flat $\mathbb{F} \subset A$ satisfying the following:

(i) $u_0(\Omega \cap \widetilde{X}_0) \subset A$. (cf. Definition 4.26).
(ii) The union of all flats parallel to $\mathbb{F}$ is a subbuilding isometric to $\mathbb{R}^m \times B$ where B itself is another Euclidean building. Then $u(\Omega) \subset \mathbb{R}^m \times B$.
(iii) Write $u_t = u|_{\mathbb{D}^n \times \{t\}} = (u_t^1, u_t^2)$ where

$$u_t^1 : \mathbb{D}^n \to \mathbb{R}^m \ \ \text{and} \ \ u_t^2 : \mathbb{D}^n \to B.$$

Here we adopt the notation that if $m = \dim(\Delta(G)_K)$, then $u_t = u_t^1$. Define

$$u^1(z_1, \ldots, z_n, t) := u_t^1(z_1, \ldots, z_n)$$

and let $u^1 = (u_1^1, \ldots, u_m^1)$ be its expression in terms of coordinate functions. Then we have the following:

(a) u_0^2 is identically constant.
(b) The partial derivatives of the coordinate functions of u^1 with respect to z_j are continuous in Ω, i.e. for each $i = 1, \ldots, m$ and $j = 1, \ldots, n$,

$$(z_1, \ldots, z_n, t) \mapsto \frac{\partial u_i^1}{\partial z_j}(z_1, \ldots, z_n, t) \in C^0(\mathbb{D}^n \times \mathbb{D}_\varepsilon).$$

(c) $(z_1, \ldots, z_n, t) \mapsto |\nabla u_t^2|^2(z_1, \ldots, z_n)$ is a continuous function on $\Omega = \mathbb{D}^n \times \mathbb{D}_\varepsilon$ and $|\nabla u_t^2|^2 \to 0$ as $t \to 0$.

Proof of Theorem 4.29. Let C be a $\varrho(\pi_1(X))$-invariant minimal closed convex subset of $\Delta(G)_K$ introduced in [BDDM22, Lemma 2.2]. The ϱ-equivariant map $u : \widetilde{\mathcal{X}} \to \Delta(G)_K$ is defined by setting $u|_{\widetilde{X}_t}$ to be the ϱ-equivariant pluriharmonic map into C, whose existence is established in [GS92], and uniqueness is shown in the proof of [BDDM22, Theorem 2.1]. This proves the existence assertion in Item (1).

Before we give a proof of Item (2), we need a preliminary result (cf. Claim 4.30 below). Let $F_t : \widetilde{X}_0 \to \widetilde{X}_t$ be a lift of a smooth family of maps $X_0 \to X_t$ depending smoothly on $t \in \mathbb{D}$ such that F_0 is the identity map. Let g_t be the pullback via F_t of the restriction to $\widetilde{X}_t$ of the Riemannian metric on $\widetilde{\mathcal{X}}$. For any ϱ-equivariant map $v : X_0 \to \Delta(G)_K$, let ${}^{g_t}E^v$ be the energy of v with respect to the metric g_t on X_0. Note that, since u_0 and $u_t \circ F_t$ are both ϱ-equivariant, the function $d^2(u_0, u_t \circ F_t)$ defined on $\widetilde{X}_0$ is $\pi_1(\mathcal{X})$-invariant, and hence descends to X_0. Same is true for the energy density functions and the directional energy density functions of u_0 and $u_t \circ F_t$.

Claim 4.30. *The function $u_t \circ F_t$ converges uniformly to u_0 on $\widetilde{X}_0$.*

Proof. Since the sequence of metrics g_t converges uniformly in C^∞ to g_0, there exists a constant $C > 0$ such that

$$(1 - Ct)^{g_0}E^{u_0} \le (1 - Ct)^{g_0}E^{u_t \circ F_t} \le {}^{g_t}E^{u_t \circ F_t} \le {}^{g_t}E^{u_0} \le (1 + Ct)^{g_0}E^{u_0}. \tag{4.9}$$

Here, the first inequality is due to u_0 being energy minimizing with respect to the metric g_0. The second inequality is due to the fact that $g_t \to g_0$. In particular, if $(x_1, \ldots, x_{2n})$ is the real coordinates on X_0 and g_t^{ij} is the expression of the co-metric of g_t with respect to these

coordinates, then $g_t^{ij} = (1 + O(t))g_0^{ij}$. Thus, the energy density function of u_t with respect to these coordinates is

$$|\nabla u_t|_{g_t}^2 = g_t^{ij} \frac{\partial u_t}{\partial x_i} \cdot \frac{\partial u_t}{\partial x_j} = (1 + O(t))g_0^{ij} \frac{\partial u_t}{\partial x_i} \cdot \frac{\partial u_t}{\partial x_j} = (1 + O(t))|\nabla u_t|_{g_0}^2$$

where $|\nabla u_t|_{g_t}^2$ (resp. $|\nabla u_t|_{g_0}^2$) is the energy density function of u_t with respect to the domain metric g_t (resp. g_0) and we use the notation $\frac{\partial u}{\partial x_i} \cdot \frac{\partial u}{\partial x_j}$ to denote the pullback inner product $\pi(\partial_{x_i}, \partial_{x_j})$ defined in [KS93, Section 2.3] with the coordinate vector fields $(\partial_{x_1}, \ldots, \partial_{x_{2n}})$. The third (fourth) inequality follows from a reason analogous to the first (second) inequality.

In particular, [GS92, Theorem 2.4] implies that there exists a constant C_0 such that, for all t is sufficiently close to 0, $u_t \circ F_t$ is Lipschitz continuous with Lipschitz constant bounded by C_0 (with respect to the distance function on $\widetilde{X}_t$ induced by the metric g_t, and hence also with respect to the metric g_0 since these metrics are uniformly equivalent).

Suppose that $u_t \circ F_t$ does not converge uniformly to u_0 as $t \to 0$. Then there exists $\varepsilon > 0$, a sequence $t_i \to 0$, $x_i \in \widetilde{X}_0$ such that $d(u_{t_i} \circ F_{t_i}(x_i), u_0(x_i)) \geq \varepsilon$. Since ϱ is a Zariski dense representation, it does not fix a point at infinity. By [KS97, Theorem 2.2.1], the action of ϱ is proper (cf. Definition 4.5). Furthermore, $\lim_{i \to \infty} {}^{g_0}E^{u_{t_i}} \circ F_{t_i} - {}^{g_0}E^{u_0} = 0$ by (4.9), which shows that $\{u_{t_i} \circ F_{t_i}\}$ is a minimizing sequence. Taking a subsequence if necessary, Lemma 4.9 implies that $\{u_{t_i} \circ F_{t_i}\}$ converges uniformly to a ϱ-equivariant energy minimizing map v. By the uniqueness of the equivariant harmonic map into C (cf. [BDDM22, Section 2.7]), we conclude $u_0 = v$. Thus, $\lim_{i \to \infty} d(u_{t_i} \circ F_{t_i}(x_i), u_0(x_i)) = 0$. This contradiction proves $u_t \circ F_t$ converges uniformly to u_0. $\qquad\square$

We now give a proof of Item (2). Since $x_0 \in \tilde{\mathcal{R}}(u_0) \subset \mathcal{R}(u_0)$, there exists a admissible coordinate system $\Omega \simeq \mathbb{D}^n \times \mathbb{D}_\varepsilon$ centered at x_0 in Definition 4.26, and an apartment A such that $u_0(\mathbb{D}^n) \subset A$. This proves Item (i). Moreover, we have $\mathrm{Ord}^{u_0}(x_0) = 1$ and the image of u_{0*} is a m-dimensional flat $\mathbb{F}$ since $x_0 \in \tilde{\mathcal{R}}(u_0)$ and $\mathrm{rank}(u_0) = m$. Here u_{0*} is defined just before Lemma 4.23. Adopting the notation for CASE 2 in the proof of Lemma 4.28, let $\mathbb{R}^m \times B \subset T$ be the subbuilding defined as the union of all apartments containing $\mathbb{F}$.

By Claim 4.30, and following the argument in the proof of Lemma 4.28, we apply Theorem 4.20 to conclude that there exist $\tau_0 > 0$ and $\delta > 0$ such that $u_t(\mathbb{D}_\delta^n) \subset \mathbb{R}^m \times B$ for $|t| < \tau_0$. In other words, by rescaling the coordinates of Ω via the transformation $(\mathbf{z}, t) \mapsto (\frac{\mathbf{z}}{\delta}, t)$ and replacing ε by τ_0, we obtain $u(\mathbb{D}^n \times \mathbb{D}_\varepsilon) \subset \mathbb{R}^m \times B$. This completes the proof of Item (ii).

Let us prove Item (iii). Since $u_t(\mathbb{D}^n) \subset \mathbb{R}^m \times B$, we can write $u_t = (u_t^1, u_t^2)$. By shrinking Ω if necessary, we can assume that the rank of du_0^1 is equal to m at all points of $\mathbb{D}^n$ which then implies that u_0^2 is identically a constant. This proves Item (a). Item (b) follows from the same argument as in CASE 2 of Lemma 4.28.

Next, analogously to (4.9), for any $t_0 \in \mathbb{D}_\varepsilon$ we have

$$(1 - Ct)^{g_{t_0}}E^{u_{t_0} \circ F_{t_0}} \leq (1 - Ct)^{g_{t_0}}E^{u_t \circ F_t} \leq {}^{g_t}E^{u_t \circ F_t} \leq {}^{g_t}E^{u_{t_0} \circ F_{t_0}} \leq (1 + Ct)^{g_{t_0}}E^{u_{t_0} \circ F_{t_0}},$$

which implies the convergence of the total energy

$$\lim_{t \to t_0} {}^{g_t}E^{u_t \circ F_t} = {}^{g_{t_0}}E^{u_{t_0} \circ F_{t_0}}. \tag{4.10}$$

Thus, following the proof of Claim 4.30, we can show that $u_t \circ F_t(\mathbf{z})$, varies uniformly in t in the sense of Lemma 4.28. Since u_t^1 varies uniformly in t (see the sentence after (4.8) in proof of Lemma 4.28), we have that

$$\lim_{t \to t_0} \int_{\mathbb{B}_r(\mathbf{z})} |\nabla u_t^1|^2 d\mathrm{Vol}_{g_t} = \int_{\mathbb{B}_r(\mathbf{z})} |\nabla u_{t_0}^1|^2 d\mathrm{Vol}_{g_{t_0}} \tag{4.11}$$

and $u_t^2(\mathbf{z})$ varies uniformly in t in the sense of Lemma 4.28.

Claim 4.31. *For any Euclidean ball $\mathbb{B}_r(\mathbf{z}) \subset \mathbb{D}^n$, we have*

$$\lim_{t \to t_0} \int_{\mathbb{B}_r(\mathbf{z})} |\nabla u_t^2|^2 dVol_{g_t} = \int_{\mathbb{B}_r(\mathbf{z})} |\nabla u_{t_0}^2|^2 dVol_{g_{t_0}}. \tag{4.12}$$

Proof. Consider the probability measure μ_t on X_0 defined by

$$d\mu_t = \frac{|\nabla(u_{t_i} \circ F_t)|^2 d\mathrm{Vol}_{g_t}}{{}^{g_t}E^{u_t \circ F_t}}.$$

Combining the weak convergence of the energy density measure of $u_t \circ F_t$ to $u_{t_0} \circ F_{t_0}$ (which follows from Proposition 4.9), the convergence of total energy (cf. (4.10)) implies the weak convergence of μ_t to μ_{t_0}. By § 4.2, μ_0 is an absolutely continuous measure. Hence $\partial \mathbb{B}_r(\mathbf{0})$ is a continuity set for μ_0, i.e. $\mu_0(\partial \mathbb{B}_r(\mathbf{0})) = 0$. By Portmanteau's theorem (cf. [Bil99, Theorem 2.1]), one has

$$\lim_{t \to t_0} \mu_t(\mathbb{B}_r(\mathbf{0})) = \mu_{t_0}(\mathbb{B}_r(\mathbf{0})).$$

Therefore, using the fact that $\lim_{t \to t_0} {}^{g_t}E^{u_t \circ F_t} = {}^{g_{t_0}}E^{u_{t_0} \circ F_{t_0}}$, we conclude

$$\lim_{t \to t_0} \int_{\mathbb{B}_r(\mathbf{z})} |\nabla u_t|^2 d\mathrm{Vol}_{g_t} = \int_{\mathbb{B}_r(\mathbf{z})} |\nabla u_{t_0}|^2 d\mathrm{Vol}_{g_{t_0}}.$$

Since $|\nabla u_t|^2 = |\nabla u_t^1|^2 + |\nabla u_t^2|^2$, (4.12) follows from (4.11). The claim is proved.   □

Recall that $u_t^2(\mathbf{z})$ varies uniformly in t in the sense of Lemma 4.28. By Claims 4.30 and 4.31, conditions in Lemma 4.28 are satisfied. Hence the map

$$(\mathbf{z}, t) = (z_1, \dots, z_n, t) \mapsto |\nabla u_t|^2(\mathbf{z})$$

is a continuous function in $\mathbb{D}^n \times \mathbb{D}_\varepsilon$. Item (c) follows from the fact that $|\nabla u_0^2(\mathbf{z})|^2 = 0$ for any $\mathbf{z} \in \mathbb{D}^n$. This completes the proof of Item (iii), and hence the proof of Item (2). The theorem is proved.   □

5. LOWER SEMI-CONTINUITY OF Γ-DIMENSION: REDUCTIVE CASE

In this section, we prove Theorem C in the case where $\pi_1(X_0)$ is reductive. We apply Theorems 2.5 and 4.29 in combination with techniques from the reductive Shafarevich conjecture [Eys04, DYK23].

5.1. **Deformation of canonical currents.** Let X be a compact Kähler manifold and let $\varrho :$ $\pi_1(X) \to G(K)$ be a Zariski dense representation, where G is a reductive group over a non-archimedean local field K. Let $u : \widetilde{X} \to \Delta(G)$ be a ϱ-equivariant pluriharmonic map.

Note that any $x \in \mathcal{R}(u)$ has an open neighborhood $\Omega_x \subset \widetilde{X}$ such that $u(\Omega_x) \subset A$ where A is an apartment of $\Delta(G)$. We fix an isometry $i_A : A \to \mathbb{R}^N$. Write $i_A \circ u|_{\Omega_x} = (u_1, \ldots, u_N)$. The pluriharmonicity of u implies that $\sqrt{-1}\partial\bar{\partial}u_i = 0$ for each u_i. We consider a smooth semi-positive $(1, 1)$-form on Ω_x defined by

$$\sqrt{-1} \sum_{i=1}^{N} \partial u_i \wedge \bar{\partial} u_i.$$

Note that such $(1, 1)$-form does not depend on the choice of the isometry i_A. By [Eys04, §3.3.2], it is also invariant under $\pi_1(X)$-action. Therefore, it descends to a smooth real closed semi-positive $(1, 1)$-form on $\mathcal{R}(u)$. It is shown in [Eys04, p. 540] that it extends to a positive closed $(1, 1)$-current T_ϱ on X with continuous potential.

Definition 5.1 (Canonical current). The above closed positive $(1, 1)$-current T_ϱ on X is called the *canonical current* of ϱ.

Let $f : \mathcal{X} \to \mathbb{D}$ be the smooth projective family, and let $\varrho : \pi_1(\mathcal{X}) \to G(K)$ be a Zariski dense representation, where G is a semi-simple algebraic group defined over a non-archimedean local field K. We apply Theorem 4.29. Then there is a ϱ-equivariant continuous map $u : \widetilde{\mathcal{X}} \to \Delta(G)_K$ such that $u_t := u|_{\widetilde{X}_t} : \widetilde{X}_t \to \Delta(G)_K$ is a ϱ-equivariant pluriharmonic for each $t \in \mathbb{D}$ satisfying the properties therein. Precisely, pick any $x_0 \in \widetilde{\mathcal{R}}(u_0)$. There exists an open neighborhood Ω of x_0 in $\widetilde{\mathcal{X}}$ such that $\pi_{\mathcal{X}} : \Omega \to \pi_{\mathcal{X}}(\Omega)$ is an isomorphism, and we have

$$u(\Omega) \subset \mathbb{R}^m \times B.$$

Here B is also a Euclidean building, and $\Omega \simeq \mathbb{D}^n \times \mathbb{D}_\varepsilon$ such that $f \circ \pi_{\mathcal{X}}(z_1, \ldots, z_n, t) = t$ within this coordinate system. Write $u_t = u|_{\mathbb{D}^n \times \{t\}} = (u_t^1, u_t^2)$ where

$$u_t^1 : \mathbb{D}^n \to \mathbb{R}^m \ \text{ and } \ u_t^2 : \mathbb{D}^n \to B.$$

Then u_0^2 is constant. We abusively write Ω for $\pi_{\mathcal{X}}(\Omega)$. Consider the canonical current T_{ϱ_t} on X_t associated with $\varrho_t : \pi_1(X) \to G(K)$. In this case, we have

$$T_{\varrho_0}|_{\Omega \cap X_0} = \sqrt{-1} \sum_{i=1}^{m} \partial u_{0,i}^1 \wedge \bar{\partial} u_{0,i}^1$$

where we write $u_t^1 := (u_{t,1}^1, \ldots, u_{t,m}^1)$. By Theorem 4.29, $\sqrt{-1} \sum_{i=1}^{m} \partial u_{t,i}^1 \wedge \bar{\partial} u_{t,i}^1$ is a smooth semi-positive $(1, 1)$-form on $\mathbb{D}^n$, that varies continuously with $t \in \mathbb{D}_\varepsilon$.

Fix any $t \in \mathbb{D}_\varepsilon$. Pick any $x \in \mathbb{D}^n \cap \mathcal{R}(u_t^2)$, where $\mathcal{R}(u_t^2)$ is the regular set of the pluriharmonic map u_t^2 defined in Definition 4.12. Then there exists a neighborhood Ω_2 of x in $\mathbb{D}^n$ such that $u_t^2(\Omega_2) \subset A_2$, where A_2 is an apartment of the building B. We fix an isometry $i_{A_2} : A_2 \to \mathbb{R}^{N-m}$. Write $i_{A_2} \circ u_t^2 = (u_{t,m+1}^2, \ldots, u_{t,N}^2)$. Then by our construction of canonical currents, we have

$$T_{\varrho_t}|_{\Omega_2} = \sqrt{-1} \sum_{i=1}^{m} \partial u_{t,i}^1 \wedge \bar{\partial} u_{t,i}^1 + \sqrt{-1} \sum_{j=m+1}^{N} \partial u_{t,j}^2 \wedge \bar{\partial} u_{t,j}^2.$$

Since x is an arbitrary point in $\mathbb{D}^n \cap \mathcal{R}(u_t^2)$, then

$$T_{\varrho_t}|_{\mathbb{D}^n \cap \mathcal{R}(u_t^2)} \geq \sqrt{-1} \sum_{i=1}^m \partial u_{t,i}^1 \wedge \bar{\partial} u_{t,i}^1.$$

Since T_{ϱ_t} has continuous potential, and $\mathbb{D}^n \backslash \mathcal{R}(u_t^2)$ is a closed subset of $\mathbb{D}^n$ of Hausdorff codimension at least two by Theorem 4.13, it follows that the above inequality holds over the whole $\mathbb{D}^n$. In conclusion, we have the following result:

Lemma 5.2. *Let $f : \mathcal{X} \to \mathbb{D}$ be a smooth projective family and $\varrho : \pi_1(X_0) \to G(K)$ be a Zariski dense representation, where G is a semi-simple algebraic group over a non-archimedean local field K. There exists a full measure open subset $X_0^\circ \subset X_0$, such that any $x_0 \in X_0^\circ$ has an admissible coordinate system $\Omega \simeq \mathbb{D}^n \times \mathbb{D}_\varepsilon$ in $\mathcal{X}$ centered at x_0, with a real $(1,1)$-form $T(z,t) = \sqrt{-1} \sum_{i,j} a_{ij}(z,t) dz_i \wedge d\bar{z}_j$ on $\mathbb{D}^n$ whose coefficients $a_{ij}(z,t)$ are continuous function, such that*

(i) *For each fixed $t \in \mathbb{D}_\varepsilon$, $T_t(z) := T(z,t)$ is a smooth semi-positive closed $(1,1)$-form on $\mathbb{D}^n$.*
(ii) *For each $t \in \mathbb{D}_\varepsilon$, one has $T_{\varrho_t}|_{\Omega \cap X_t} \geq T_t$.*
(iii) *$T_{\varrho_0}|_{\Omega \cap X_0} = T_0$.*

Proof. Within the notions above, we set $T(z,t) := \sqrt{-1} \sum_{i=1}^m \partial u_{t,i}^1 \wedge \bar{\partial} u_{t,i}^1$. The above arguments together with Theorem 4.29 yield the lemma. $\square$

When G is a tori, we have a much simpler result.

Lemma 5.3. *Let $f : \mathcal{X} \to \mathbb{D}$ be the smooth projective family. Let $\varrho : \pi_1(\mathcal{X}) \to G(K)$ be a representation, where G is a tori over a non-archimedean local field K. Then the canonical current T_{ϱ_t} on X_t induced by $\varrho_t : \pi_1(X_t) \to G(K)$ is a smooth semi-positive $(1,1)$-form on X_t that varies smoothly in t.*

Proof. The Bruhat-Tits building of G is a real Euclidean space $V := \mathbb{R}^N$ such that $G(K)$ acts on V by translation (cf. [KP23]). For the action of $G(K)$ on V, it induces a representation $\tau : \pi_1(\mathcal{X}) \to (V, +)$. Let the homomorphism $\mathrm{pr}_i : (\mathbb{R}^N, +) \to (\mathbb{R}, +)$ be the projection into i-th factor, and let $\tau_i = \mathrm{pr}_i \circ \tau$. Then $\tau_i : \pi_1(\mathcal{X}) \to (\mathbb{R}, +)$ can be identified with an element $\lambda_i \in H^1(\mathcal{X}, \mathbb{R}) \simeq H^1(X_0, \mathbb{R})$ since $H^1(\mathcal{X}, \mathbb{R}) \simeq \mathrm{Hom}(H_1(\mathcal{X}, \mathbb{Z}), \mathbb{R})$. Let g be a Kähler metric on $\mathcal{X}$. We fix a smooth trivialization $F : \mathcal{X} \to X_0 \times \mathbb{D}$, and let $F_t : X_t \to X_0$ be the induced diffeomorphism. Then F_t induces a smooth family of Riemannian metrics g_t on X_0. Therefore, there exists a family of smooth 1-forms $\xi_{i,t}$ on X_0, each is harmonic with respect to the metric g_t and varying smoothly in t, such that their cohomology class $\{\xi_{i,t}\} = \lambda_i$. In this case, $F_t^* \xi_{i,t}$ is a harmonic 1-form on $(X_t, g|_{X_t})$. Let $\eta_{i,t} = (F_t^* \xi_{i,t})^{1,0}$ be the $(1,0)$-part of $\xi_{i,t}$, which is moreover a holomorphic 1-form on X_t by the Hodge theory. Then $\eta_{i,t}$ varies smoothly with respect to t. In other words, for any $x_0 \in X$ and any admissible coordinate system $\Omega \simeq \mathbb{D}^n \times \mathbb{D}_\varepsilon$ centered at x_0 in $\mathcal{X}$, one has

$$\eta_{i,t}|_{\mathbb{D}^n} = \sum_{j=1}^n a_j(z,t) dz_j,$$

such that $a_j(z,t)$ are smooth functions on $\mathbb{D}^n \times \mathbb{D}_\varepsilon$. By [DM24], the canonical current of ϱ_t is given by

$$T_{\varrho_t}|_{\Omega \cap X_t} = \sqrt{-1} \sum_{i=1}^{n} \eta_{i,t} \wedge \overline{\eta_{i,t}}.$$

Consequently, T_{ϱ_t} is a smooth closed semi-positive $(1,1)$-form on X_t varying smoothly with t. □

Lemmas 5.2 and 5.3 imply the following stronger result than Lemma 5.2.

Theorem 5.4. *Let $f : \mathcal{X} \to \mathbb{D}$ be a smooth projective family and $\varrho : \pi_1(X_0) \to G(K)$ be a Zariski dense representation, where G is a reductive algebraic group over a non-archimedean local field K. There exists a full measure open subset $X_0^\circ \subset X_0$ such that any $x_0 \in X_0^\circ$ has an admissible coordinate system $\Omega \simeq \mathbb{D}^n \times \mathbb{D}_\varepsilon$ in $\mathcal{X}$ centered at x_0, with a real $(1,1)$-form $T(z,t) = \sqrt{-1} \sum_{i,j} a_{ij}(z,t)dz_i \wedge d\bar{z}_j$ on $\mathbb{D}^n$ satisfying*

(i)   *the coefficents $a_{ij}(z,t)$ are continuous function on $\mathbb{D}^n \times \mathbb{D}_\varepsilon$.*
(ii)   *For each fixed $t \in \mathbb{D}_\varepsilon$, $T_t(z) := T(z,t)$ is a smooth semi-positive closed $(1,1)$-form on $\mathbb{D}^n$.*
(iii) *For each $t \in \mathbb{D}_\varepsilon$, one has $T_{\varrho_t}|_{\Omega \cap X_t} \geq T_t$.*
(iv)   *$T_{\varrho_0}|_{\Omega \cap X_0} = T_0$.*

Proof. Consider the enlarged Bruhat-Tits building $\Delta(G)$. It is indeed the product of the Bruhat-Tits building of $\Delta(\mathscr{D}G)$ where $\mathscr{D}G$ is the derived group of G, with a real Euclidean space $V := \mathbb{R}^N$ such that $G(K)$ acts on V by translation (cf. [KP23]). Note that there is a natural action of $\mathscr{D}G(K)$ on $\Delta(\mathscr{D}G)$. The action of $G(K)$ on $\Delta(\mathscr{D}G)$ is given by the composition of $G(K) \to \mathscr{D}G(K)$ and the action of of $\mathscr{D}G(K)$ on $\Delta(\mathscr{D}G)$.

We consider the representation $\sigma : \pi_1(X) \to \mathscr{D}G(K)$ induced by ϱ, which is Zariski dense. Let T_{σ_t} be the canonical currents associated with $\sigma_t : \pi_1(X_t) \to \mathscr{D}G(K)$. Then it satisfies the properties in Theorem 5.4.

On the other hand, for the action of $G(K)$ on V, it induces a representation $\tau_t : \pi_1(X_t) \to (V,+)$ for each $t \in \mathbb{D}$. By Lemma 5.3, the canonical current T_{τ_t} forms a family of smooth semi-positive closed $(1,1)$-forms on X_t that vary smoothly in t. Observe that $T_{\varrho_t} = T_{\tau_t} + T_{\sigma_t}$ is the canonical current associated with ϱ_t. Lemma 5.2 together with Lemma 5.3 establish the theorem. □

5.2. **Proof of Theorem C.(i).** In this subsection, we prove Theorem C.(i), which is sufficient for establishing Theorem D. This will also imply Theorem C in the case where $\pi_1(X_0)$ is reductive.

Definition 5.5 (*m*-positive form)**.** Let X be a complex manifold. A semi-positive $(1,1)$-form ω on X is said to be *m-positive* if at each point of X, ω has at least m strictly positive eigenvalues.

Theorem 5.6. *Let $f : \mathcal{X} \to \mathbb{D}$ be a smooth projective family. Let $M_t := M_B(\pi_1(X_t), \mathrm{GL}_N)(\mathbb{C})$ denote the Betti moduli space of $\pi_1(X_t)$. Define $H_t \triangleleft \pi_1(X_t)$ as the intersection of the kernels of all reductive representations $\varrho : \pi_1(X_t) \to \mathrm{GL}_N(\mathbb{C})$. Then the function $t \mapsto \gamma d(X_t, H_t)$ on $\mathbb{D}$ is* lower semicontinuous.

Proof. We write X for X_0. Consider the Betti moduli space $M := M_B(\pi_1(X), \mathrm{GL}_N)(\mathbb{C})$. By [Eys04], [DYK23, Theorems 3.29] and [DYK23, Proof of Theorem 4.31] there exist Zariski dense representations $\{\tau_i : \pi_1(X) \to G_i(K_i)\}_{i=1,\ldots,k}$, where each $G_i \subset \mathrm{GL}_N$ is a

reductive group over a non-archimedean local field K_i of characteristic zero, and $\mathbb{C}$-VHS $\{\sigma_i : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})\}_{i=1,\ldots,\ell}$ such that the Shafarevich morphism $\mathrm{sh}_M : X \to \mathrm{Sh}_M(X)$ of M defined in Definition 3.3 exists and satisfies the following properties:

(i) there exist closed positive $(1, 1)$-currents $S_1, \ldots, S_k$ and closed semi-positive $(1, 1)$-forms $\omega_1, \ldots, \omega_\ell$ on $\mathrm{Sh}_M(X)$ such that $\mathrm{sh}_M^* S_i = T_{\tau_i}$ and $\mathrm{Sh}_M^* \omega_j = \omega_{\sigma_j}$. Here T_{τ_i} is the canonical current associated with $\tau_i : \pi_1(X) \to G_i(K_i)$ defined in Definition 5.1 and ω_{σ_j} is the canonical form associated with $\sigma_j : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ defined in Definition 2.4.

(ii) There is a Kähler form $\omega_{\mathrm{Sh}_M(X)}$ on $\mathrm{Sh}_M(X)$ such that

$$\sum_{i=1}^{k} S_i + \sum_{j=1}^{\ell} \omega_j = \omega_{\mathrm{Sh}_M(X)} + \sqrt{-1}\partial\bar{\partial}\psi,$$

where ψ is a continuous function on $\mathrm{Sh}_M(X)$.

(iii) There exists a Zariski open subset of $\mathrm{Sh}_M(X)$ over which each S_i is smooth.

(iv) For each fiber Z of sh_M and any reductive representation $\tau : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$, the image $\tau(\mathrm{Im}[\pi_1(Z) \to \pi_1(X)])$ is a finite group,

Since each S_i has continuous potential, it follows that the positive closed $(1, 1)$-current $(\sum_{i=1}^{k} S_i + \sum_{j=1}^{\ell} \omega_j)$ has *minimal singularity* in the sense of Demailly, i.e. it is less singular than any other positive current in its cohomology class (cf. [BEGZ10, §1.4] for the definition). Let $\mathrm{Sh}_M(X)^\circ$ be the regular locus of $\mathrm{Sh}_M(X)$. Set $m := \dim \mathrm{Sh}_M(X)$. By [BEGZ10, Definition 1.17] and Item (ii), the *non-pluripolar product* of $(\sum_{i=1}^{k} S_i + \sum_{j=1}^{\ell} \omega_j)$ is equal to

$$\int_{\mathrm{Sh}_M(X)^\circ} \left(\sum_{i=1}^{k} S_i + \sum_{j=1}^{\ell} \omega_j \right)^m = \int_{\mathrm{Sh}_M(X)^\circ} (\omega_{\mathrm{Sh}_M(X)})^m > 0.$$

This implies that, there exists an open subset U of $\mathrm{Sh}_M(X)^\circ$ such that

(1) $\mathrm{sh}_M^{-1}(U) \to U$ is a proper holomorphic submersion.

(2) $\sum_{i=1}^{k} S_i + \sum_{j=1}^{\ell} \omega_j$ is smooth and strictly positive over U.

We define $\{\tau_{i,t} : \pi_1(X_t) \to G_i(K_i)\}_{i=1,\ldots,k}$ to be the representation as the compositions of τ_i with the natural isomorphism $\pi_1(X_t) \to \pi_1(X)$. Similarly, the representations $\{\sigma_{i,t} : \pi_1(X_t) \to \mathrm{GL}_N(\mathbb{C})\}_{i=1,\ldots,\ell}$ are defined as the compositions of σ_i and $\pi_1(X_t) \to \pi_1(X)$. Note that $\sigma_{i,t}$ might not necessarily underlie a $\mathbb{C}$-VHS for $t \neq 0$. Let $T_{\tau_{i,t}}$ be the canonical current on X_t of $\tau_{i,t}$. We apply Theorem 5.4. Then there exists a point $x_0 \in \mathrm{sh}_M^{-1}(U)$ such that it has an admissible coordinate system $(\Omega \simeq \mathbb{D}^n \times \mathbb{D}_\varepsilon; (z, t))$ in $\mathscr{X}$ centered at x_0 together with a continuous $(1, 1)$-form $T_i(z, t) = \sqrt{-1} \sum_{i,j} a_{ij}(z, t) dz_i \wedge d\bar{z}_j$ on $\mathbb{D}^n$ satisfying the following properties:

(a) $\Omega \cap X_0 \subset \mathrm{sh}_M^{-1}(U)$.

(b) the coefficents $a_{ij}(z, t)$ of $T_i(z, t)$ are continuous function on $\mathbb{D}^n \times \mathbb{D}_\varepsilon$.

(c) For each fixed $t \in \mathbb{D}_\varepsilon$, $T_{i,t}(z) := T_i(z, t)$ is a smooth semi-positive closed $(1, 1)$-form on $\mathbb{D}^n$.

(d) For each t, we have $T_{\tau_{i,t}}(z)|_{\Omega \cap X_t} \geq T_{i,t}(z)$.

(e) For $t = 0$, we have $T_{\tau_{i,0}}(z)|_{\Omega \cap X_0} = T_{i,0}(z)$.

Let $T_{\sigma_{i,t}}$ be the canonical form on X_t associated with $\sigma_{i,t}$. By Theorem 2.5, the canonical forms $\omega_{\sigma_{i,t}}$ on X_t vary continuously in t. Recall that

$$\sum_{i=1}^{k} T_{\tau_{i,0}} + \sum_{j=1}^{\ell} \omega_{\sigma_{j,0}} = \sum_{i=1}^{k} T_{\tau_i} + \sum_{j=1}^{\ell} \omega_{\sigma_j} = \text{sh}_M^*(\sum_{i=1}^{k} S_i + \sum_{j=1}^{\ell} \omega_j).$$

Hence by Items (1) and (2), Items (a), (b) and (e), $\left(\sum_{i=1}^{k} T_{i,0} + \sum_{j=1}^{\ell} \omega_{\sigma_{j,0}}\right) |_{\Omega \cap X_0}$ is m-semipositive. Note that the matrix continuous function has lower semicontinuous rank. This implies that, after we shrink the neighborhood Ω of x_0 in $\mathscr{X}$, there exists some $\varepsilon > 0$ such that for each $t \in \mathbb{D}_\varepsilon$, we have $X_t \cap \Omega \neq \varnothing$, and $\left(\sum_{i=1}^{k} T_{i,t} + \sum_{j=1}^{\ell} \omega_{\sigma_{j,t}}\right) |_{\Omega \cap X_t}$ is m-positive. Therefore, by Item (d), we conclude the following

Claim 5.7. *For each $t \in \mathbb{D}_\varepsilon$, the positive $(1,1)$-current $\left(\sum_{i=1}^{k} T_{\tau_{i,t}} + \sum_{j=1}^{\ell} \omega_{\sigma_{j,t}}\right) |_{\Omega \cap X_t}$ is bounded from below by the smooth closed m-positive $(1,1)$-form $\left(\sum_{i=1}^{k} T_{i,t} + \sum_{j=1}^{\ell} \omega_{\sigma_{j,t}}\right) |_{\Omega \cap X_t}$.*

Note that $\pi_1(X_t) \simeq \pi_1(X)$. Hence $M_t := M_{\text{B}}(\pi_1(X_t), \text{GL}_N)(\mathbb{C})$ is naturally identified with M. For each $t \in \mathbb{D}_\varepsilon$, we consider the Shafarevich morphism $\text{sh}_{M_t} : X_t \to \text{Sh}_{M_t}(X_t)$, whose existence is also proven in [Eys04, DYK23]. Let Z be any smooth fiber of sh_{M_t}. By Definition 3.3, $\tau_{i,t}(\text{Im}[\pi_1(Z) \to \pi_1(X_t)])$ is finite. By the unicity of $\tau_{i,t}$-equivariant harmonic maps in [DM24, Theorem B], it follows that $T_{\tau_{i,t}}|_Z$ is trivial. By the unicity of harmonic bundles in [Cor88], $\omega_{\sigma_{j,t}}|_Z$ is also trivial. Let $\text{Sh}_{M_t}(X_t)^\circ$ be a Zariski dense open subset of the regular locus of $\text{Sh}_{M_t}(X_t)$ such that $\text{sh}_{M_t} : X_t \to \text{Sh}_{M_t}(X_t)$ is smooth over $\text{Sh}_{M_t}(X_t)^\circ$. Set $X_t^\circ = \text{sh}_{M_t}^{-1}(\text{Sh}_{M_t}(X_t)^\circ)$, and $m_t := \dim \text{Sh}_{M_t}(X_t)$. The above arguments implies that $\left(\sum_{i=1}^{k} T_{\tau_{i,t}} + \sum_{j=1}^{\ell} \omega_{\sigma_{j,t}}\right)$ is *at most m_t-positive* over X_t°.

Note that $X_t^\circ \cap \Omega_t$ is non-empty. By Claim 5.7 we conclude that $m_t \geq m$. It yields

Claim 5.8. *For each $t \in \mathbb{D}_\varepsilon$, we have $\dim \text{Sh}_{M_t}(X_t) \geq \dim \text{Sh}_M(X_0)$.*

By Remark 3.5, we note that the Γ-reduction $\gamma_{(X_t, H_t)} : X_t \dashrightarrow \Gamma_{H_t}(X_t)$ for (X_t, H_t) is bimeromorphic to $\text{sh}_{M_t} : X_t \to \text{Sh}_{M_t}(X_t)$. Hence we have $\gamma d(X_t, H_t) = \dim \text{Sh}_{M_t}(X_t)$ for each t. Theorem C.(i) then follows from Claim 5.8.

Let us prove Theorem C.(ii) if $\pi_1(X_0)$ is reductive. If there exists an almost faithful reductive representation $\varrho : \pi_1(X) \to \text{GL}_N(\mathbb{C})$, then $\varrho_t : \pi_1(X_t) \to \text{GL}_N(\mathbb{C})$ is also almost faithful and reductive for each $t \in \mathbb{D}$. By Lemma 3.4, $\text{sh}_{M_t} : X_t \to \text{Sh}_{M_t}(X_t)$ is the Shafarevich morphism of X_t. From Remark 3.5, one has $\dim \text{Sh}_{M_t}(X_t) = \gamma d(X_t)$. By Claim 5.8, this directly implies the second claim of Theorem C. $\qquad\square$

6. Pseudo-Brody hyperbolicity in families

As an application of Theorem C.(i), we prove a result on the openness of pseudo-Brody hyperbolicity.

Theorem 6.1 (=Theorem D). *Let $f : \mathscr{X} \to \mathbb{D}$ be a smooth projective family. Assume that there is a big and reductive representation $\varrho : \pi_1(X_0) \to \text{GL}_N(\mathbb{C})$. If X_0 is pseudo-Brody hyperbolic, then X_t is also pseudo-Brody hyperbolic for sufficiently small t.*

Proof. In [CDY22, Theorem C], we prove Conjecture 1.10 if there exists a big and reductive representation $\pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$. Thus, by our assumption for X_0, we apply [CDY22, Theorem C] to conclude that X_0 is of general type. By Siu's invariance of plurigenera [Siu98, Pă u07], X_t is of general type for any $t \in \mathbb{D}$.

Set $M := M_{\mathrm{B}}(\pi_1(X_0), \mathrm{GL}_N)(\mathbb{C})$. Since $[\varrho] \in M$, the Shafarevich morphism $\mathrm{sh}_M : X_0 \to \mathrm{Sh}_M(X_0)$ is a birational morphism. By [DYK23, Lemma 3.25], there exists another reductive representation $\tau : \pi_1(X_0) \to \mathrm{GL}_N(\mathbb{C})$ such that for any reductive representation $\sigma : \pi_1(X_0) \to \mathrm{GL}_N(\mathbb{C})$, we have

$$\ker \tau \subset \ker \sigma. \tag{6.1}$$

Let $\tau_t : \pi_1(X_t) \to \mathrm{GL}_N(\mathbb{C})$ be the composite map of τ and the natural isomorphism $\pi_1(X_t) \to \pi_1(X_0)$ induced by f.

Claim 6.2. *There exists $\varepsilon > 0$ such that τ_t is a big and reductive representation for any $t \in \mathbb{D}_\varepsilon$.*

Proof. By (6.1), we have

$$\ker \tau_t \subset \ker \sigma, \quad \forall \text{ reductive } \sigma : \pi_1(X_t) \to \mathrm{GL}_N(\mathbb{C}). \tag{6.2}$$

This implies that, for any closed subvariety Z of X_t, if $\tau_t(\mathrm{Im}[\pi_1(Z) \to \pi_1(X_t)])$ is finite, then $\sigma(\mathrm{Im}[\pi_1(Z) \to \pi_1(X_t)])$ is finite for any reductive representation $\sigma : \pi_1(X_0) \to \mathrm{GL}_N(\mathbb{C})$. Set $M_t := M_{\mathrm{B}}(\pi_1(X_t), \mathrm{GL}_N)(\mathbb{C})$. By Definition 3.3, for every $t \in \mathbb{D}$, $\mathrm{sh}_{M_t} : X_t \to \mathrm{Sh}_{M_t}(X_t)$ coincides with the Shafarevich morphism $\mathrm{sh}_{\tau_t} : X_t \to \mathrm{Sh}_{\tau_t}(X_t)$ of τ_t.

By Theorem C.(i), there exists $\varepsilon > 0$ such that for any $t \in \mathbb{D}_\varepsilon$, we have

$$\dim \mathrm{Sh}_{\tau_t}(X_t) = \dim \mathrm{Sh}_{M_t}(X_t) \geq \dim \mathrm{Sh}_M(X_0) = \dim X_0.$$

Therefore, τ_t is a big and reductive representation for any $t \in \mathbb{D}_\varepsilon$. $\qquad\square$

Since X_t is of general type, by Claim 6.2 we apply [CDY22, Theorem C] again to conclude that X_t is pseudo-Brody hyperbolic for any $t \in \mathbb{D}_\varepsilon$. The theorem is proved. $\qquad\square$

From Claim 6.2, we obtain the following result, which may be of independent interest.

Proposition 6.3. *Let $f : \mathcal{X} \to \mathbb{D}$ be a smooth projective family. Assume that there is a big and reductive representation $\varrho : \pi_1(X_0) \to \mathrm{GL}_N(\mathbb{C})$. Then $\tau : \pi_1(X_t) \to \mathrm{GL}_N(\mathbb{C})$ is big and reductive for sufficiently small t.* $\qquad\square$

Note that it is unclear whether $\varrho_t : \pi_1(X_t) \to \mathrm{GL}_N(\mathbb{C})$, induced by the big representation ϱ in Proposition 6.3, is big for sufficiently small t.

We conclude this section with the proof of Corollary E.

Proof of Corollary E. In Case (a), by the work of Griffiths-Schmid [GS69], the period domain $\mathscr{D}$ is equipped with a natural metric that has negative holomorphic sectional curvature along the horizontal direction. Since the period map of ϱ is assumed to be generically finite onto its image, it follows from the Ahlfors-Schwarz lemma that X is pseudo-Brody hyperbolic.

On the other hand, by Lemma 3.6, ϱ is big. By [Sim92], ϱ is also reductive. Hence, the conditions in Theorem D are fulfilled, allowing us to conclude that a small deformation of X is also pseudo-Brody hyperbolic.

In Case (b), by [CDY22, Theorem A], X is pseudo-Brody hyperbolic. We apply Theorem D to conclude that a small deformation of X is also pseudo-Brody hyperbolic. The corollary is proved. $\qquad\square$

7. LOWER SEMI-CONTINUITY OF Γ-DIMENSION: LINEAR CASE

In this section, we will prove Theorem C.(ii).

7.1. Regularity of harmonic maps to symmetric spaces in families.

Lemma 7.1. *Let Γ be a finitely generated group. Consider the surjective isogeny $p : \mathrm{SL}_N(\mathbb{C}) \times \mathbb{C}^* \to \mathrm{GL}_N(\mathbb{C})$ defined by $p(A, t) := tA$. Then for any representation $\varrho : \Gamma \to \mathrm{GL}_N(\mathbb{C})$, there exists a finite index subgroup Γ' of Γ such that $\varrho|_{\Gamma'}$ lifts to a representation $\Gamma' \to \mathrm{SL}_N(\mathbb{C}) \times \mathbb{C}^*$.*

Proof. Set $G := \varrho(\Gamma)$ and define $G_1 := p^{-1}(G)$. Note that $\ker p = \{(\zeta_i I, \zeta_i^{-1})\}_{i=1,\dots,N}$, where $\zeta_1, \dots, \zeta_N$ are N-th roots of unity. In particular, these elements are all torsion elements.

By Selberg's theorem, G_1 is virtually torsion-free, and thus there exists a finite index subgroup G_2 of G_1 such that $G_2 \cap \ker p = \{e\}$. Hence, $p|_{G_2} : G_2 \to p(G_2)$ is an isomorphism. Note that $p(G_2)$ is a finite index subgroup of G.

It follows that $\Gamma' := \varrho^{-1}(p(G_2))$ is a finite index subgroup of Γ. We define $\varrho' = (p|_{G_2})^{-1} \circ \varrho|_{\Gamma'} : \Gamma' \to G_2$. Then we have

$$p_2 \circ \varrho' = \varrho|_{\Gamma'}.$$

This completes the proof of the lemma. □

Remark 7.2. Since the Γ-dimension of a variety is invariant under étale covers, we can apply Lemma 7.1 to reduce the proof of Theorems A and C by replacing $\mathscr{X}$ with a suitable finite étale cover. This allows us to assume that the representation $\pi_1(\mathscr{X}) \to \mathrm{GL}_N(\mathbb{C})$ lifts to $\pi_1(\mathscr{X}) \to \mathrm{SL}_N(\mathbb{C}) \times \mathbb{C}^*$.

Proposition 7.3. *Let $f : \mathscr{X} \to \mathbb{D}$ be a smooth projective family, and let $\varrho : \pi_1(\mathscr{X}) \to \mathrm{SL}_N(\mathbb{C})$ be a simple representation. There exists a ϱ-equivariant continuous map $u : \widetilde{\mathscr{X}} \to \mathcal{S}$, where $\mathcal{S} := \mathrm{SL}_N(\mathbb{C})/\mathrm{SU}_N$ is the Riemannian symmetric space of non-compact type associated with $\mathrm{SL}_N(\mathbb{C})$ such that*

(i)   for each $t \in \mathbb{D}$, $u|_{\widetilde{X}_t} : \widetilde{X}_t \to \mathcal{S}$ is the unique ϱ_t-equivariant harmonic map.
(ii)   the map u vary smoothly in t.

Proof. We equip $\mathscr{X}$ with a Kähler metric g. We endow X_t with the Kähler metric $g_t := g|_{X_t}$ for any $t \in \mathbb{D}$, and $\mathcal{S}$ the canonical Riemannian metric for Riemannian symmetric space. By [Cor88], there exists a unique ϱ_t-equivariant harmonic map $u_t : \widetilde{X}_t \to \mathcal{S}$, which is moreover pluriharmonic. Let $u : \widetilde{X} \to \mathcal{S}$ be the defined by setting $u|_{\widetilde{X}_t} = u_t$. This proves Item (i).

Claim 7.4. *Let $F_t : \widetilde{X}_0 \to \widetilde{X}_t$ be the diffeomorphism induced by the C^∞-trivialization $\mathscr{X} \to X_0 \times \mathbb{D}$. Then the function $u_t \circ F_t : \widetilde{X}_0 \to \mathcal{S}$ converges uniformly to u_0. In particular, $u : \widetilde{\mathscr{X}} \to \mathcal{S}$ is continuous.*

Proof. The proof is close to that of Claim 4.30. We abusively denote by g_t the induced Riemannian metric on X_0 via the natural diffeomorphism $X_0 \to X_t$. By the same arguments as in (4.9), the energy function $t \mapsto \{{}^{g_t} E^{u_t \circ F_t}\}$ is continuous. Fix any $\varepsilon \in (0, 1)$. By a classical result of Eells-Sampson [ES64], $\{u_t \circ F_t\}_{t \in \mathbb{D}_\varepsilon}$ is uniformly Lipschitz continuous.

By (4.9) again, the function $t \mapsto \{{}^{g_0} E^{u_t \circ F_t}\}$ is also continuous and converges to $\{{}^{g_0} E^{u_0}\}$ as $t \to 0$. Hence, $u_t \circ F_t : \widetilde{X}_0 \to \mathcal{S}$ is an energy-minimizing sequence in the sense of Definition 4.8 with X_0 endowed with the fixed Riemannian metric g_0.

Since ϱ is semi-simple, the Zariski closure G of $\varrho(\pi_1(\mathscr{X}))$ is a reductive group. Then $\varrho(\pi_1(\mathscr{X}))$ does not fix a point at infinity of $\mathcal{S}$. Otherwise, G would be contained in $P(\mathbb{C})$, where P is a proper parabolic subgroup, which is impossible. By Lemma 4.6, the action of $\varrho(\pi_1(\mathscr{X}))$ on $\mathcal{S}$ is proper. Hence, we can apply Proposition 4.9 to conclude that $u_t \circ F_t$ converges uniformly to a ϱ_0-equivariant harmonic map $v : \widetilde{X_0} \to \mathcal{S}$ (with respect to the metric g_0) as $t \to 0$. By the uniqueness theorem in [Cor88], one has $v = u_0$.

We thus conclude that $u_t \circ F_t$ converges uniformly to u_0 as $t \to 0$. The same argument also shows that for any $t_0 \in \mathbb{D}$, $u_t \circ F_t$ converges uniformly to u_{t_0} as $t \to t_0$. The claim is proved. $\square$

The exponential map of $\mathcal{S}$ provides a diffeomorphism between $\mathbb{R}^m$ and $\mathcal{S}$, thereby inducing a global coordinate system $(x_1, \ldots, x_m)$ for $\mathcal{S}$. Let $\Gamma^\gamma_{\alpha\beta} \in C^\infty(\mathcal{S})$ denote the Christoffel symbols of the Levi-Civita connection on $\mathcal{S}$ with respect to this global coordinate system.

We express $u = (u^1, \ldots, u^m)$ in terms of these global coordinates on $\mathcal{S}$. Let $\Omega \simeq \mathbb{D}^n \times \mathbb{D}_\epsilon$ be any admissible coordinate as described in Definition 4.26. We use the notation

$$u_t^\alpha(z_1, \ldots, z_n) := u^\alpha(z_1, \ldots, z_n, t), \quad \forall \alpha = 1, \ldots, m.$$

Since each u_t is harmonic, over Ω we have

$$\Delta_t u_t^\alpha + \Gamma^\alpha_{\beta\gamma}(u_t) g_t^{i\bar{j}} \frac{\partial u_t^\beta}{\partial z_i} \frac{\partial u_t^\gamma}{\partial \bar{z}_j} = 0. \tag{7.1}$$

Here, Δ_t denotes the Laplacian operator with respect to the metric g_t, and $[g_t^{i\bar{j}}]_{i,j=1,\ldots,n}$ is the inverse of the $n \times n$-matrix $[\langle \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j} \rangle_{g_t}]_{i,j=1,\ldots,n}$.

Since the metrics g_t vary smoothly with t, and $\{^{g_t}E^{u_t}\}_{t\in\mathbb{D}_\varepsilon}$ is uniformly bounded, it follows from [ES64] that there exists a constant $C_1 > 0$ such that

$$\left| \frac{\partial u_t^\alpha}{\partial \bar{z}_i}(\mathbf{z}) \right| \le C_1, \quad \left| \frac{\partial u_t^\alpha}{\partial z_i}(\mathbf{z}) \right| \le C_1, \tag{7.2}$$

for each $\alpha = 1, \ldots, m$, $i = 1, \ldots, n$, $t \in \mathbb{D}_\varepsilon$, and $\mathbf{z} \in \mathbb{D}^n$.

Since u is continuous, there exists a constant $C_2 > 0$ such that $|\Gamma^\alpha_{\beta\gamma}(u_t(\mathbf{z}))| \le C_2$ for any $\alpha, \beta, \gamma \in \{1, \ldots, m\}$, $t \in \mathbb{D}_\epsilon$, and $\mathbf{z} \in \mathbb{D}^n$. Fix any $\mu \in (0, 1)$.

We apply the interior elliptic estimates for the elliptic equations (7.1) (cf. [Sim96, Section 1.7, Lemma 3]) to conclude that there exists a constant $C_3 > 0$ such that for each $t \in \mathbb{D}_\varepsilon$ and any $\alpha \in \{1, \ldots, m\}$,

$$\left| u_t^\alpha \right|_{C^{1,\mu};\mathbb{D}^n_{\frac{3}{4}}} := \sup_{\mathbf{z}\in\mathbb{D}_{\frac{3}{4}}} |u_t^\alpha(\mathbf{z})| + \sum_{j=1}^{2n} \sup_{\mathbf{z}\in\mathbb{D}_{\frac{3}{4}}} \left| \frac{\partial u_t^\alpha}{\partial x_j}(\mathbf{z}) \right| + \sum_{j=1}^{2n} \sup_{\mathbf{z}_1,\mathbf{z}_2\in\mathbb{D}^n_{\frac{3}{4}}, \, \mathbf{z}_1\neq\mathbf{z}_2} \frac{\left| \frac{\partial u_t^\alpha}{\partial x_j}(\mathbf{z}_1) - \frac{\partial u_t^\alpha}{\partial x_j}(\mathbf{z}_2) \right|^\mu}{|\mathbf{z}_1 - \mathbf{z}_2|} \le C_3 \tag{7.3}$$

where $(z_1 = x_1 + ix_2, \ldots, z_n = x_{2n-1} + ix_{2n})$ are the coordinates in $\mathbb{D}^n$. Let $\mu' \in (\frac{\mu}{2}, \mu)$ and any $t_0 \in \mathbb{D}_\varepsilon$. Note that the Hölder space $C^{1,\mu}(\mathbb{D}^n_{\frac{3}{4}})$ is compactly contained in $C^{1,\mu'}(\mathbb{D}^n_{\frac{3}{4}})$. Then for any sequence $t_i \to t_0$, after taking a subsequence if necessary, $u_{t_i}^\alpha$ converges to some v^α in $C^{1,\mu'}(\mathbb{D}^n_{\frac{3}{4}})$. Note that u_t^α converges uniformly to $u_{t_0}^\alpha$ as $t \to t_0$. It follows that $v^\alpha = u_{t_0}^\alpha$. Hence $u_{t_i}^\alpha$ converges to $u_{t_0}^\alpha$ in $C^{1,\mu'}(\mathbb{D}^n_{\frac{3}{4}})$. By (7.1) together with a bootstrap argument, for any

positive integer $k > 0$, $u_{t_i}^\alpha$ converges to $u_{t_0}^\alpha$ in $C^{k, \frac{\mu}{2}}(\mathbb{D}_{\frac{1}{2}}^n)$ as $i \to \infty$. This proves Item (ii). The proposition is proved.

□

Remark 7.5. Note that results similar to Proposition 7.3 were obtained in [EL80, Sle22] under an additional assumption that $\Gamma := \varrho(\pi_1(\mathcal{X}))$ is a *uniform and torsion free lattice* in $\mathrm{SL}_N(\mathbb{C})$, i.e., $\mathcal{S}/\Gamma$ is a compact Riemannian locally symmetric manifold.

Lemma 7.6. *Let* $\tau : \pi_1(\mathcal{X}) \to \mathbb{C}^*$ *be a representation. Consider the natural action* $\mathbb{C}^*$ *on* $\mathbb{R}$ *by* $z.t := t + \log |z|$. *Then there exists a* τ-*equivariant harmonic map* $u : \widetilde{\mathcal{X}} \to \mathbb{R}$, *which vary smoothly in* t.

Proof. Note that there exists an isomorphism $\mathbb{R} \times \mathrm{U}(1) \to \mathbb{C}^*$ defined by $(t, z) \mapsto \exp(t).z$. Denote by $\tau_1 : \pi_1(\mathcal{X}) \to \mathbb{R}$ and $\tau_2 : \pi_1(\mathcal{X}) \to \mathrm{U}(1)$ be the projections of τ according to the above isomorphism. Then τ_1 corresponds to a cohomology class $\lambda \in H^1(\mathcal{X}, \mathbb{R})$. Fix a Kähler metric g on $\mathcal{X}$. By the classical Hodge theory, there exists 1-forms ξ_t on X_t, each is harmonic with respect to the metric g_t and varying smoothly in t, such that their cohomology class $\{\xi_t\} = \iota_t^* \lambda$, where $\iota_t : X_t \to \mathcal{X}$ denotes the inclusion map.

We take a smooth section $s : \mathbb{D} \to \widetilde{\mathcal{X}}$ of $\widetilde{\mathcal{X}} \to \mathscr{D}$. Then for each $t \in \mathbb{D}$, we define a smooth map $u_t(x) := \int_{s(t)}^x \pi_{X_t}^* \xi_t$. It follows that the union $u : \widetilde{\mathcal{X}} \to \mathbb{R}$ of u_t is a continuous map, which vary smoothly in t. The lemma is proved.   □

Theorem 7.7. *Let* $f : \mathcal{X} \to \mathbb{D}$ *be a smooth projective family, and let* $\sigma : \pi_1(\mathcal{X}) \to \mathrm{GL}_N(\mathbb{C})$ *be a semi-simple representation. Then after we replace* $\mathcal{X}$ *by a suitable finite étale cover, for the flat bundle* $(\mathcal{V}_\sigma, D)$ *on* $\mathcal{X}$ *induced by* σ, *there exists a continuous metric* h *on* $\mathcal{V}_\sigma$ *satisfying the following properties:*

(i)   *the restriction* $(\mathcal{V}_\sigma, D, h)|_{X_t}$ *is a harmonic bundle for each* $t \in \mathbb{D}$.
(ii) *Writing* $V_t := \mathcal{V}_\sigma|_{X_t}$, *the metric* $h|_{V_t}$ *varies smoothly in* t.

Proof. We might decompose σ into a direct sum of simple representations, and it suffices to prove the theorem for the case when σ is simple. By Lemma 7.1, after we replace $\mathcal{X}$ by a suitable finite étale cover, we may assume that σ lifts to a simple representation $\sigma' : \pi_1(\mathcal{X}) \to \mathrm{SL}_N(\mathbb{C}) \times \mathbb{C}^*$. Let $\varrho : \pi_1(\mathcal{X}) \to \mathrm{SL}_N(\mathbb{C})$ and $\tau : \pi_1(\mathcal{X}) \to \mathbb{C}^*$ denote the compositions of σ' with the first and second projections, respectively, i.e., $\mathrm{SL}_N(\mathbb{C}) \times \mathbb{C}^* \to \mathrm{SL}_N(\mathbb{C})$ and $\mathrm{SL}_N(\mathbb{C}) \times \mathbb{C}^* \to \mathbb{C}^*$.

By Proposition 7.3 and Lemma 7.6, there exist a continuous ϱ-equivariant map $u_1 : \widetilde{\mathcal{X}} \to \mathcal{S}$ and continuous τ-equivariant map $u_2 : \widetilde{\mathcal{X}} \to \mathbb{R}$ such that $u_{1,t} = u_1|_{\widetilde{X_t}} : \widetilde{X_t} \to \mathcal{S}$ is pluriharmonic and $u_{2,t} = u_2|_{\widetilde{X_t}} : \widetilde{X_t} \to \mathbb{R}$ is also pluriharmonic.

Let $\mathscr{P}_N(\mathbb{C})$ be the space of positive definite hermitian symmetric $N \times N$ matrices with Riemannian metric defined by

$$\langle X, Y \rangle_H = \mathrm{tr}(H^{-1} X H^{-1} Y), \quad \forall X, Y = T_H \mathscr{P}_N(\mathbb{C}).$$

Note that there is a natural isometric identification $\mathscr{P}_N(\mathbb{C}) \simeq \mathrm{GL}_N(\mathbb{C})/\mathrm{U}_N$. Indeed, $\mathrm{GL}_N(\mathbb{C})$ acts isometrically and transitively on $\mathscr{P}_N(\mathbb{C})$ where the action of $g \in \mathrm{GL}_N(\mathbb{C})$ is given by $H \mapsto gHg^\dagger$. Since U_N is the stabilizer of the identity, the map $\mathrm{GL}_N(\mathbb{C})/\mathrm{U}_N \to \mathscr{P}_N(\mathbb{C})$ defined by $[g] \mapsto gg^\dagger$ is an isometry. The subspace $\mathscr{P}_N(\mathbb{C})_1$ consisting of $H \in \mathscr{P}_N(\mathbb{C})$ with

$\det H = 1$ is a totally geodesic submanifold of $\mathscr{P}_N(\mathbb{C})$. Moreover, $\mathscr{P}_N(\mathbb{C})_1$ is invariant by the action of $\mathrm{SL}_N(\mathbb{C})$ with SU_N being the stabilizer of the identity. Thus, we also have the isometric identification $\mathscr{P}_N(\mathbb{C})_1 \simeq \mathcal{S}$.

Observe that

$$\psi : \mathscr{P}_N(\mathbb{C})_1 \times \mathbb{R} \to \mathscr{P}_N(\mathbb{C})$$
$$(H, t) \mapsto e^{\frac{t}{N}} H.$$

is an isometry. We define the action of $\mathrm{SL}_N(\mathbb{C}) \times \mathbb{C}^*$ on $\mathscr{P}_N(\mathbb{C})_1 \times \mathbb{R}$ by setting $(g, z).(H, t) :=$ $(gAg^\dagger, t + 2N \log |z|)$. Then ψ is equivariant, i.e. for any $(g, z) \in \mathrm{SL}_N(\mathbb{C}) \times \mathbb{C}^*$, we have

$$\psi((g, z).(H, t)) = \psi((gHg^\dagger, t + 2N \log |z|)) = gHg^\dagger e^{\frac{t}{N}} |z|^2 = (zg).e^{\frac{t}{N}} H \qquad (7.4)$$

Identifying $u_1(x)$ as an element in $\mathscr{P}_N(\mathbb{C})_1$, we have that $u(x) := \psi(u_1(x), 2Nu_2(x)) =$ $e^{2u_2(x)} u_1(x) \in \mathscr{P}_N(\mathbb{C}) \simeq \mathrm{GL}_N(\mathbb{C})/\mathrm{U}_N$. Since $u_1|_{\widetilde{X}_t}$ and $u_2|_{\widetilde{X}_t}$ are pluriharmonic and ψ is an isometry, $u|_{\widetilde{X}_t}$ is also pluriharmonic. Note that u is σ-equivariant by (7.4). By Proposition 7.3 and Lemma 7.6, $u|_{\widetilde{X}_t}$ varies smoothly in t.

Consider the flat bundle $\mathscr{V}_\sigma := \widetilde{\mathcal{X}} \times_\sigma \mathbb{C}^N$ induced by σ. Then for any $x \in \mathcal{X}$, select any $\tilde{x} \in \pi_{\mathcal{X}}^{-1}(x)$ and and define the Hermitian metric h on $\mathscr{V}_\sigma$ by setting $h_x := u(x)$. Since u is σ-equivariant, this definition provides a well-defined Hermitian metric h on $\mathscr{V}_\sigma$. Moreover, $h|_{X_t}$ is the harmonic metric for $V_t := \mathscr{V}_\sigma|_{X_t}$. Since u varies smoothly in t, the restriction $h|_{V_t}$ also varies smoothly in t. Thus, the theorem is proved. $\qquad \square$

Remark 7.8. We can use Theorem 7.7 to strengthen Theorem 2.5, showing that ω_{ϱ_t} varies *smoothly* in t. Indeed, let $(V_t, D_t, h_t) := (\mathscr{V}_\sigma, D, h)|_{X_t}$. Decompose D_t as $D_t = \nabla_t + \Phi_t$, where ∇_t is the metric connection associated with (V_t, h_t), and Φ_t is self-adjoint. By Theorem 7.7.(ii), Φ_t varies smoothly with t.

Next, decompose Φ_t as $\Phi_t = \Phi_t' + \Phi_t''$, where Φ_t' and Φ_t'' are the $(1, 0)$- and $(0, 1)$-components of Φ_t, respectively. Since both Φ_t' and Φ_t'' vary smoothly with t, it follows that $\omega_{\varrho_t} = \sqrt{-1}\mathrm{tr}(\Phi_t' \wedge \Phi_t'')$ also varies smoothly with t.

7.2. **Semi-canonical form.** Let (V, D, h) be a harmonic bundle on a compact Kähler n-fold X. We define the Laplacian by setting

$$\Delta := DD^* + D^*D, \qquad (7.5)$$

where D^* is the adjoint of D with respect to h. Any k-form $\eta \in A^k(X, V)$ with value in V is called harmonic if $\Delta(\eta) = 0$. The cohomology $H^\bullet_{\mathrm{DR}}(X, V)$ is defined as the cohomology of the following complex

$$0 \to A^0(X, V) \xrightarrow{D} A^1(X, V) \xrightarrow{D} \cdots \xrightarrow{D} A^{2n}(X, V).$$

In [Sim92, Lemma 2.2], it is shown that the natural inclusion

$$\mathscr{H}^k(X, V) \to H^k_{\mathrm{DR}}(X, V) \qquad (7.6)$$

is an isomorphism, where $\mathscr{H}^k(X, V)$ denotes the space of V-valued harmonic k-forms.

We assume further that V underlies a real structure, i.e. it is a complexification of a real local system $V_\mathbb{R}$. Let $\eta \in \mathscr{H}^1(X, V)$ be a harmonic 1-form. We decompose $\eta = \eta^{1,0} + \eta^{0,1}$ into $(1, 0)$ and $(0, 1)$-parts, i.e. $\eta^{1,0} \in A^{1,0}(X, V)$ and $\eta^{0,1} \in A^{0,1}(X, V)$.

Lemma 7.9. *If $\alpha := \{\eta\} \in H^1_{\mathrm{DR}}(X, V_{\mathbb{R}})$, then we have $\overline{\eta^{1,0}} = \eta^{0,1}$. Here $\overline{\eta^{1,0}}$ is the complex conjugate of $\eta^{1,0}$ with respect to the real structure by $V_{\mathbb{R}}$.*

Proof. Let $\sigma : V \to V$ be the complex conjugate of V with respect to the $\mathbb{R}$-structure by $V_{\mathbb{R}}$. Then $\sigma D = D\sigma$. It follows that $D(\overline{\eta}) = 0$. By the definition of $H^{\bullet}_{\mathrm{DR}}(X, V_{\mathbb{R}})$, there exists a D-closed $V_{\mathbb{R}}$-valued real 1-form β such that $\{\beta\} = \alpha$. It follows that

$$\{\overline{\eta}\} = \{\overline{\beta}\} = \{\beta\} = \alpha.$$

We decompose $D = \partial + \bar{\partial} + \theta + \theta^*$ such that $(\partial + \bar{\partial}, h)$ is a metric connection, and $\theta + \theta^*$ is self-adjoint. Then by [Sim92, Lemma 2.12], $\sigma D' = D''$, where $D' := \partial + \theta^*$ and $D'' := \bar{\partial} + \theta$. Since η is harmonic, by [Sim92, Lemma 2.2] one has

$$D''\eta = D'\eta = 0.$$

This implies that

$$D''(\overline{\eta}) = D'(\overline{\eta}) = 0.$$

By [Sim92, Lemma 2.2] again, $\overline{\eta} \in \mathscr{H}^1(X, V)$. It follows from (7.6) that $\overline{\eta} = \eta$. This implies the lemma.   □

Definition 7.10 (Semi-canonical form). Let (V, D, h) be a harmonic bundle on a compact Kähler manifold X. For any $\alpha \in H^1_{\mathrm{DR}}(X, V_{\mathbb{R}})$, let $\eta \in \mathscr{H}^1(X, V)$ be the harmonic representative in α as above. The *semi-canonical form* associated with α is a smooth real $(1, 1)$-form on X defined by

$$S_{\alpha} := \sqrt{-1}\{\eta^{1,0}, \eta^{1,0}\}_h, \tag{7.7}$$

where $\{\bullet, \bullet\}_h$ denotes the operation of taking the wedge product of differential forms combined with the inner product of sections of V induced by the metric h.

Note, however, that S_{α} is not necessarily closed or semi-positive. Moreover, it depends on the rescaling of the harmonic metric h and is therefore referred to as *semi-canonical*.

Lemma 7.11. *Let (V, D, h) be a harmonic bundle on a compact Kähler manifold X such that the corresponding local system is defined over $\mathbb{R}$, denoted by $V_{\mathbb{R}}$. Let $g : Z \to X$ be a holomorphic map from another compact Kähler manifold Z. If $g^* V_{\mathbb{R}}$ is a trivial local system, then for any $\alpha \in H^1_{\mathrm{DR}}(X, V_{\mathbb{R}})$, the $(1, 1)$-form $g^* S_{\alpha}$ is closed and semi-positive.*

Proof. Let $\underline{\mathbb{R}^r} := Z \times \mathbb{R}^r$ be the trivial real vector bundle of rank $r := \mathrm{rank}\, V$ on Z. Let h_0 be the standard metric on $\underline{\mathbb{R}^r}$, and let D_0 be the canonical connection on $\underline{\mathbb{R}^r}$. Note that the pullback $g^*(V, D, h)$ is still a harmonic bundle over Z. By the unicity of harmonic metrics in [Cor88], $g^* V_{\mathbb{R}}$ is isomorphic to $(\underline{\mathbb{R}^r}, D_0, h_0)$, up to rescaling h_0 by a constant factor. Note that for harmonic 1-forms, we have $g^* \mathscr{H}^1(X, V) \subset \mathscr{H}^1(Z, g^* V)$ (that is not true for general harmonic k-forms!). Since $g^* V_{\mathbb{R}}$ is a trivial local system, it follows that $g^* \eta \simeq (\xi_1, \ldots, \xi_r)$, where $\xi_i \in \mathscr{H}^1(Z, \mathbb{R})$. Hence, we have $\eta^{1,0} \simeq (\xi_1^{1,0}, \ldots, \xi_r^{1,0})$, which implies that

$$g^* \sqrt{-1}\{\eta^{1,0}, \eta^{1,0}\}_h = \sqrt{-1}\{g^* \eta^{1,0}, g^* \eta^{1,0}\}_h = \sqrt{-1} \sum_{i=1}^{r} \xi_i^{1,0} \wedge \xi_i^{0,1}.$$

The classical Hodge theory implies that $\xi_i^{1,0}$ is a holomorphic 1-form on X that is d-closed. Hence $g^* \sqrt{-1}\{\eta^{1,0}, \eta^{1,0}\}_h$ is semi-positive and closed.   □

7.3. **Semi-canonical forms in families.**

Lemma 7.12. *Let $f : \mathscr{X} \to \mathscr{D}$ be a smooth projective family and let $\varrho : \pi_1(\mathscr{X}) \to \mathrm{GL}_N(\mathbb{C})$ be a semi-simple representation. We apply Lemma 7.1 to replace $\mathscr{X}$ by a finite étale cover such that the conditions for ϱ in Theorem 7.7 are fulfilled. Let $(\mathscr{V}, D)$ be the flat bundle on $\mathscr{X}$ induced by ϱ, and let h be the metric for $\mathscr{V}$ constructed in Theorem 7.7. Fix some $\alpha \in H^1_{\mathrm{DR}}(X_0, V_0)$. Let $\alpha_t \in H^1_{\mathrm{DR}}(X_t, V_t)$ be the image of α under the natural isomorphism $H^1_{\mathrm{DR}}(X_0, V_0) \to H^1_{\mathrm{DR}}(X_t, V_t)$. Then for the V_t-valued harmonic 1-form $\eta_t \in \mathscr{H}^1(X_t, V_t)$ such that $\{\eta_t\} = \alpha_t$, we have that $F_{t,*}(\eta_t) \in A^1(X_0, V_0)$ varies smoothly in t, where $F_t : X_t \to X_0$ is the diffeomorphism induced by the C^∞-trivialization $\mathscr{X} \overset{C^\infty}{\to} X_0 \times \mathbb{D}$. In particular, if ϱ has image in $\mathrm{GL}_N(\mathbb{R})$, the semi-canonical forms S_{α_t} vary smoothly in t.*

Proof. By Theorem 7.7, $(\mathscr{V}, D, h)|_{X_t}$ is a harmonic bundle corresponding to the representation ϱ_t. Consider the family of Laplacian operators Δ_t for (V_t, D_t, h_t) defined in (7.5). By Theorem 7.7, Δ_t is a smooth family of formally self-adjoint elliptic operators. Since the function $t \mapsto \dim H^1_{\mathrm{DR}}(X_t, V_t)$ is constant, we apply [Kod05, Theorem 7.4] to conclude that $\ker \Delta_t = \mathscr{H}^1(X_t, V_t)$ varies smoothly. It follows that η_t, thus $\eta_t^{1,0}$, varies smoothly in t. By Theorem 7.7.(ii), $S_{\alpha_t} := \sqrt{-1}\{\eta_t^{1,0}, \eta_t^{1,0}\}_{h_t}$ varies smoothly in t. The lemma is proved. $\square$

7.4. **1-step $\mathbb{R}$-VMHS.**

We begin by recalling the definition of an $\mathbb{R}$-VMHS of weight length 1. For further properties of this type of $\mathbb{R}$-VMHS, we refer readers to [Car87, Pea00].

Definition 7.13 ($\mathbb{R}$-VMHS). A $\mathbb{R}$-VMHS $\mathcal{M}$ of weight length 1 on a complex manifold X consists of

(i)   A real local system $\mathcal{M}_X$ of finite type;
(ii) An increasing filtration $\{0\} = \mathcal{W}_{-2} \subset \mathcal{W}_{-1} \subset \mathcal{W}_0 = \mathcal{M}_X$ of by real sub-local systems;
(iii) A finite decreasing filtration $\mathcal{F}^\bullet$ by locally free subsheaves of $\mathcal{M} := \mathcal{M}_X \otimes_\mathbb{R} O_X$ which satisfies the Griffiths transversality condition:

$$\nabla(\mathcal{F}^p) \subset \Omega^1_X \otimes_{O_X} \mathcal{F}^{p-1},$$

and such that $\mathcal{W}$ and $\mathcal{F}$ define an $\mathbb{R}$-MHS on each fiber $(\mathcal{M}(x), \mathcal{W}_\bullet(x), \mathcal{F}^\bullet(x))$ at x of the vector bundle $\mathcal{M}$. Here ∇ is the flat connection on $\mathcal{M}$ inherited from the local system $\mathcal{M}_X$.

In what follows, we will denote $\mathcal{L}_i := \mathcal{W}_i/\mathcal{W}_{i-1}$. Let us denote by $\mathscr{M}$ the mixed period domain of $\mathcal{M}$, and $\mathscr{D}_i$ the period domain for $\mathscr{L}_i$. Then the $\mathbb{R}$-VMHS $\mathcal{M}$ induces an $\mathbb{R}$-VHS on $\mathcal{L}_i$ of weight i for $i = -1, 0$. Note that there exists a natural affine bundle $\mathscr{M} \to \mathscr{D}_{-1} \times \mathscr{D}_0$. Consider the $\mathbb{R}$-VHS $\mathcal{L} := \mathcal{L}_0^* \otimes \mathcal{L}_{-1}$ on X, which is of weight -1.

Lemma 7.14. *Let $\mathcal{M}$ be an $\mathbb{R}$-VMHS of weight length 1 on a smooth projective variety X. Assume that the connection on $\mathcal{L}_{-1} \oplus \mathcal{L}_0$ induced by $\mathcal{M}$ is given by*

$$D := \begin{bmatrix} D_{-1} & \eta \\ 0 & D_0 \end{bmatrix},$$

where $\eta \in \mathscr{H}^1(X, \mathcal{L})$ and $\eta(x) \in \Omega^{1,0}_{X,x} \otimes \mathcal{L}^{-1,0} \oplus \Omega^{0,1}_{X,x} \otimes \mathcal{L}^{0,-1}$, with $\mathcal{L}^{p,q}$ denoting the Hodge (p,q)-subspace of $\mathcal{L}$.

Let $f : Z \to X$ be a morphism from another smooth projective variety. Let Y be a connected component of $Z \times_X \widetilde{X}$, and let $g : Y \to \widetilde{X}$ be the natural holomorphic map. Let $p : \widetilde{X} \to \mathscr{M}$ be the mixed period map, and let $\pi_Z : Y \to Z$ be the covering map. If $f^* \mathcal{L}_i$ is trivial for $i = -1, 0$, then

$$d(p \circ g) = \pi_Z^* f^* \eta^{1,0}.$$

For the semi-canonical form S_η associated with η, $\pi_Z^* f^* S_\eta$ is strictly positive at the points of Y where $p \circ g$ is immersive.

Proof. Since $f^* \mathcal{L}_i$ is trivial for $i = -1, 0$, the composite $p \circ g(Y) \subset V(H_{-1}, H_0)$, where $V(H_{-1}, H_0)$ is the fiber of the natural fibration $\mathscr{M} \to \mathscr{D}_{-1} \times \mathscr{D}_0$ at some point $H_0 \oplus H_{-1} \in \mathscr{D}_{-1} \times \mathscr{D}_0$. Here H_i is a real Hodge structure of weight i.

Fix a base point $y_0 \in Y$. Then for any $y \in Y$, the parallel transport of $\pi_Z^* f^* \nabla$ along any smooth path γ connecting y_0 and y induces a $\mathbb{R}$-linear isomorphism of $H_{-1} \oplus H_0$ given by

$$\begin{bmatrix} 1 & \int_\gamma \pi_Z^* f^* \eta \\ 0 & 1 \end{bmatrix}.$$

One can check that this does not depend on the choice of γ.

Let $\eta = \eta^{1,0} + \eta^{0,1}$ be the decomposition of η with respect to $\Omega_X^{1,0} \otimes \mathcal{L}^{-1,0} \oplus \Omega_X^{0,1} \otimes \mathcal{L}^{0,-1}$. Then $\overline{\eta^{1,0}} = \eta^{0,1}$. We have

$$p \circ g(y) - p \circ g(y_0) = \left[\int_\gamma \pi_Z^* f^* \eta \right] = \left[\int_\gamma \pi_Z^* f^* \eta^{1,0} \right],$$

where $\left[\int_\gamma \pi_Z^* f^* \eta^{1,0} \right]$ denotes the image of $\int_\gamma \pi_Z^* f^* \eta^{1,0}$ under the quotient map

$$\mathrm{Hom}(H_0, H_{-1})_{\mathbb{C}} \to \mathrm{Hom}(H_0, H_{-1})_{\mathbb{C}}/F^0 \mathrm{Hom}(H_0, H_{-1})_{\mathbb{C}} \simeq \oplus_{p \leq -1} \mathrm{Hom}(H_0, H_{-1})_{\mathbb{C}}^{p, -1-p}.$$

This implies that

$$dp \circ g = \pi_Z^* f^* \eta^{1,0}. \tag{7.8}$$

Since $f^* \mathcal{L}$ is a trivial local system, by the same proof as in Lemma 7.11, $f^* \eta \simeq (\xi_1, \dots, \xi_r)$, where each ξ_i is a real harmonic 1-form on Z. It follows that

$$f^* S_\eta = \sqrt{-1} \sum_{i=1}^r \xi^{1,0} \wedge \xi^{0,1}.$$

If $p \circ g$ is immersive at some $y \in Y$, then by (7.8), we have $\pi_Z^* f^* \eta^{1,0}(v) \neq 0$ for any non-zero $v \in T_y Y$. This is equivalent to that $f^* S_\eta$ is strictly positive at $\pi_Z(y)$. The second assertion follows. The lemma is proved. $\qquad\square$

7.5. **Proof of Theorems A and C.(ii).** We will apply the tools from the linear Shafarevich conjecture [EKPR12], combined with the strategy outlined in § 5.2, to prove Theorem C.(ii).

Proposition 7.15. *Let X be a smooth projective variety. Define $H_N \triangleleft \pi_1(X)$ as the intersection of the kernels of all* linear *representations $\pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$. Then after replacing X by a finite étale cover, Shafarevich morphism $\mathrm{sh}_{H_N} : X \to \mathrm{Sh}_{H_N}(X)$ exists, and there exist*

(a) *a family of Zariski dense representations $\{\tau_i : \pi_1(X) \to G(K_i)\}_{i=1,\dots,\ell}$ where each $G_i \subset \mathrm{GL}_N$ is a reductive group over a non-archimedean local field K_i of characteristic zero;*

(b) *a reductive representation $\varrho_1 : \pi_1(X) \to \mathrm{GL}_{N_1}(\mathbb{C})$ underlying a $\mathbb{C}$-VHS $\mathcal{L}_1$, that is a direct sum of representations $\pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$.*

(c) *A real $\mathbb{R}$-VMHS M on X, that is an extension of a $\mathbb{R}$-VHS $\mathcal{L}_0$ by another $\mathbb{R}$-VHS $\mathcal{L}_{-1}$, with the extension class denoted by $\alpha \in H^1_{\mathrm{DR}}(X, \mathcal{L}_0^* \otimes \mathcal{L}_{-1})$. The weight of $\mathcal{L}_i$ is i for $i = -1, 0$.*

(d) *Set $\mathcal{L} := \mathcal{L}_0^* \otimes \mathcal{L}_{-1}$. Let $\eta \in \mathscr{H}^1(X, \mathcal{L})$ be the harmonic representative in α. Then $\eta(x) \in \Omega_{X,x}^{1,0} \otimes \mathcal{L}^{-1,0} \oplus \Omega_{X,x}^{0,1} \otimes \mathcal{L}^{0,-1}$, with $\mathcal{L}^{p,q}$ denoting the Hodge (p,q)-subspace of $\mathcal{L}$.*

(e) *A $(1,1)$-current T on $\mathrm{Sh}_{H_N}(X)$ such that $T|_U$ is a strictly positive smooth $(1,1)$-form for some open subset U of $\mathrm{Sh}_{H_N}(X)^{\mathrm{reg}}$,*

such that for some $\varepsilon > 0$, the sum $\sum_{i=1}^{\ell} T_{\tau_i} + \omega_{\varrho_1} + \varepsilon S_\alpha = \mathrm{sh}_{H_N}^* T$. Here T_{τ_i} is the canonical current of τ_i, ω_{ϱ_1} is the canonical form of ϱ_1, and S_α is the semi-canonical form of α.

Proof. Let $M := M_{\mathrm{B}}(\pi_1(X), \mathrm{GL}_N)(\mathbb{C})$ be the character variety of $\pi_1(X)$. By [DYK23, Proof of Theorem 3.29], there exist

(a) a family of Zariski dense representations $\{\tau_i : \pi_1(X) \to G(K_i)\}_{i=1,\dots,\ell}$ where each $G_i \subset \mathrm{GL}_N$ is a reductive group over a non-archimedean local field K_i of characteristic zero;

(b) a reductive representation $\varrho_1 : \pi_1(X) \to \mathrm{GL}_{N_1}(\mathbb{C})$ underlying a $\mathbb{C}$-VHS $\mathcal{L}_1$, that is a direct sum of representations $\pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$;

such that the following properties hold: we define $\widetilde{H_M^0}$ to be the intersection of the kernels of all semisimple representations $\pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$. Denote by $\widetilde{X_M^0} := \widetilde{X}/\widetilde{H_M^0}$ and $\pi_0 : \widetilde{X_M^0} \to X$ the covering map. Then the period map of $\mathcal{L}_1$ descends to $\phi : \widetilde{X_M^0} \to \mathscr{D}_1$, where $\mathscr{D}_1$ is the period map of the $\mathbb{R}$-VHS $\mathcal{L}_1$. For the holomorphic map
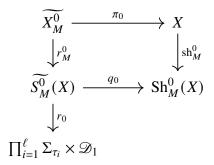
$$\Phi_0 : \widetilde{X_M^0} \to \prod_{i=1}^{\ell} \Sigma_{\tau_i} \times \mathscr{D}_1$$

$$x \mapsto (s_{\tau_1} \circ \pi_0, \dots, s_{\tau_\ell} \circ \pi_0, \phi(x)),$$

each connected component of the fiber of Φ_0 is compact. Here $s_{\tau_i} : X \to \Sigma_{\tau_i}$ is the Katzarkov-Eyssidieux reduction for τ_i (cf. [DYK23] for the definition). By [DYK23, Proof of Theorem 3.29], Φ_0 factors through

$$\widetilde{X_M^0} \xrightarrow{r_M^0} \widetilde{S_M^0}(X) \xrightarrow{r_0} \prod_{i=1}^{\ell} \Sigma_{\tau_i} \times \mathscr{D}_1$$

where r_M^0 is a proper surjective holomorphic fibration and r_0 is holomorphic map with each fiber being a discrete set. Moreover, $\widetilde{S_M^0}(X)$ does not contain compact subvarieties. Therefore, the Galois group $\mathrm{Aut}(\widetilde{X_M^0}/X)$ induces an action on $\widetilde{S_M^0}(X)$ which is properly discontinuous, and such that r_M^0 is equivariant with respect to the action by $\mathrm{Aut}(\widetilde{X_M^0}/X)$. By [DYK23, Lemma 3.28], replacing X by a finite étale cover, we can assume that such an action on $\widetilde{S_M^0}(X)$ is free.

Taking the quotient of r_M^0 by $\mathrm{Aut}(\widetilde{X_M^0}/X)$, we obtain

$$
\begin{array}{ccc}
\widetilde{X_M^0} & \xrightarrow{\ \pi_0\ } & X \\
\downarrow{\scriptstyle r_M^0} & & \downarrow{\scriptstyle \mathrm{sh}_M^0} \\
\widetilde{S_M^0}(X) & \xrightarrow{\ q_0\ } & \mathrm{Sh}_M^0(X) \\
\downarrow{\scriptstyle r_0} & & \\
\prod_{i=1}^{\ell} \Sigma_{\tau_i} \times \mathscr{D}_1 & &
\end{array}
$$

Here $\mathrm{Sh}_M^0(X)$ is called the *reductive Shafarevich morphism* associated with M.

By [EKPR12, §5.2], there is an $\mathbb{R}$-VMHS $\mathcal{M}$ of weight length 1 with the mixed period domain $\mathscr{M}$ (cf. [EKPR12, Lemma 5.4]) and an infinite Galois étale cover $\pi_1 : \widetilde{X_M^1} \to X$ (cf. [EKPR12, p. 1575] for the definition) factorizing through $\pi_0 : \widetilde{X_M^0} \to X$ such that

(a)   the mixed period domain descends to $\psi : \widetilde{X_M^1} \to \mathscr{M}$;
(b)   for the holomorphic map

$$
\Phi_1 : \widetilde{X_M^1} \to \prod_{i=1}^{\ell} \Sigma_{\tau_i} \times \mathscr{D}_1 \times \mathscr{M}
$$
$$
x \mapsto (s_{\tau_1} \circ \pi_1, \ldots, s_{\tau_\ell} \circ \pi_1, \phi(x), \psi(x)),
$$

each connected component of the fiber of Φ is compact.

Here we abusively use $\phi : \widetilde{X_M^1} \to \mathscr{D}_1$ to denote by the composite of $\phi : \widetilde{X_M^0} \to \mathscr{D}_1$ with $\widetilde{X_M^1} \to \widetilde{X_M^0}$. By [EKPR12, p. 1576], Φ_1 factors through

$$
\widetilde{X_M^1} \xrightarrow{r_M^1} \widetilde{S_M^1}(X) \xrightarrow{r_1} \prod_{i=1}^{\ell} \Sigma_{\tau_i} \times \mathscr{D}_1 \times \mathscr{M}
$$

where r_M^1 is a proper surjective holomorphic fibration and r_1 is holomorphic map with each fiber being a discrete set.

By [EKPR12, Lemma 5.7], $\widetilde{S_M^1}(X)$ does not contain compact subvarieties. Therefore, the Galois group $\mathrm{Aut}(\widetilde{X_M^1}/X)$ induces an action on $\widetilde{S_M^1}(X)$ which is properly discontinuous, and such that r_M^1 is equivariant with respect to the action by $\mathrm{Aut}(\widetilde{X_M^1}/X)$. Replacing X by a finite étale cover, we assume that such an action is free. Taking the quotient of r_M^1 by $\mathrm{Aut}(\widetilde{X_M^1}/X)$,

we obtain:

$$
\begin{array}{ccccccc}
\widetilde{X^1_M} & \xrightarrow{\;\;\pi_1\;\;} & X & = & X & \xleftarrow{\;\;\pi_0\;\;} & \widetilde{X^0_M} \\
\downarrow{r^1_M} & & \downarrow{\mathrm{sh}^1_M} & & \downarrow{\mathrm{sh}^0_M} & & \downarrow{r^0_M} \\
\widetilde{S^1_M}(X) & \xrightarrow[\text{étale}]{q_1} & \mathrm{Sh}^1_M(X) & \xrightarrow{g} & \mathrm{Sh}^0_M(X) & \xleftarrow[\text{étale}]{q_0} & \widetilde{S^0_M}(X) \\
\downarrow{r_1} & & & & & & \downarrow{r_0} \\
\prod_{i=1}^{\ell} \Sigma_{\tau_i} \times \mathscr{D}_1 \times \mathscr{M} & & & & & & \prod_{i=1}^{\ell} \Sigma_{\tau_i} \times \mathscr{D}_1
\end{array}
\qquad (7.9)
$$

with Φ_1 on the left and Φ_0 on the right.

By [EKPR12, p. 1549 & Proposition 3.10], Sh^1_M is the Shafarevich morphism associated with (X, H_N). We apply Selberg's theorem to replace X by a further finite étale cover such that the monodromy representation of $\mathscr{M}$ has torsion free image. By the construction of sh^1_M, for each fiber Z of sh^1_M, $\mathscr{M}|_Z$ has finite, thus trivial monodromy. Then $\mathscr{M}$ descends to a $\mathbb{R}$-VMHS of weight length 1 on Sh^1_M, which we denote by $\mathscr{M}'$.

Let Z be an irreducible component of any fiber of $g : \mathrm{Sh}^1_M(X) \to \mathrm{Sh}^0_M(X)$. Let $\widetilde{Z^1_M}$ be any connected component of the inverse image $q_1^{-1}(Z)$. Let $\mathscr{D}_{-1} \times \mathscr{D}_0$ be the graded period domain of $\mathscr{M}$. Note that the natural map $\mathscr{M} \to \mathscr{D}_{-1} \times \mathscr{D}_0$ is a holomorphic vector bundle (cf. [Car87]). Then there exists some $P \in \mathscr{D}_{-1} \times \mathscr{D}_0$ such that for the fiber V of $\mathscr{M} \to \mathscr{D}_{-1} \times \mathscr{D}_0$ at $P \in \mathscr{D}_{-1} \times \mathscr{D}_0$, we have $\psi|_{\widetilde{Z^1_M}} : \widetilde{Z^1_M} \to V$. Moreover, $q := \psi|_{\widetilde{Z^1_M}}$ factors as

$$
\begin{array}{ccc}
\widetilde{Z^1_M}^{\,\mathrm{sn}} & \xrightarrow{\;p\;} & V' \\
\downarrow & \searrow & \downarrow \\
\widetilde{Z^1_M} & \xrightarrow{\;q\;} & V
\end{array}
$$

where $\widetilde{Z^1_M}^{\,\mathrm{sn}}$ is the semi-normalization of $\widetilde{Z^1_M}$, $V' \to V$ is a linear injective map from a complex vector space V' and p is a proper holomorphic map (cf. [EKPR12, p. 1576]). Since r_1 has discrete fibers, and the image of $\widetilde{Z^1_M}$ under the composite map

$$
\widetilde{S^1_M}(X) \xrightarrow{r_1} \prod_{i=1}^{\ell} \Sigma_{\tau_i} \times \mathscr{D}_1 \times \mathscr{M} \to \prod_{i=1}^{\ell} \Sigma_{\tau_i} \times \mathscr{D}_1
$$

is constant, it follows that p is a finite map. Here $\prod_{i=1}^{\ell} \Sigma_{\tau_i} \times \mathscr{D}_1 \times \mathscr{M} \to \prod_{i=1}^{\ell} \Sigma_{\tau_i} \times \mathscr{D}_1$ is the natural projection map.

By [EKPR12, Example 2.2.2], the $\mathbb{R}$-VMHS $\mathscr{M}$ satisfies the conditions in Lemma 7.14. Let $\alpha \in H^1_{\mathrm{DR}}(X, \mathcal{L}_0^* \otimes \mathcal{L}_{-1})$ be the extension class defining $\mathscr{M}$, where $\mathcal{L}_{-1} \oplus \mathcal{L}_0$ is the graded $\mathbb{R}$-VHS of $\mathscr{M}$. Let S_α be the semi-canonical form associated with α defined in (7.7). Since $\mathscr{M} = (\mathrm{sh}^1_M)^* \mathscr{M}'$, there exists a real $(1,1)$-form S'_α on $\mathrm{Sh}^1_M(X)$ such that $S_\alpha = (\mathrm{sh}^1_M)^* S'_\alpha$. Then by Lemma 7.14 $S'_\alpha|_Z$ is closed and semi-positive and is strictly positive over the locus where $q : \widetilde{Z^1_M} \to V$ is immersive.

By § 5.2, there exists a semi-positive closed $(1,1)$-current T_0 with continuous potential on $\mathrm{Sh}^0_M(X)$ such that

(1) One has

$$T_{\tau_1} + \cdots + T_{\tau_\ell} + \omega_{\varrho_1} = (\mathrm{sh}_M^0)^* T_0$$

(2) $T_0|_\Omega$ is smooth and strictly positive for some open subset $\Omega \subset \mathrm{Sh}_M^0(X)^{\mathrm{reg}}$.

Here T_{τ_i} is the canonical current for τ_i defined in Definition 5.1, and $\omega_{\mathcal{L}}$ is the canonical form associated with the representation induced by the $\mathbb{C}$-VHS $\mathcal{L}$.

Let Z be a general fiber of $g : \mathrm{Sh}_M^1(X) \to \mathrm{Sh}_M^0(X)$ over Ω. Then $S'_\alpha|_Z$ is semi-positive and strictly positive at general points of Z. This implies that, for a general point $z \in Z$, it has a neighborhood $U \subset \mathrm{Sh}_M^1(X)^{\mathrm{reg}}$ such that $(g^*T_0 + \varepsilon S'_\alpha)|_U$ is a strictly positive smooth $(1,1)$-form (but not necessarily d-closed). Let $T := g^*T_0 + \varepsilon S'_\alpha$, which is the desired $(1,1)$-current in the proposition. The proof is accomplished. □

Theorem 7.16. *Let $f : \mathcal{X} \to \mathbb{D}$ be a smooth projective family.*

(i) *If there exists an almost faithful linear representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$, then $t \mapsto \gamma d(X_t)$ is a lower semicontinuous function on $\mathbb{D}$.*

(ii) *If there exists a big representation $\sigma : \pi_1(X_0) \to \mathrm{GL}_N(\mathbb{C})$, then X_t has big fundamental group for sufficiently small t.*

Proof. We write X for X_0. We now apply Proposition 7.15 and use the notations therein without recalling it. Then after replacing $\mathcal{X}$ by some suitable finite étale cover, there exist

(a) a family of Zariski dense representations $\{\tau_i : \pi_1(X) \to G_i(K_i)\}_{i=1,\ldots,\ell}$ where each $G_i \subset \mathrm{GL}_N$ is a reductive group over a non-archimedean local field K_i of characteristic zero,

(b) a reductive representation $\varrho_1 : \pi_1(X) \to \mathrm{GL}_{N_1}(\mathbb{C})$ underlying a $\mathbb{C}$-VHS $\mathcal{L}_1$, that is a direct sum of representations $\pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$.

(c) A real $\mathbb{R}$-VMHS $\mathcal{M}$ on X with torsion free monodromy, that is an extension of a $\mathbb{R}$-VHS $\mathcal{L}_0$ by another $\mathbb{R}$-VHS $\mathcal{L}_{-1}$, with the extension class denoted by $\alpha \in H^1_{\mathrm{DR}}(X, \mathcal{L}_0^* \otimes \mathcal{L}_{-1})$. The weight of $\mathcal{L}_i$ is i for $i = -1, 0$. Let r_N be the rank of $\mathcal{M}$.

(d) Set $\mathcal{L} := \mathcal{L}_0^* \otimes \mathcal{L}_{-1}$. Let $\eta \in \mathscr{H}^1(X, \mathcal{L})$ be the harmonic representative in α. Then $\eta(x) \in \Omega_{X,x}^{1,0} \otimes \mathcal{L}^{-1,0} \oplus \Omega_{X,x}^{0,1} \otimes \mathcal{L}^{0,-1}$ for any $x \in X$, with $\mathcal{L}^{p,q}$ denoting the Hodge (p,q)-subspace of $\mathcal{L}$.

(e) A $(1,1)$-current T on $\mathrm{Sh}_{H_N}(X)$ such that $T|_U$ is a strictly positive smooth $(1,1)$-form for some open subset U of $\mathrm{Sh}_{H_N}(X)^{\mathrm{reg}}$, and for some $\varepsilon > 0$, one has $\sum_{i=1}^\ell T_{\tau_i} + \omega_{\varrho_1} + \varepsilon S_\alpha = \mathrm{sh}_{H_N}^* T$.

Here T_{τ_i} is the canonical current of τ_i, ω_{ϱ_1} is the canonical form of ϱ_1, and S_α is the semi-canonical form of α.

Let $F_t : X_t \to X$ be the natural diffeomorphism induced by the C^∞ trivialization $\mathcal{X} \to X \times \mathbb{D}$. Define $\mathcal{L}_{i,t} := F_t^*(\mathcal{L}_i)$ as the local systems on X_t for $i = -1, 0, 1$. Let α_t denote the image of α under the natural isomorphism

$$F_t^* : H^1_{\mathrm{DR}}(X, \mathcal{L}_0^* \otimes \mathcal{L}_{-1}) \to H^1_{\mathrm{DR}}(X_t, \mathcal{L}_{0,t}^* \otimes \mathcal{L}_{-1,t}).$$

Note that α_t is the extension class defining $\mathcal{M}_t := F_t^* \mathcal{M}$, which is the extension of $\mathcal{L}_{0,t}$ by $\mathcal{L}_{-1,t}$.

We apply Lemma 7.1 and Theorem 7.7 to replace $\mathcal{X}$ by a finite étale cover such that for the semi-simple local system $\mathcal{V}$ on $\mathcal{X}$ induced by $\mathcal{L}_0^* \otimes \mathcal{L}_{-1}$, there is a metric h for $\mathcal{V}$ such that

$h|_{\mathscr{V}|_{X_t}}$ varies smoothly in t. Note that $\mathscr{V}|_{X_t} = \mathcal{L}_{0,t}^* \otimes \mathcal{L}_{-1,t}$. By Lemma 7.12, the semi-canonical form S_{α_t} associated with α_t vary smoothly in t.

Set $m := \dim \mathrm{Sh}_{H_N}(X)$. By Item (e), there exists an open subset U_0 of X such that $(\sum_{i=1}^{\ell} T_{\tau_i} + \omega_{\varrho_1} + \varepsilon S_\alpha)|_{U_0}$ is m-positive in the sense of Definition 5.5. By Claim 5.7, there exists an open subset Ω of $\mathscr{X}$ such that for some $\delta > 0$, one has $\Omega_t := \Omega \cap X_t \neq \varnothing$ for $t \in \mathbb{D}_\delta$, and

$$(\sum_{i=1}^{\ell} T_{\tau_{i,t}} + \omega_{\varrho_{1,t}} + \varepsilon S_{\alpha,t})|_{\Omega_t}$$

is at least m-positive. Here $T_{\tau_{i,t}}$ is the canonical current of $\tau_{i,t} : \pi_1(X_t) \to G_i(K_i)$, $\omega_{\varrho_{1,t}}$ is the canonical form of $\varrho_{1,t} : \pi_1(X_t) \to \mathrm{GL}_{N_1}(\mathbb{C})$.

Denote by $H_{N,t}$ the intersection of kernels of all linear representation $\pi_1(X_t) \to \mathrm{GL}_N(\mathbb{C})$. Consider the Shafarevich morphism $\mathrm{sh}_{H_{N,t}} : X_t \to \mathrm{Sh}_{H_{N,t}}(X_t)$, whose existence is ensured by Proposition 7.15. By the arguments below Claim 5.7, for any smooth fiber Z of $\mathrm{sh}_{H_{N,t}}$, $(\sum_{i=1}^{\ell} T_{\tau_{i,t}} + \omega_{\varrho_{1,t}})|_Z$ vanishes identically.

Proof of Theorem 7.16.(i): Since there exists an almost faithful linear representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$, it follows that $\varrho_t : \pi_1(X_t) \to \mathrm{GL}_N(\mathbb{C})$ is also almost faithful. Then $H_{N,t}$ is finite, which implies that $\mathrm{sh}_{H_{N,t}} : X_t \to \mathrm{Sh}_{H_{N,t}}(X_t)$ is the Shafarevich morphism of X_t for each $t \in \mathbb{D}$. Therefore, for each fiber Z of $\mathrm{sh}_{H_{N,t}}$, $\mathrm{Im}[\pi_1(Z) \to \pi_1(X_t)]$ is finite. It follows that $\mathcal{M}_t|_Z$ has finite, thus trivial monodromy as $\mathcal{M}$ has torsion free monodromy.

Recall that $\alpha_t \in H^1(X_t, \mathcal{L}_{0,t}^* \otimes \mathcal{L}_{-1,t})$ is the extension class defining $\mathcal{M}_t := F_t^*\mathcal{M}$, that is the extension of $\mathcal{L}_{0,t}$ by $\mathcal{L}_{-1,t}$. Thus, $\alpha_t|_Z$ is also trivial. Hence, by (7.7), $S_{\alpha_t}|_Z$ is trivial for each smooth fiber Z of $\mathrm{sh}_{H_{N,t}}$.

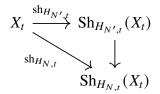Fix any $t \in \mathbb{D}_\delta$. We choose an open subset U_t of $\mathrm{Sh}_{H_{N,t}}(X_t)^{\mathrm{reg}}$ such that

(1) $\mathrm{sh}_{H_{N,t}}^{-1}(U_t) \to U_t$ is a proper holomorphic submersion.
(2) $W := \Omega_t \cap \mathrm{sh}_{H_{N,t}}^{-1}(U_t)$ is non-empty.

It follows that, $(\sum_{i=1}^{\ell} T_{\tau_{i,t}} + \omega_{\varrho_{1,t}} + \varepsilon S_{\alpha,t})|_W$ is at most m_t-positive, where $m_t := \dim \mathrm{Sh}_{H_{N,t}}(X_t)$. In conclusion, we have $m_t \geq m$. This yields $\dim \mathrm{Sh}_{H_{N,t}}(X_t) \geq \dim \mathrm{Sh}_{H_{N,0}}(X_0)$ for any $t \in \mathbb{D}_\delta$. Recall that $\mathrm{sh}_{H_{N,t}}$ is the Shafarevich morphism of X_t. This implies that $\gamma d(X_t) \geq \gamma d(X_0)$ for any $t \in \mathbb{D}_\delta$. Theorem 7.16.(i) is proved.

Proof of Theorem 7.16.(ii): Set $N' := r_N + N$, where r_N is the rank of the $\mathbb{R}$-VMHS $\mathcal{M}$ in Item (c). Let $H_{N',t}$ be the intersection of the kernels of all linear representations $\pi_1(X_t) \to \mathrm{GL}_{N'}(\mathbb{C})$. Define a representation $\sigma' : \pi_1(X) \to \mathrm{GL}_{N'}(\mathbb{C})$, which is the direct sum of σ and the trivial representation $\pi_1(X) \to \mathrm{GL}_{r_N}(\mathbb{C})$. By our definition of $H_{N'}$, we have $H_{N'} \subset \ker \sigma' = \ker \sigma$.

Since σ is big, it follows that $\mathrm{sh}_{H_{N'}} : X \to \mathrm{Sh}_{H_{N'}}(X)$ and $\mathrm{sh}_{H_N} : X \to \mathrm{Sh}_{H_N}(X)$ are both birational morphisms. The same argument shows that $H_{N',t} \subset H_{N,t}$ for each $t \in \mathbb{D}$.

Thus, for each t, the following factorization map holds, due to the properties of the Shafarevich morphism and the universal property of Stein factorization:

$$X_t \xrightarrow{\mathrm{sh}_{H_{N',t}}} \mathrm{Sh}_{H_{N',t}}(X_t)$$

(with $\mathrm{sh}_{H_{N,t}}$ and diagonal/vertical arrows to $\mathrm{Sh}_{H_{N,t}}(X_t)$)

Note that we might have $\dim \mathrm{Sh}_{H_{N',t}}(X_t) < \dim \mathrm{Sh}_{H_{N,t}}(X_t)$ for some t.

Let $\varrho_{\mathcal{M}} : \pi_1(X) \to \mathrm{GL}_{r_N}(\mathbb{C})$ be the monodromy representation of $\mathcal{M}$, which has a torsion-free image. Let $\varrho_{\mathcal{M},t} : \pi_1(X_t) \to \mathrm{GL}_{r_N}(\mathbb{C})$ be the induced representation via the diffeomorphism $F_t : X_t \to X$. Then we have $H_{N',t} \subset \ker \varrho_{\mathcal{M},t}$. This implies that for each fiber Z of $\mathrm{sh}_{H_{N',t}}$, the image $\varrho_{\mathcal{M},t}(\mathrm{Im}[\pi_1(Z) \to \pi_1(X_t)])$ is finite, and thus trivial.

Since $\alpha_t \in H^1(X_t, \mathcal{L}_{0,t}^* \otimes \mathcal{L}_{-1,t})$ is the extension class defining the local system $\mathcal{M}_t := F_t^* \mathcal{M}$, corresponding to $\varrho_{\mathcal{M},t}$ as an extension of $\mathcal{L}_{0,t}$ by $\mathcal{L}_{-1,t}$, we deduce that $\alpha_t|_Z$ is trivial. By (7.7), $S_{\alpha_t}|_Z$ is also trivial for each smooth fiber Z of $\mathrm{sh}_{H_{N',t}}$. In conclusion, for each smooth fiber Z of $\mathrm{sh}_{H_{N',t}}$,

$$\left(\sum_{i=1}^{\ell} T_{\tau_{i,t}} + \omega_{\varrho_{1,t}} + \varepsilon S_{\alpha_t} \right) \Big|_Z \text{ is trivial.} \tag{7.10}$$

Fix any $t \in \mathbb{D}_\delta$. We choose an open subset U_t of $\mathrm{Sh}_{H_{N',t}}(X_t)^{\mathrm{reg}}$ such that:

(1) $\mathrm{sh}_{H_{N',t}}^{-1}(U_t) \to U_t$ is a proper holomorphic submersion.

(2) The open subset $W := \Omega_t \cap \mathrm{sh}_{H_{N',t}}^{-1}(U_t)$ is non-empty.

Then by (7.10), $\left(\sum_{i=1}^{\ell} T_{\tau_{i,t}} + \omega_{\varrho_{1,t}} + \varepsilon S_{\alpha_t} \right)\Big|_W$ is at most m_t'-positive, where $m_t' := \dim \mathrm{Sh}_{H_{N',t}}(X_t)$. Recall that

$$(\sum_{i=1}^{\ell} T_{\tau_{i,t}} + \omega_{\varrho_{1,t}} + \varepsilon S_{\alpha,t})|_{\Omega_t}$$

is at least m-positive, where $m := \dim \mathrm{Sh}_{H_N}(X) = \dim X$. This implies that $m_t' = m = \dim X_t$. Hence $\mathrm{sh}_{H_{N',t}}$ is a birational morphism for each $t \in \mathbb{D}_\delta$. This proves that X_t has a big fundamental group for $t \in \mathbb{D}_\delta$. Theorem 7.16.(ii) is proved.

$\square$

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