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(with an appendix by Dan Abramovich)

On the hyperbolicity of base spaces for maximally variational families of smooth projective varieties

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Abstract. For maximal variational smooth families of projective manifolds whose general fibers have semi-ample canonical bundle, the Viehweg hyperbolicity conjecture states that the base spaces of such families are of log-general type. This deep conjecture was recently proved by Campana–Păun and was later generalized by Popa–Schnell. In this paper we prove that those base spaces are pseudo Kobayashi hyperbolic, as predicted by the Lang conjecture: any complex quasi-projective manifold is pseudo Kobayashi hyperbolic if it is of log-general type. As a consequence, we prove the Brody hyperbolicity of moduli spaces of polarized manifolds with semi-ample canonical bundle. This proves a 2003 conjecture by Viehweg–Zuo. We also prove the Kobayashi hyperbolicity of base spaces for effectively parametrized families of minimal projective manifolds of general type. This generalizes previous work by To–Yeung, who further assumed that these families are canonically polarized.

Keywords. Pseudo Kobayashi hyperbolicity, Brody hyperbolicity, moduli spaces, Viehweg–Zuo question, polarized variation of Hodge structures, Viehweg–Zuo Higgs bundles, Finsler metric, positivity of direct images, Griffiths curvature formula for Hodge bundles

0. Introduction

0.1. Main theorems

A complex space X is *Brody hyperbolic* if there is no non-constant holomorphic map $\gamma : \mathbb{C} \rightarrow X$. The first result in this paper is the affirmative answer to a conjecture by Viehweg–Zuo [63, Question 0.2] on the Brody hyperbolicity of moduli spaces for polarized manifolds with semi-ample canonical sheaf.

Theorem A (Brody hyperbolicity of moduli spaces). *Consider the moduli functor \mathcal{P}_h of polarized manifolds with semi-ample canonical sheaf introduced by Viehweg [61, §7.6],*

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where h is the Hilbert polynomial associated to the polarization \mathcal{H} . Assume that for some quasi-projective manifold V there exists a smooth family $(f_U : U \rightarrow V, \mathcal{H}) \in \mathcal{P}_h(V)$ for which the induced moduli map $\varphi_U : V \rightarrow P_h$ is quasi-finite over its image, where P_h denotes the quasi-projective¹ coarse moduli scheme for \mathcal{P}_h . Then the base space V is Brody hyperbolic.

A complex space X is called *pseudo Kobayashi hyperbolic* if X is hyperbolic modulo a proper Zariski closed subset $\Delta \subsetneq X$, that is, the Kobayashi pseudo-distance $d_X : X \times X \rightarrow [0, +\infty[$ of X satisfies $d_X(p, q) > 0$ for any distinct $p, q \in X$ not both in Δ . In particular, X is *pseudo Brody hyperbolic*: any non-constant holomorphic map $\gamma : \mathbb{C} \rightarrow X$ has image $\gamma(\mathbb{C}) \subset \Delta$. When Δ is an empty set, this definition reduces to the usual definition of *Kobayashi hyperbolicity*, and the Kobayashi pseudo-distance d_X is a distance.

In this paper we indeed prove a stronger result than Theorem A.

Theorem B. *Let $f_U : U \rightarrow V$ be a smooth projective morphism between complex quasi-projective manifolds with connected fibers. Assume that the general fiber of f_U has semi-ample canonical bundle, and f_U is of maximal variation, that is, the general fiber of f_U can only be birational to at most countably many other fibers. Then the base space V is pseudo Kobayashi hyperbolic.*

As a byproduct, we reduce the pseudo Kobayashi hyperbolicity of varieties to the existence of certain negatively curved Higgs bundles (which we call *Viehweg–Zuo Higgs bundles* in Definition 1.1). This provides a main building block for our recent work [26] on the hyperbolicity of bases of log Calabi–Yau pairs.

Another aim of the paper is to confirm a folklore conjecture on the *Kobayashi hyperbolicity* for moduli spaces of minimal projective manifolds of general type, which can be thought of as an analytic refinement of Theorem A in the case when fibers have big and nef canonical bundle.

Theorem C. *Let $f_U : U \rightarrow V$ be a smooth projective family of minimal projective manifolds of general type over a quasi-projective manifold V . Assume that f_U is effectively parametrized, that is, the Kodaira–Spencer map*

$$\rho_y : \mathcal{T}_{V,y} \rightarrow H^1(U_y, \mathcal{T}_{U_y}) \quad (0.1.1)$$

is injective for each point $y \in V$, where \mathcal{T}_{U_y} denotes the tangent bundle of the fiber $U_y := f_U^{-1}(y)$. Then the base space V is Kobayashi hyperbolic.

0.2. Previous related results

Theorem B is closely related to the *Viehweg hyperbolicity conjecture*: if $f_U : U \rightarrow V$ is a maximally variational smooth projective family of projective manifolds with semi-ample canonical bundle over a quasi-projective manifold V , then the base V must be of log-general type. In the series of works [28, 62, 63], Viehweg–Zuo constructed in a first

¹The quasi-projectivity of P_h was proved by Viehweg [61].

step a big subsheaf of symmetric log differential forms of the base (so-called *Viehweg–Zuo sheaves*). Building on this result, the Viehweg hyperbolicity conjecture was shown by Kebekus–Kovács [35–37] when V is a surface or threefold, by Patakfalvi [44] when V is compact or admits a non-uniruled compactification, and it was completely solved by Campana–Păun [16], who proved a vast generalization of the famous generic semipositivity result of Miyaoka (see also [14, 15, 52] for other proofs). More recently, using a deep theory of Hodge modules, Popa–Schnell [47] constructed Viehweg–Zuo sheaves on the base space V of the smooth family $f_U : U \rightarrow V$ of projective manifolds whose geometric generic fiber admits a good minimal model. Combining this with the aforementioned theorem of Campana–Păun, they proved that such base space V is of log-general type. Therefore, Theorem B is predicted by a famous conjecture of Lang (cf. [42, Chapter VIII, Conjecture 1.4]), which stipulates that a complex quasi-projective manifold is pseudo Kobayashi hyperbolic if and only if it is of log-general type. To our knowledge, Lang’s conjecture is by now known for the trivial case of curves, for general hypersurfaces X in the complex projective space $\mathbb{C}P^n$ of high degrees [10, 22, 56] as well as their complements $\mathbb{C}P^n \setminus X$ [11], for projective manifolds whose universal cover carries a bounded strictly plurisubharmonic function [9], for quotients of bounded (symmetric) domains [12, 13, 49], and for subvarieties of abelian varieties [65]. Theorem B therefore provides some new evidence for Lang’s conjecture.

Theorem A was first proved by Viehweg–Zuo [63, Theorem 0.1] for moduli spaces of *canonically polarized* manifolds. Combining the approaches by Viehweg–Zuo [63] with those by Popa–Schnell [47], very recently Popa–Taji–Wu [48, Theorem 1.1] proved Theorem A for moduli spaces of *polarized* manifolds with big and semi-ample canonical bundles. As we will see below, our work owes a lot to the general strategies and techniques in their work [48, 63].

The Kobayashi hyperbolicity of the moduli spaces \mathcal{M}_g of compact Riemann surfaces of genus $g \geq 2$ has long been known owing to the work of Royden and Wolpert [50, 64]. The first important breakthrough on higher-dimensional generalizations was made by To–Yeung [58], who proved Kobayashi hyperbolicity of the base V considered in Theorem C when the canonical bundle K_{U_y} of each fiber $U_y := f_U^{-1}(y)$ of $f_U : U \rightarrow V$ is further assumed to be ample (see also [7, 54] for alternative proofs). Differently from the approaches in [48, 63], their strategy is to study the curvature of the generalized Weil–Petersson metric for families of canonically polarized manifolds, along the approaches initiated by Siu [55] and later developed by Schumacher [53]. For smooth families of Calabi–Yau manifolds (resp. orbifolds), Berndtsson–Păun–Wang [7] and Schumacher [54] (resp. To–Yeung [59]) proved the Kobayashi hyperbolicity of the base once the family is assumed to be effectively parametrized.

Recently, Lu, Sun, Zuo and the author [29] proved a big Picard-type theorem for moduli spaces of polarized manifolds with semi-ample canonical sheaf. A crucial step of the proof relies on the “generic local Torelli-type theorem” in Theorem D. Theorem D also inspired us a lot in our more recent work [27] on the big Picard theorem for varieties admitting variations of Hodge structure.

0.3. Strategy of the proof

For the smooth family $f_U : U \rightarrow V$ of canonically polarized manifolds with maximal variation, Viehweg–Zuo [63] constructed certain negatively twisted Higgs bundles (which we call *Viehweg–Zuo Higgs bundles* in Definition 1.1) $(\tilde{\mathcal{E}}, \tilde{\theta}) := (\bigoplus_{q=0}^n \mathcal{L}^{-1} \otimes E^{n-q,q}, \bigoplus_{q=0}^n \mathbb{1} \otimes \theta_{n-q,q})$, over some smooth projective compactification Y of a certain birational model \tilde{V} of V , where \mathcal{L} is some big and nef line bundle on Y , and $(\bigoplus_{q=0}^n E^{n-q,q}, \bigoplus_{q=0}^n \theta_{n-q,q})$ is a Higgs bundle induced by a polarized variation of Hodge structure defined over a Zariski open set of \tilde{V} . In a recent paper [48], Popa–Taji–Wu introduced several new inputs to develop Viehweg–Zuo’s strategy in [63], which enables them to construct those Higgs bundles on base spaces of smooth families whose geometric generic fiber admits a good minimal model (see also Theorem 1.2 for a weaker statement as well as a slightly different proof following the original construction by Viehweg–Zuo). As we will see in the main content, Viehweg–Zuo Higgs bundles (VZ Higgs bundles for short) are crucial tools in proving our main results.

When all fibers $U_y := f_U^{-1}(y)$ of the smooth family $f_U : U \rightarrow V$ considered in Theorem B have ample or big and nef canonical bundles, let us briefly recall the general strategy of proving the *pseudo Brody hyperbolicity* of V in [48, 63]. A certain Higgs subbundle (\mathcal{F}, η) of $(\tilde{\mathcal{E}}, \tilde{\theta})$ with log poles contained in the divisor $D := Y \setminus \tilde{V}$ gives rise to a morphism

$$\tau_{\gamma,k} : \mathcal{T}_{\mathbb{C}}^{\otimes k} \rightarrow \gamma^*(\mathcal{L}^{-1} \otimes E^{n-k,k}) \quad (0.3.1)$$

for any entire curve $\gamma : \mathbb{C} \rightarrow \tilde{V}$. If $\gamma : \mathbb{C} \rightarrow \tilde{V}$ is Zariski dense, then by the Kodaira–Nakano vanishing (when K_{U_y} is ample) and the Bogomolov–Sommese vanishing theorems (when K_{U_y} is big and nef), one can verify that $\tau_{\gamma,1}(\mathbb{C}) \neq 0$. Hence there is some $m > 0$ (depending on γ) such that $\tau_{\gamma,m}$ factors through $\gamma^*(\mathcal{L}^{-1} \otimes N^{n-m,m})$, where $N^{n-m,m}$ is the kernel of the Higgs field $\theta_m : E^{n-m,m} \rightarrow E^{n-m-1,m+1} \otimes \Omega_Y(\log D)$. Applying Zuo’s theorem [67] on the negativity of $N^{n-m,m}$, a certain positively curved metric for \mathcal{L} can produce a singular hermitian metric on $\mathcal{T}_{\mathbb{C}}$ with *Gaussian curvature* bounded from above by a negative constant, which contradicts the (Demailly’s) Ahlfors–Schwarz lemma [21, Lemma 3.2]. However, this approach did not provide enough information on the Kobayashi pseudo-distance of the base V . Moreover, the use of a vanishing theorem cannot show $\tau_{\gamma,1}(\mathbb{C}) \neq 0$ when fibers of $f_U : U \rightarrow V$ are not minimal manifolds of general type.

One of the main results in the present paper is to apply the VZ Higgs bundle to construct a (possibly degenerate) Finsler metric F on some birational model \tilde{V} of the base V , whose holomorphic sectional curvature is bounded above by a negative constant (*negatively curved Finsler metric* in Definition 2.3(ii)). A bimeromorphic criterion for pseudo Kobayashi hyperbolicity in Lemma 2.4 states that the base is pseudo Kobayashi hyperbolic if F is *positive definite* over a Zariski dense open set. Let us now briefly explain our idea of the constructions. By factorizing through some Higgs subsheaf $(\mathcal{F}, \eta) \subseteq (\tilde{\mathcal{E}}, \tilde{\theta})$ with logarithmic poles *only* along the boundary divisor $D := Y \setminus \tilde{V}$, one can define, for

any $k = 1, \dots, n$, a morphism

$$\tau_k : \text{Sym}^k \mathcal{T}_Y(-\log D) \rightarrow \mathcal{L}^{-1} \otimes E^{n-k,k}, \quad (0.3.2)$$

where \mathcal{L} is some big line bundle over Y equipped with a *positively curved* singular hermitian metric $h_{\mathcal{L}}$. Then for each k , the hermitian metric h_k on $\tilde{\mathcal{E}}_k := \mathcal{L}^{-1} \otimes E^{n-k,k}$ induced by the Hodge metric as well as $h_{\mathcal{L}}$ (see Proposition 1.3 for details) will give rise to a Finsler metric F_k on $\mathcal{T}_Y(-\log D)$ by taking the k -th root of the pull-back $\tau_k^* h_k$. However, the holomorphic sectional curvature of F_k might not be negatively curved. Inspired by the aforementioned work of Schumacher, To–Yeung and Berndtsson–Päun–Wang [7, 53, 54, 58] on curvature computations for generalized Weil–Petersson metrics on families of canonically polarized manifolds, we define a convex sum of Finsler metrics

$$F := \left(\sum_{k=1}^n \alpha_k F_k^2 \right)^{1/2} \quad \text{with } \alpha_1, \dots, \alpha_n \in \mathbb{R}^+ \quad (0.3.3)$$

on $\mathcal{T}_Y(-\log D)$, to offset the unwanted positive terms in the curvature $\Theta_{\tilde{\mathcal{E}}_k}$ by negative contributions from the $\Theta_{\tilde{\mathcal{E}}_{k+1}}$ (the last order term $\Theta_{\tilde{\mathcal{E}}_n}$ is always semi-negative by the Griffiths curvature formula). We prove in Proposition 2.14 that for proper $\alpha_1, \dots, \alpha_n > 0$, the holomorphic sectional curvature of F is negative and bounded away from zero. To summarize, we establish an *algorithm* for the construction of Finsler metrics via VZ Higgs bundles.

To prove Theorem B, we first note that VZ Higgs bundles over some birational model \tilde{V} of the base space V were constructed by Popa–Taji–Wu [48]. Let Y be some smooth projective compactification \tilde{V} with simple normal crossing boundary $D := Y \setminus \tilde{V}$. By our construction of a negatively curved Finsler metric F defined in (0.3.3) via VZ Higgs bundles, to show that F is *positive definite* over some Zariski open set, it suffices to prove that $\tau_1 : \mathcal{T}_Y(-\log D) \rightarrow \mathcal{L}^{-1} \otimes E^{n-1,1}$ defined in (0.3.2) is *generically injective* (which we call *generic local Torelli* for VZ Higgs bundles in §1.1). This is proved in Theorem D, by using the degeneration of the Hodge metric and the curvature properties of Hodge bundles. In particular, we show that the generic injectivity of τ_1 is indeed an intrinsic feature of all VZ Higgs bundles (unrelated to the Kodaira dimension of fibers of f !). By a standard inductive argument in [48, 63], one can easily show that Theorem B implies Theorem A.

Now we will explain the strategy to prove Theorem C. Note that VZ Higgs bundles are only constructed over some birational model \tilde{V} of V , which is not Kobayashi hyperbolic in general. This motivates us first to establish a *bimeromorphic criterion for Kobayashi hyperbolicity* in Lemma 2.5. Based on this criterion, in order to apply VZ Higgs bundles to prove the Kobayashi hyperbolicity of the base V in Theorem C, it suffices to show that

- ♠ for any given point y on the base V , there exists a VZ Higgs bundle $(\tilde{\mathcal{E}}, \tilde{\theta})$ constructed over some birational model $\nu : \tilde{V} \rightarrow V$ such that $\nu^{-1} : V \dashrightarrow \tilde{V}$ is defined at y ;
- ♣ the negatively curved Finsler metric F on \tilde{V} defined in (0.3.3) induced by $(\tilde{\mathcal{E}}, \tilde{\theta})$ is positive definite at the point $\nu^{-1}(y)$.

Roughly speaking, the idea is to produce an abundant supply of *fine* VZ Higgs bundles to construct sufficiently many negatively curved Finsler metrics, which are obstructions to the degeneracy of Kobayashi pseudo-distance d_V of V . This is much more demanding than the Brody hyperbolicity and Viehweg hyperbolicity of V , which can be shown by the existence of *only one* VZ Higgs bundle on an arbitrary birational model of V , as mentioned in [28, 47, 48, 63].

Let us briefly explain how we achieve both \spadesuit and \clubsuit .

As far as we see in [48, 63], in their construction of VZ Higgs bundles, one has to blow up the base several times (indeed twice). Recall that the basic setup in [48, 63] is the following: after passing to some smooth birational model $f_{\tilde{U}} : \tilde{U} = U \times_V \tilde{V} \rightarrow \tilde{V}$ of $f_U : U \rightarrow V$, one can find a smooth projective compactification $f : X \rightarrow Y$ of $\tilde{U}^r \rightarrow \tilde{V}$,

$$\begin{array}{ccccc}
 U^r & \xleftarrow{\text{bir}} & \tilde{U}^r & \xrightarrow{\subseteq} & X \\
 \downarrow & & \downarrow & & \downarrow f \\
 V & \xleftarrow[\nu]{\text{bir}} & \tilde{V} & \xrightarrow{\subseteq} & Y
 \end{array} \tag{0.3.4}$$

so that there exists (at least) one hypersurface

$$H \in |\ell K_{X/Y} - \ell f^* \mathcal{L}| \quad \text{for some } \ell \gg 0 \tag{0.3.5}$$

which is *transverse* to the general fibers of f . Here \mathcal{L} is some big and nef line bundle over Y , and $U^r := U \times_V \cdots \times_V U$ (resp. \tilde{U}^r) is the r -fold fiber product of $f_U : U \rightarrow V$ (resp. $f_{\tilde{U}} : \tilde{U} \rightarrow \tilde{V}$). The VZ Higgs bundle is indeed the logarithmic Higgs bundle associated to the Hodge filtration of an auxiliary variation of polarized Hodge structure constructed by taking the middle-dimensional relative de Rham cohomology on the cyclic cover of X ramified along H .

In order to find an H as in (0.3.5), a crucial step in [48, 63] is the use of *weakly semi-stable reduction* by Abramovich–Karu [2] so that, after changing the birational model $U \rightarrow V$ by performing a certain (uncontrollable) base change $\tilde{U} := U \times_V \tilde{V} \rightarrow \tilde{V}$, one can find a “good” compactification $X \rightarrow Y$ of $\tilde{U}^r \rightarrow \tilde{V}$ and a finite dominant morphism $W \rightarrow Y$ from a smooth projective manifold W such that the base change $X \times_Y W \rightarrow W$ is birational to a *mild morphism* $Z \rightarrow W$, which is in particular flat with reduced fibers (even functorial under fiber products). For our goal \spadesuit , we need a more refined control of the *alteration for the base* in the weakly semistable reduction [2, Theorem 0.3], which remains unknown at the moment. Fortunately, as was suggested to us and proved in Appendix A by Abramovich, using moduli of Alexeev stable maps one can establish a \mathbb{Q} -*mild reduction* for the family $U \rightarrow V$ in place of the *mild reduction* in [63], so that we can also find a “good” compactification $X \rightarrow Y$ of $U^r \rightarrow V$ without passing through the birational models $\tilde{V} \rightarrow V$ as in (0.3.4). This is the main theme of Appendix A.

Even if we can apply \mathbb{Q} -mild reduction to avoid the first blow-up of the base as in [48, 63], the second blow-up is in general inevitable. Indeed, the *discriminant* of the new family $Z_H \rightarrow Y \supset V$ obtained by taking the cyclic cover along H in (0.3.5) is in general not normal crossing. One has thus to blow up this discriminant locus of $Z_H \rightarrow Y$ to make

it normal crossing as in [48]. Therefore, to ensure (\spadesuit) , it then suffices to show that there exists a compactification $f : X \rightarrow Y$ of the smooth family $U' \rightarrow V$ such that for some sufficiently ample line bundle \mathcal{A} over Y ,

$$f_*(mK_{X/Y}) \otimes \mathcal{A}^{-m} \text{ is globally generated over } V \text{ for some } m \gg 0. \quad (*)$$

Indeed, for any given point $y \in V$, by $(*)$ one can find H transverse to the fiber $X_y := f^{-1}(y)$, and thus the new family $Z_H \rightarrow Y$ will be smooth over an open set containing y . To the best of our knowledge, $(*)$ was only known when the moduli is canonically polarized [28, Proposition 3.4]. §3.2 is devoted to the proof of $(*)$ for the family $U \rightarrow V$ in Theorem C (see Theorem 3.7(iii) below). This in turn achieves (\spadesuit) .

To get (\clubsuit) , our idea is to take *different cyclic coverings* by “moving” H in (0.3.5), to produce different “fine” VZ Higgs bundles. For any given point $y \in V$, by (\spadesuit) , one can take a birational model $\nu : \tilde{V} \rightarrow V$ such that ν is isomorphic at y , and there exists a VZ Higgs bundle $(\tilde{\mathcal{E}}, \tilde{\theta})$ on the normal crossing compactification $Y \supset \tilde{V}$. To prove that the induced negatively curved Finsler metric F is positive definite at $\tilde{y} := \nu^{-1}(y)$, by our definition of F in (0.3.3), it suffices to show that τ_1 defined in (0.3.2) is *injective* at \tilde{y} in the sense of \mathbb{C} -linear map between complex vector spaces

$$\tau_{1, \tilde{y}} : \mathcal{T}_{\tilde{V}, \tilde{y}} \xrightarrow{\cong} \mathcal{T}_Y(-\log D)_{\tilde{y}} \xrightarrow{\rho_{\tilde{y}}} H^1(X_{\tilde{y}}, \mathcal{T}_{X_{\tilde{y}}}) \xrightarrow{\varphi_{\tilde{y}}} \tilde{\mathcal{E}}_{1, \tilde{y}}.$$

As we will see in §3.4, when H in (0.3.5) is properly chosen (indeed, transverse to the fiber X_y) which is ensured by $(*)$, $\varphi_{\tilde{y}}$ is injective at \tilde{y} . Hence $\tau_{1, \tilde{y}}$ is injective by our assumption of *effective parametrization* (hence $\rho_{\tilde{y}}$ is injective) in Theorem C. This is our strategy to prove Theorem C.

Notations and conventions

Throughout this article we will work over the complex number field \mathbb{C} .

- An *algebraic fiber space*² (or *fibration* for short) $f : X \rightarrow Y$ is a surjective projective morphism between projective manifolds with connected geometric fibers. A \mathbb{Q} -divisor E in X is said to be *f-exceptional* if $f(E)$ is an algebraic variety of codimension at least 2 in Y .
- We say that a morphism $f_U : U \rightarrow V$ is a *smooth family* if f_U is a surjective smooth projective morphism with connected fibers between quasi-projective varieties.
- For any surjective morphism $Y' \rightarrow Y$, and the algebraic fiber space $f : X \rightarrow Y$, we denote by $(X \times_Y Y')^\sim$ the (unique) irreducible component (say the *main component*) of $X \times_Y Y'$ which dominates Y' .
- Let $\mu : X' \rightarrow X$ be a birational morphism from a projective manifold X' to a singular variety X . The morphism μ is called a *strong desingularization* if $\mu^{-1}(X^{\text{reg}}) \rightarrow X^{\text{reg}}$ is an isomorphism. Here X^{reg} denotes the smooth locus of X .

²Here we follow the definition in [43].

- For any birational morphism $\mu : X' \rightarrow X$, the *exceptional locus* denoted by $\text{Ex}(\mu)$ is the inverse image of the smallest closed set of X outside of which μ is an isomorphism.
- Denote by $X^r := X \times_Y \cdots \times_Y X$ the r -fold fiber product of the fibration $f : X \rightarrow Y$, by $(X^r)^{\sim}$ the *main component* of X^r dominating Y , and by $X^{(r)}$ a *strong desingularization* of $(X^r)^{\sim}$.
- For any quasi-projective manifold Y , a Zariski open subset $Y_0 \subset Y$ is called a *big open set* of Y if $\text{codim}_{Y \setminus Y_0}(Y) \geq 2$.
- A singular hermitian metric h on the line bundle L is said to be *positively curved* if the curvature current satisfies $\Theta_h(L) \geq 0$.

1. Brody hyperbolicity of the base

To begin, let us introduce the definition of *Viehweg–Zuo Higgs bundles* over quasi-projective manifolds in an abstract way following [48, 63]. Then we prove a generic local Torelli theorem for VZ Higgs bundles. We will show that based on the previous work by Viehweg–Zuo and Popa–Taji–Wu, this generic local Torelli theorem suffices to prove Theorem A.

1.1. Abstract Viehweg–Zuo Higgs bundles

The definition we present below follows from the formulation in [28, 63] and [48, Proposition 2.7].

Definition 1.1 (Abstract Viehweg–Zuo Higgs bundles). Let V be a quasi-projective manifold, and let $Y \supset V$ be a projective compactification of V with the boundary $D := Y \setminus V$ simple normal crossing. A *Viehweg–Zuo Higgs bundle on V* is a logarithmic Higgs bundle $(\tilde{\mathcal{E}}, \tilde{\theta})$ over Y consisting of the following data:

- a divisor S on Y such that $D + S$ is simple normal crossing,
- a big and nef line bundle \mathcal{L} over Y with $\mathbf{B}_+(\mathcal{L}) \subset D \cup S$,
- a Higgs bundle $(\mathcal{E}, \theta) := (\bigoplus_{q=0}^n E^{n-q,q}, \bigoplus_{q=0}^n \theta_{n-q,q})$ induced by the lower canonical extension of a polarized VHS defined over $Y \setminus (D \cup S)$,
- a Higgs subsheaf $(\mathcal{F}, \eta) \subset (\tilde{\mathcal{E}}, \tilde{\theta})$,

which satisfy the following properties:

- (1) $(\tilde{\mathcal{E}}, \tilde{\theta}) := (\mathcal{L}^{-1} \otimes \mathcal{E}, \mathbb{1} \otimes \theta)$. In particular, $\tilde{\theta} : \tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{E}} \otimes \Omega_Y(\log(D + S))$, and $\tilde{\theta} \wedge \tilde{\theta} = 0$.
- (2) The Higgs subsheaf (\mathcal{F}, η) has log poles only on the boundary D , that is, $\eta : \mathcal{F} \rightarrow \mathcal{F} \otimes \Omega_Y(\log D)$.
- (3) Write $\tilde{\mathcal{E}}_k := \mathcal{L}^{-1} \otimes E^{n-k,k}$, and denote $\mathcal{F}_k := \tilde{\mathcal{E}}_k \cap \mathcal{F}$. Then the first stage \mathcal{F}_0 of \mathcal{F} is an *effective line bundle*. In other words, there exists a non-trivial morphism $\mathcal{O}_Y \rightarrow \mathcal{F}_0$.

As shown in [28], by iterating η k times, we obtain

$$\mathcal{F}_0 \xrightarrow{\overbrace{\eta \circ \dots \circ \eta}^{k \text{ times}}} \mathcal{F}_k \otimes (\Omega_Y(\log D))^{\otimes k}.$$

Since $\eta \wedge \eta = 0$, the above morphism factors through $\mathcal{F}_k \otimes \text{Sym}^k \Omega_Y(\log D)$, and by (3) one thus obtains

$$\mathcal{O}_Y \rightarrow \mathcal{F}_0 \rightarrow \mathcal{F}_k \otimes \text{Sym}^k \Omega_Y(\log D) \rightarrow \mathcal{L}^{-1} \otimes E^{n-k,k} \otimes \text{Sym}^k \Omega_Y(\log D).$$

Equivalently, we have a morphism

$$\tau_k : \text{Sym}^k \mathcal{T}_Y(-\log D) \rightarrow \mathcal{L}^{-1} \otimes E^{n-k,k}. \quad (1.1.1)$$

It was proven in [28, Corollary 4.5] that τ_1 is always non-trivial. We say that a VZ Higgs bundle satisfies the *generic local Torelli* if $\tau_1 : \mathcal{T}_Y(-\log D) \rightarrow \mathcal{L}^{-1} \otimes E^{n-1,1}$ in (1.1.1) is generically injective. As we will see in §1.4 (in Theorem D) the generic local Torelli holds for any VZ Higgs bundle.

1.2. A quick tour of Viehweg–Zuo’s construction

For smooth families $U \rightarrow V$ of Theorems A and B, it was shown in [28] and [48, Proposition 2.7] that there is a VZ Higgs bundle over some birational model \tilde{V} of V . Indeed, using the deep theory of mixed Hodge modules, Popa–Taji–Wu [48] can even construct VZ Higgs bundles over the bases of maximal variational smooth families whose geometric generic fiber admits a good minimal model. Since in the proof of Theorem C we need to study the precise loci where τ_1 is injective, in this subsection we recollect Viehweg–Zuo’s construction of VZ Higgs bundles over the base space V (up to a birational model and a projective compactification) from the smooth projective family $f_U : U \rightarrow V$ of Theorem B. We refer the readers to [28] and [48] for more details. In §3.4, we show how to refine this construction to prove Theorem C.

Theorem 1.2. *Let $U \rightarrow V$ be the smooth family of Theorem B. Then after replacing V by a birational model \tilde{V} , there is a smooth compactification $Y \supset \tilde{V}$ and a VZ Higgs bundle over \tilde{V} .*

Proof. By [48, 63], one can take a birational morphism $\nu : \tilde{V} \rightarrow V$ and a smooth compactification $f : X \rightarrow Y$ of $U^r \times_V \tilde{V} \rightarrow \tilde{V}$ such that there exists a hypersurface

$$H \in |\ell \Omega_{X/Y}^n(\log \Delta) - \ell f^* \mathcal{L} + \ell E|, \quad n := \dim X - \dim Y, \quad (1.2.1)$$

with \mathcal{L} a big and nef line bundle over Y such that

- (1) the complement $D := Y \setminus \tilde{V}$ is simple normal crossing;
- (2) the hypersurface H is smooth over some Zariski open set $V_0 \subset \tilde{V}$ with $D + S := Y \setminus V_0$ simple normal crossing;
- (3) the divisor E is effective and f -exceptional with $f(E) \cap V_0 = \emptyset$;
- (4) $\mathbf{B}_+(\mathcal{L}) \cap V_0$ is empty, where $\mathbf{B}_+(\mathcal{L})$ is the augmented base locus of \mathcal{L} .

Here we denote $\Delta := f^{-1}(D)$ so that $(X, \Delta) \rightarrow (Y, D)$ is a log morphism. Within this basic setup, let us now construct a VZ Higgs bundle over \hat{V} following [28]. Leaving out a codimension 2 subvariety of Y supported on $D + S$, we assume that

- the morphism f is flat, and E in (1.2.1) disappears;
- the divisor $D + S$ is smooth, and both Δ and $\Sigma = f^{-1}S$ are relative normal crossing.

Set $\mathcal{L} := \Omega_{X/Y}^n(\log \Delta)$. Let $\delta : W \rightarrow X$ be a blow-up of X with centers in $\Delta + \Sigma$ such that $\delta^*(H + \Delta + \Sigma)$ is a normal crossing divisor. One thus obtains a cyclic covering of δ^*H by taking the ℓ -th root out of δ^*H . Let Z be a strong desingularization of this covering, which is smooth over V_0 by (2). We denote the compositions by $h : W \rightarrow Y$ and $g : Z \rightarrow Y$; their restrictions to V_0 are both smooth. Write $\Pi := g^{-1}(S \cup D)$, which can be assumed to be normal crossing. Leaving out a further codimension 2 subvariety supported on $D + S$, we assume that h and g are also flat, and both $\delta^*(H + \Delta + \Sigma)$ and Π are relative normal crossing. Set

$$F^{n-q,q} := R^q h_* \left(\delta^* (\Omega_{X/Y}^{n-q}(\log \Delta)) \otimes \delta^* \mathcal{L}^{-1} \otimes \mathcal{O}_W \left(\left[\frac{\delta^* H}{\ell} \right] \right) \right) / \text{torsion}.$$

It was shown in [28, §4] that there exists a natural edge morphism

$$\tau_{n-q,q} : F^{n-q,q} \rightarrow F^{n-q-1,q+1} \otimes \Omega_Y(\log D), \quad (1.2.2)$$

which gives rise to the first Higgs bundle $(\bigoplus_{q=0}^n F^{n-q,q}, \bigoplus_{q=0}^n \tau_{n-q,q})$ defined over a big open subset of Y containing V_0 .

Write $Z_0 := Z \setminus \Pi$. Then the local system $R^n g_* \mathbb{C}_{\uparrow Z_0}$ extends to a locally free sheaf \mathcal{V} on Y (here Y is projective rather than a big open set!) equipped with a logarithmic connection

$$\nabla : \mathcal{V} \rightarrow \mathcal{V} \otimes \Omega_Y(\log(D + S)),$$

whose eigenvalues of the residues lie in $[0, 1)$ (the so-called *lower canonical extension*). By Schmid's *nilpotent orbit theorem* [51], the Hodge filtration of $R^n g_* \mathbb{C}_{\uparrow Z_0}$ extends to a filtration $\mathcal{V} := \mathcal{F}^0 \supset \mathcal{F}^1 \supset \dots \supset \mathcal{F}^n$ of *subbundles* so that their graded sheaves $E^{n-q,q} := \mathcal{F}^{n-q} / \mathcal{F}^{n-q+1}$ are also locally free, and there exists

$$\theta_{n-q,q} : E^{n-q,q} \rightarrow E^{n-q-1,q+1} \otimes \Omega_Y(\log(D + S)).$$

This defines the second Higgs bundle $(\bigoplus_{q=0}^n E^{n-q,q}, \theta_{n-q,q})$. As observed in [28, 63], $E^{n-q,q} = R^q g_* \Omega_{Z/Y}^{n-q}(\log \Pi)$ over a big open subset of Y by the theorem of Steenbrink [57, 66]. By the construction of the cyclic cover Z , this in turn implies the following commutative diagram over a big open subset of Y :

$$\begin{array}{ccc} \mathcal{L}^{-1} \otimes E^{n-q,q} & \xrightarrow{\mathbb{1} \otimes \theta_{n-q,q}} & \mathcal{L}^{-1} \otimes E^{n-q-1,q+1} \otimes \Omega_Y(\log(D + S)) \\ \uparrow \rho_{n-q,q} & & \uparrow \rho_{n-q-1,q+1} \otimes \iota \\ F^{n-q,q} & \xrightarrow{\tau_{n-q,q}} & F^{n-q-1,q+1} \otimes \Omega_Y(\log D) \end{array} \quad (1.2.3)$$

as shown in [63, Lemma 6.2] (cf. also [28, Lemma 4.4]). Note that all the objects are defined on a big open set of Y except for $(\bigoplus_{q=0}^n E^{n-q,q}, \theta_{n-q,q})$, which is defined on the whole Y . Following [63, §6], for every $q = 0, \dots, n$, we define $F^{n-q,q}$ to be the reflexive hull, and the morphisms $\tau_{n-q,q}$ and $\rho_{n-q,q}$ extend naturally.

To conclude that $(\bigoplus_{q=0}^n \mathcal{L}^{-1} \otimes E^{n-q,q}, \bigoplus_{q=0}^n \mathbb{1} \otimes \theta_{n-q,q})$ is a VZ Higgs bundle as in Definition 1.1, we have to introduce a Higgs subsheaf with log poles supported on D . Write $\tilde{\theta}_{n-q,q} := \mathbb{1} \otimes \theta_{n-q,q}$ for short. Following [28, Corollary 4.5] (cf. also [48]), for each $q = 0, \dots, n$ we define a coherent torsion-free sheaf $\mathcal{F}_q := \rho_{n-q,q}(F^{n-q,q}) \subset E^{n-q,q}$. Since $F^{n,0} \supset \mathcal{O}_Y$, one has $\mathcal{F}_0 \supset \mathcal{O}_Y$. By (1.2.2) and (1.2.3),

$$\tilde{\theta}_{n-q,q}|_{\mathcal{F}_q} : \mathcal{F}_q \rightarrow \mathcal{F}_{q+1} \otimes \Omega_Y(\log D);$$

let η_q be the restriction of $\tilde{\theta}_{n-q,q}$ to \mathcal{F}_q . Then $(\mathcal{F}, \eta) := (\bigoplus_{q=0}^n \mathcal{F}_q, \bigoplus_{q=0}^n \eta_q)$ is a Higgs subbundle of $(\tilde{\mathcal{E}}, \tilde{\theta}) := (\bigoplus_{q=0}^n \mathcal{L}^{-1} \otimes E^{n-q,q}, \bigoplus_{q=0}^n \tilde{\theta}_{n-q,q})$. ■

1.3. Proper metrics for logarithmic Higgs bundles

We adopt the same notations as Definition 1.1 in the rest of §1. As is well-known, \mathcal{E} can be endowed with the Hodge metric h induced by the polarization, which may blow up around the simple normal crossing boundary $D + S$. However, according to the work of Schmid, Cattani–Schmid–Kaplan and Kashiwara [20, 33, 51], h has *mild singularities* (at most logarithmic), and as proved in [63, §7] (for unipotent monodromies) and [48, §3] (for quasi-unipotent monodromies), one can take a proper singular metric g_α on \mathcal{L} such that the induced singular hermitian metric $g_\alpha^{-1} \otimes h$ on $\tilde{\mathcal{E}} := \mathcal{L}^{-1} \otimes \mathcal{E}$ is locally bounded from above. Before we summarize the above-mentioned results in [48, §3], we introduce some notations from *loc. cit.*

Write the simple normal crossing divisor $D = D_1 + \dots + D_k$ and $S = S_1 + \dots + S_\ell$. Let $f_{D_i} \in H^0(Y, \mathcal{O}_Y(D_i))$ and $f_{S_i} \in H^0(Y, \mathcal{O}_Y(S_i))$ be the canonical sections defining D_i and S_i . We fix smooth hermitian metrics g_{D_i} and g_{S_i} on $\mathcal{O}_Y(D_i)$ and $\mathcal{O}_Y(S_i)$. Set

$$r_{D_i} := -\log \|f_{D_i}\|_{g_{D_i}}^2, \quad r_{S_i} := -\log \|f_{S_i}\|_{g_{S_i}}^2,$$

and define

$$r_D := \prod_{i=1}^k r_{D_i}, \quad r_S := \prod_{i=1}^{\ell} r_{S_i}.$$

Let g be a singular hermitian metric with analytic singularities on the big and nef line bundle \mathcal{L} such that g is smooth on $Y \setminus \mathbf{B}_+(\mathcal{L}) \supset Y \setminus (D \cup S)$, and the curvature current satisfies $\sqrt{-1} \Theta_g(\mathcal{L}) \geq \omega$ for some smooth Kähler form ω on Y . For $\alpha \in \mathbb{N}$, define

$$g_\alpha := g \cdot (r_D \cdot r_S)^\alpha.$$

The following proposition is a slight variant of [48, Lemma 3.1, Corollary 3.4].

Proposition 1.3 ([48]). *When $\alpha \gg 0$, after rescaling f_{D_i} and f_{S_i} , there exists a continuous, positive definite hermitian form ω_α on $\mathcal{T}_Y(-\log D)$ such that*

(i) over $V_0 := Y \setminus D \cup S$, the curvature form

$$\sqrt{-1} \Theta_{g_\alpha}(\mathcal{L})|_{V_0} \geq r_D^{-2} \cdot \omega_\alpha|_{V_0};$$

(ii) the singular hermitian metric $h_g^\alpha := g_\alpha^{-1} \otimes h$ on $\mathcal{L}^{-1} \otimes \mathcal{E}$ is locally bounded on Y , and smooth outside $D + S$; moreover, h_g^α is degenerate on $D + S$;

(iii) the singular hermitian metric $r_D^2 h_g^\alpha$ on $\mathcal{L}^{-1} \otimes \mathcal{E}$ is also locally bounded on Y . ■

Remark 1.4. It follows from Proposition 1.3 that both h_g^α and $r_D^2 h_g^\alpha$ can be seen as Finsler metrics on $\mathcal{L}^{-1} \otimes \mathcal{E}$ which are degenerate on $\text{Supp}(D + S)$ and positive definite on V_0 .

Although the last statement of Proposition 1.3(ii) is not explicitly stated in [48], it can be easily seen from the proof of [48, Corollary 3.4]. Proposition 1.3 mainly relies on the asymptotic behavior of the Hodge metric for *lower canonical extension* of a variation of Hodge structure (cf. Theorem 1.5 below) when the local monodromies around the boundaries are only quasi-unipotent.

Theorem 1.5 ([48, Lemma 3.2]). *Let $\mathcal{H} = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \dots \supset \mathcal{F}^N \supset 0$ be a variation of Hodge structure defined over $(\Delta^*)^p \times \Delta^q$, where Δ (resp. Δ^*) is the unit disk (resp. punctured unit disk). Consider the lower canonical extension ${}^l\mathcal{F}^\bullet$ over $\Delta^{p+q} \supset (\Delta^*)^p \times \Delta^q$, and denote by (\mathcal{E}, θ) the associated Higgs bundle. Then for any holomorphic section $s \in \Gamma(U, \mathcal{E})$, where $U \subsetneq \Delta^{p+q}$ is a relatively compact open set containing the origin, one has the norm estimate*

$$|s|_{\text{hod}} \leq C \left((-\log |t_1|) \cdot \dots \cdot (-\log |t_p|) \right)^\alpha, \quad (1.3.1)$$

where α is some positive constant independent of s , and $t = (t_1, \dots, t_{p+q})$ denotes the coordinates of Δ^{p+q} .

Let us mention that the estimates of the Hodge metric for *upper canonical extension* were obtained by Peters [46] in one variable, and by Catanese–Kawamata [19] in several variables, based on [20, 51]. We provide a slightly different proof of Theorem 1.5 for completeness sake, following closely the approaches in [19, 46].

Proof of Theorem 1.5. The fundamental group $\pi_1((\Delta^*)^p \times \Delta^q)$ is generated by elements $\gamma_1, \dots, \gamma_p$, where γ_j may be identified with the counterclockwise oriented generator of the fundamental group of the j -th copy of Δ^* in $(\Delta^*)^p$. Set T_j to be the monodromy transformation with respect to γ_j ; they pairwise commute and are known to be quasi-unipotent, that is, for any multivalued section $\underline{v}(t_1, \dots, t_{p+q})$ of \mathcal{H} , one has

$$\underline{v}(t_1, \dots, e^{2\pi i} t_j, \dots, t_{p+q}) = T_j \cdot \underline{v}(t_1, \dots, t_{p+q}),$$

and $[T_j, T_k] = 0$ for any $j, k = 1, \dots, p$. Set $T_j = D_j \cdot U_j$ to be the (unique) Jordan–Chevalley decomposition, so that D_j is diagonalizable and U_j is unipotent with $[D_j, U_j] = 0$. Since T_j is quasi-unipotent by the theorem of Borel, all the eigenvalues of D_j are thus roots of unity. Set $N_j := \frac{1}{2\pi i} \sum_{k>0} (I - U_j)^k / k$. If $D_j = \text{diag}(d_{j\ell})$ then we set

$S_j = \text{diag}(\lambda_{j\ell})$ with $\lambda_{j\ell} \in (-2\pi i, 0]$ and $\exp(\lambda_{j\ell}) = d_{j\ell}$. Since $[T_j, T_k] = 0$, Jordan–Chevalley decomposition implies that

$$[S_j, S_k] = [S_j, N_k] = [N_j, N_k] = 0. \quad (1.3.2)$$

Fix a point $t_0 \in (\Delta^*)^p \times \Delta^q$, and take a basis $v_1, \dots, v_r \in V_{t_0}$ such that S_1, \dots, S_p are simultaneously diagonal, that is,

$$S_j(v_\ell) = \lambda_{j\ell}. \quad (1.3.3)$$

Define $\underline{v}_1(t), \dots, \underline{v}_r(t)$ to be the induced multivalued flat sections. Then

$$e_j(t) := \exp\left(-\frac{1}{2\pi i} \sum_{i=1}^p (S_i + N_i) \cdot \log t_i\right) \underline{v}_j(t)$$

are single-valued and form a basis of holomorphic sections for the lower canonical extension ${}^l\mathcal{H}$.

Recall that the $d_{j\ell}$ are all roots of unity. One can thus take a positive integer m such that $m_{j\ell} := -m\lambda_{j\ell}/(2\pi i)$ are all *non-negative integers*. Equivalently, each T_j^m is unipotent. Define a ramified cover

$$\pi : \Delta^{p+q} \rightarrow \Delta^{p+q}, \quad (w_1, \dots, w_{p+q}) \mapsto (w_1^m, \dots, w_p^m, w_{p+1}, \dots, w_{p+q}),$$

and set π' to be the restriction of π to $(\Delta^*)^p \times \Delta^q$. Then $\pi'^* \mathcal{F}^\bullet$ is a variation of Hodge structure on $(\Delta^*)^p \times \Delta^q$ with unipotent monodromy, and we define ${}^c\pi'^* \mathcal{H}$ to be the canonical extension of $\pi'^* \mathcal{H}$. Set $\underline{u}_j(w) := \pi'^* \underline{v}_j$, which is a multivalued section for the local system $\pi'^* \mathcal{H}$. Then

$$\underline{u}_j(w_1, \dots, e^{2\pi i} w_j, \dots, w_{p+q}) = T_j^m \cdot \underline{u}_j(w_1, \dots, w_{p+q}).$$

Define

$$\tilde{e}_j(w) := \exp\left(-\frac{1}{2\pi i} \sum_{i=1}^p m N_i \cdot \log w_i\right) \underline{u}_j(w); \quad (1.3.4)$$

these elements form a basis of ${}^c\pi'^* \mathcal{H}$. Based on the work of [20, 51], it was shown in [63, Claim 7.8] that

$$|\tilde{e}_j(w)|_{\text{hod}} \leq C_0 ((-\log |w_1|) \cdots (-\log |w_p|))^\alpha \quad (1.3.5)$$

for some positive constants C_0 and α . On the other hand, we have

$$\begin{aligned} \pi'^* e_j(w) &= \exp\left(-\frac{1}{2\pi i} \sum_{i=1}^p (S_i + N_i) \cdot \log w_i^m\right) \pi'^* \underline{v}_j(w) \\ &\stackrel{(1.3.2)}{=} \exp\left(-\frac{1}{2\pi i} \sum_{i=1}^p m N_i \cdot \log w_i\right) \cdot \exp\left(-\frac{1}{2\pi i} \sum_{i=1}^p m S_i \log w_i\right) \pi'^* \underline{v}_j(w) \end{aligned}$$

$$\begin{aligned}
&\stackrel{(1.3.3)}{=} \exp\left(-\frac{1}{2\pi i} \sum_{i=1}^p m N_i \cdot \log w_i\right) \cdot \exp\left(-\frac{1}{2\pi i} \sum_{i=1}^p m \lambda_{ij} \log w_i\right) \pi'^* \underline{v}_j(w) \\
&= \prod_{i=1}^p w_i^{m_{ij}} \cdot \exp\left(-\frac{1}{2\pi i} \sum_{i=1}^p m N_i \cdot \log w_i\right) \cdot \underline{u}_j(w) \\
&\stackrel{(1.3.4)}{=} \prod_{i=1}^p w_i^{m_{ij}} \cdot \tilde{e}_j(w).
\end{aligned}$$

By the definition of lower canonical extension, the m_{ij} are all non-negative integers, and thus

$$\begin{aligned}
\pi'^* |e_j|_{\text{hod}}(w) &= |\pi'^* e_j(w)|_{\text{hod}} = \prod_{i=1}^p |w_i|^{m_{ij}} |\tilde{e}_j(w)|_{\text{hod}} \\
&\stackrel{(1.3.5)}{\leq} C_0 ((-\log |w_1|) \cdot \dots \cdot (-\log |w_p|))^\alpha.
\end{aligned}$$

Hence

$$|e_j|_{\text{hod}}(t) \leq \frac{C_0}{m^p} ((-\log |t_1|) \cdot \dots \cdot (-\log |t_p|))^\alpha.$$

Note that ${}^l\mathcal{H} \stackrel{\infty}{\simeq} \mathcal{E}$. Therefore, for any holomorphic section $s \in \Gamma(U, \mathcal{E})$, there exist smooth functions $f_1, \dots, f_r \in \mathcal{O}(U)$ such that $s = \sum_{j=1}^r f_j e_j$. This shows the estimate (1.3.1). \blacksquare

Remark 1.6. For the Hodge metric of upper canonical extension, one makes the choice $\lambda_{j\ell} \in [0, 2\pi i)$ instead of $\lambda_{j\ell} \in (-2\pi i, 0]$ in the proof of Theorem 1.5. Then the same computation as above easily shows that

$$|e_j|_{\text{hod}}(t) \leq \prod_{i=1}^p |t_i|^{-\frac{\lambda_{ij}}{2\pi i}} \frac{C}{m^p} ((-\log |t_1|) \cdot \dots \cdot (-\log |t_p|))^\alpha,$$

which was obtained in [19].

1.4. Generic local Torelli for VZ Higgs bundles

In this section we prove that the generic local Torelli holds for any VZ Higgs bundle, which is a crucial step in the proofs of Theorems A and B.

Theorem D (Generic local Torelli). *For all abstract Viehweg–Zuo Higgs bundles defined in Definition 1.1, the morphism $\tau_Y : \mathcal{T}_Y(-\log D) \rightarrow \mathcal{L}^{-1} \otimes E^{n-1,1}$ defined in (1.1.1) is generically injective.*

Proof. By Definition 1.1, the non-zero morphism $\mathcal{O}_Y \rightarrow \mathcal{F}_0 \rightarrow \mathcal{L}^{-1} \otimes E^{n,0}$ induces a global section $s \in H^0(Y, \mathcal{L}^{-1} \otimes E^{n,0})$, which is generically non-vanishing over $V_0 := Y \setminus (D \cup S)$. Set

$$V_1 := \{y \in V_0 \mid s(y) \neq 0\}, \quad (1.4.1)$$

which is a non-empty Zariski open set of V_0 . For the first stage of the VZ Higgs bundle $\mathcal{L}^{-1} \otimes E^{n,0}$, we equip it with a singular metric $h_g^\alpha := g_\alpha^{-1} \otimes h$ as in Proposition 1.3, so that items 1.3(i) and 1.3(ii) of that proposition are satisfied. Note that h_g^α is smooth over V_0 . Let D' be the $(1, 0)$ -part of its Chern connection over V_0 , and Θ_0 its curvature form. Then by the Griffiths curvature formula for Hodge bundles (see [31]), over V_0 we have

$$\begin{aligned} \Theta_0 &= -\Theta_{\mathcal{L}, g_\alpha} \otimes \mathbb{1} + \mathbb{1} \otimes \Theta_h(E^{n,0}) = -\Theta_{\mathcal{L}, g_\alpha} \otimes \mathbb{1} - \mathbb{1} \otimes (\theta_{n,0}^* \wedge \theta_{n,0}) \\ &= -\Theta_{\mathcal{L}, g_\alpha} \otimes \mathbb{1} - \tilde{\theta}_{n,0}^* \wedge \tilde{\theta}_{n,0}, \end{aligned} \quad (1.4.2)$$

where we set $\tilde{\theta}_{n-k,k} := \mathbb{1} \otimes \theta_{n-k,k} : \mathcal{L}^{-1} \otimes E^{n-k,k} \rightarrow \mathcal{L}^{-1} \otimes E^{n-k-1,k+1} \otimes \Omega_Y(\log(D+S))$, and define $\tilde{\theta}_{n,0}^*$ to be the adjoint of $\tilde{\theta}_{n,0}$ with respect to the metric h_g^α . Hence over V_1 one has

$$\begin{aligned} &-\sqrt{-1} \partial \bar{\partial} \log |s|_{h_g^\alpha}^2 \\ &= \frac{\{\sqrt{-1} \Theta_0(s), s\}_{h_g^\alpha}}{|s|_{h_g^\alpha}^2} + \frac{\sqrt{-1} \{D's, s\}_{h_g^\alpha} \wedge \{s, D's\}_{h_g^\alpha}}{|s|_{h_g^\alpha}^4} - \frac{\sqrt{-1} \{D's, D's\}_{h_g^\alpha}}{|s|_{h_g^\alpha}^2} \\ &\leq \frac{\{\sqrt{-1} \Theta_0(s), s\}_{h_g^\alpha}}{|s|_{h_g^\alpha}^2} \end{aligned} \quad (1.4.3)$$

thanks to the Lagrange inequality

$$\sqrt{-1} |s|_{h_g^\alpha}^2 \cdot \{D's, D's\}_{h_g^\alpha} \geq \sqrt{-1} \{D's, s\}_{h_g^\alpha} \wedge \{s, D's\}_{h_g^\alpha}.$$

Putting (1.4.2) into (1.4.3), over V_1 one has

$$\begin{aligned} \sqrt{-1} \Theta_{\mathcal{L}, g_\alpha} - \sqrt{-1} \partial \bar{\partial} \log |s|_{h_g^\alpha}^2 &\leq -\frac{\{\sqrt{-1} \tilde{\theta}_{n,0}^* \wedge \tilde{\theta}_{n,0}(s), s\}_{h_g^\alpha}}{|s|_{h_g^\alpha}^2} \\ &= \frac{\sqrt{-1} \{\tilde{\theta}_{n,0}(s), \tilde{\theta}_{n,0}(s)\}_{h_g^\alpha}}{|s|_{h_g^\alpha}^2} \end{aligned} \quad (1.4.4)$$

where $\tilde{\theta}_{n,0}(s) \in H^0(Y, \mathcal{L}^{-1} \otimes E^{n-1,1} \otimes \Omega_Y(\log(D+S)))$. By Proposition 1.3(ii), for any $y \in D \cup S$, one has

$$\lim_{y' \in V_0, y' \rightarrow y} |s|_{h_g^\alpha}^2(y') = 0.$$

Therefore, it follows from the compactness of Y that there exists $y_0 \in V_0$ such that $|s|_{h_g^\alpha}^2(y_0) \geq |s|_{h_g^\alpha}^2(y)$ for any $y \in V_0$. Hence $|s|_{h_g^\alpha}^2(y_0) > 0$, and by (1.4.1), $y_0 \in V_1$. Since $|s|_{h_g^\alpha}^2$ is smooth over V_0 , $\sqrt{-1} \partial \bar{\partial} \log |s|_{h_g^\alpha}^2(y_0)$ is semi-negative. By Proposition 1.3(i), $\sqrt{-1} \Theta_{\mathcal{L}, g_\alpha}$ is strictly positive at y_0 . By (1.4.4) and $|s|_{h_g^\alpha}^2(y_0) > 0$, we conclude that

$\sqrt{-1} \{\tilde{\theta}_{n,0}(s), \tilde{\theta}_{n,0}(s)\}_{h_g^g}$ is strictly positive at y_0 . In particular, $\tilde{\theta}_{n,0}(s)(\xi) \neq 0$ for any non-zero $\xi \in \mathcal{T}_{Y,y_0}$. Over V_0 the map

$$\tau_1 : \mathcal{T}_Y(-\log D) \rightarrow \mathcal{L}^{-1} \otimes E^{n-1,1}$$

in (1.1.1) is defined by $\tau_1(\xi) := \tilde{\theta}_{n,0}(s)(\xi)$, so it is *injective at* $y_0 \in V_1$. Hence τ_1 is *generically injective*. The theorem is thus proved. ■

Remark 1.7. Viehweg–Zuo [28] showed that $\tau_1 : \mathcal{T}_Y(-\log D) \rightarrow \mathcal{L}^{-1} \otimes E^{n-1,1}$ defined in (1.1.1) does not vanish on V using a global argument relying on the Griffiths curvature computation for the Hodge metric and the bigness of direct image sheaves due to Kawamata and Viehweg. Moreover, by the work of Viehweg–Zuo [63] and Popa–Taji–Wu [48], it has already been known that when fibers in Theorem A have big and semi-ample canonical bundle, the VZ Higgs bundles constructed in Theorem 1.2 over the base always satisfy Theorem D.

Though Theorem A follows from our more general result in Theorem B, we are able to prove Theorem A by directly applying the results of Viehweg–Zuo [63] and Popa–Taji–Wu [48]. Since we need some efforts to prove Theorem B, let us quickly show how to combine their work with Theorem D to prove Theorem A.

Proof of Theorem A. By the stratified arguments of Viehweg–Zuo [63], it suffices to prove that there cannot exist a Zariski dense entire curve. Assume for contradiction that there exists such a $\gamma : \mathbb{C} \rightarrow V$. The existence of a VZ Higgs bundle on some birational model \tilde{V} of V is known by Theorem 1.2. Let $\tilde{\gamma} : \mathbb{C} \rightarrow \tilde{V}$ be the lift of γ , which is also Zariski dense. In [48, 63], the authors proved that the restriction of τ_1 defined in (1.1.1) on \mathbb{C} , say $\tau_1|_{\mathbb{C}} : \mathcal{T}_{\mathbb{C}} \rightarrow \tilde{\gamma}^*(\mathcal{L}^{-1} \otimes E^{n-1,1})$, has to vanish identically, or else, they can construct a pseudo-hermitian metric on \mathbb{C} with strictly negative Gaussian curvature which violates the Ahlfors–Schwarz lemma. By Theorem D, this cannot happen since $\tilde{\gamma} : \mathbb{C} \rightarrow \tilde{V}$ is Zariski dense. The theorem is proved. ■

2. Pseudo Kobayashi hyperbolicity of the base

In this section we first establish an algorithm to construct Finsler metrics whose holomorphic sectional curvatures are bounded above by a negative constant via VZ Higgs bundles. By our construction and generic local Torelli Theorem D, those Finsler metrics are positive definite over a Zariski open set, and by the Ahlfors–Schwarz lemma, we prove that a quasi-projective manifold is pseudo Kobayashi hyperbolic once it is equipped with a VZ Higgs bundle, and thus prove Theorem B.

2.1. Finsler metric and (pseudo) Kobayashi hyperbolicity

Throughout this subsection, X will denote a complex manifold of dimension n .

Definition 2.1 (Finsler metric). Let \mathcal{E} be a holomorphic vector bundle on X . A *Finsler metric*³ on \mathcal{E} is a real non-negative continuous function $F : \mathcal{E} \rightarrow [0, +\infty[$ such that

$$F(av) = |a|F(v)$$

for any $a \in \mathbb{C}$ and $v \in \mathcal{E}$. The Finsler metric F is *positive definite* on some subset $S \subset X$ if for any $x \in S$ and any non-zero vector $v \in \mathcal{E}_x$, $F(v) > 0$.

When F is a Finsler metric on \mathcal{T}_X , we also say that F is a *Finsler metric on X* .

Let \mathcal{E} and \mathcal{G} be locally free sheaves on X , and suppose that there is a morphism

$$\varphi : \text{Sym}^m \mathcal{E} \rightarrow \mathcal{G}.$$

Then for any Finsler metric F on \mathcal{G} , φ induces a pseudo-metric $(\varphi^* F)^{1/m}$ on \mathcal{E} defined by

$$(\varphi^* F)^{1/m}(e) := F(\varphi(e^{\otimes m}))^{1/m} \quad (2.1.1)$$

for any $e \in \mathcal{E}$. It is easy to verify that $(\varphi^* F)^{1/m}$ is also a Finsler metric on \mathcal{E} . Moreover, if over some open set U , φ is an injection as a morphism between vector bundles, and F is positive definite over U , then $(\varphi^* F)^{1/m}$ is also positive definite over U .

Definition 2.2. (i) The *Kobayashi–Royden infinitesimal pseudo-metric* of X is a length function $\kappa_X : \mathcal{T}_X \rightarrow [0, +\infty[$, defined by

$$\kappa_X(\xi) = \inf_{\gamma} \{ \lambda > 0 \mid \exists \gamma : \mathbb{D} \rightarrow X, \gamma(0) = x, \lambda \cdot \gamma'(0) = \xi \} \quad (2.1.2)$$

for any $x \in X$ and $\xi \in \mathcal{T}_X$, where \mathbb{D} denotes the unit disk in \mathbb{C} .

(ii) The *Kobayashi pseudo-distance* of X , denoted by $d_X : X \times X \rightarrow [0, +\infty[$, is

$$d_X(p, q) = \inf_{\ell} \int_0^1 \kappa_X(\ell'(\tau)) d\tau$$

for any $p, q \in X$, where the infimum is taken over all differentiable curves $\ell : [0, 1] \rightarrow X$ joining p to q .

(iii) Let $\Delta \subsetneq X$ be a closed subset. A complex manifold X is *Kobayashi hyperbolic modulo Δ* if $d_X(p, q) > 0$ for any distinct $p, q \in X$ not both in Δ . When Δ is an empty set, the manifold X is *Kobayashi hyperbolic*; when Δ is proper and Zariski closed, the manifold X is *pseudo Kobayashi hyperbolic*.

By definition it is easy to show that if X is Kobayashi hyperbolic (resp. pseudo Kobayashi hyperbolic), then X is Brody hyperbolic (resp. algebraically degenerate). Brody's theorem says that when X is compact, X is Kobayashi hyperbolic if it is Brody hyperbolic. However unlike the case of Kobayashi hyperbolicity, no criterion is known for pseudo Kobayashi hyperbolicity of a compact complex space in terms of entire curves. Moreover, there are many examples of complex (quasi-projective) manifolds which are Brody hyperbolic but not Kobayashi hyperbolic.

³This definition is a bit different from the definition in [38], which requires *convexity* or *triangle inequality*, and the Finsler metric there can be upper semi-continuous.

For any holomorphic map $\gamma : \mathbb{D} \rightarrow X$, the Finsler metric F induces a continuous hermitian pseudo-metric on \mathbb{D} ,

$$\gamma^* F^2 = \sqrt{-1} \lambda(t) dt \wedge d\bar{t},$$

where $\lambda(t)$ is a non-negative continuous function on \mathbb{D} . The *Gaussian curvature* $K_{\gamma^* F^2}$ of the pseudo-metric $\gamma^* F^2$ is defined to be

$$K_{\gamma^* F^2} := -\frac{1}{\lambda} \frac{\partial^2 \log \lambda}{\partial t \partial \bar{t}}. \quad (2.1.3)$$

Definition 2.3. Let X be a complex manifold endowed with a Finsler metric F .

- (i) For any $x \in X$ and $v \in \mathcal{T}_{X,x}$, let $[v]$ denote the complex line spanned by v . We define the *holomorphic sectional curvature* $K_{F,[v]}$ in the direction of $[v]$ by

$$K_{F,[v]} := \sup K_{\gamma^* F^2}(0)$$

where the supremum is taken over all $\gamma : \mathbb{D} \rightarrow X$ such that $\gamma(0) = x$ and $[v]$ is tangent to $\gamma'(0)$.

- (ii) We say that F is *negatively curved* if there is a positive constant c such that $K_{F,[v]} \leq -c$ for all $v \in \mathcal{T}_{X,x}$ for which $F(v) > 0$.
- (iii) A point $x \in X$ is a *degeneracy point* of F if $F(v) = 0$ for some non-zero $v \in \mathcal{T}_{X,x}$, and the set of such points is denoted by Δ_F .

As mentioned in §0, our negatively curved Finsler metrics are only constructed on birational models of the base spaces in Theorems B and C, we thus have to establish bimeromorphic criteria for (pseudo) Kobayashi hyperbolicity to prove the main theorems.

Lemma 2.4 (Bimeromorphic criteria for pseudo Kobayashi hyperbolicity). *Let $\mu : X \rightarrow Y$ be a bimeromorphic morphism between complex manifolds. If there exists a Finsler metric F on X which is negatively curved in the sense of Definition 2.3(ii), then X is Kobayashi hyperbolic modulo Δ_F , and Y is Kobayashi hyperbolic modulo $\mu(\text{Ex}(\mu) \cup \Delta_F)$, where $\text{Ex}(\mu)$ is the exceptional locus of μ . In particular, when Δ_F is a proper analytic subvariety of X , both X and Y are pseudo Kobayashi hyperbolic.*

Proof. The first statement is a slight variant of [38, Theorem 3.7.4]. By normalizing F we may assume that $K_F \leq -1$. By the Ahlfors–Schwarz lemma, one has $F \leq \kappa_X$. Let $\delta_F : X \times X \rightarrow [0, +\infty[$ be the distance function on X defined by F in a similar way to d_X :

$$\delta_F(p, q) := \inf_{\ell} \int_0^1 F(\ell'(\tau)) d\tau$$

for any $p, q \in X$, where the infimum is over all differentiable curves $\ell : [0, 1] \rightarrow X$ joining p to q . Since F is continuous and positive definite over $X \setminus \Delta_F$, for any $p \in X \setminus \Delta_F$, one has $d_X(p, q) \geq \delta_F(p, q) > 0$ for any $q \neq p$, which proves the first statement.

Let $\text{Hol}(Y, y)$ be the set of holomorphic maps $\gamma : \mathbb{D} \rightarrow Y$ with $\gamma(0) = y$. Pick any point $y \in U := Y \setminus \mu(\text{Ex}(\mu))$. Then there is a unique point $x \in X$ with $\mu(x) = y$. Hence

μ induces a bijection of sets

$$\mathrm{Hol}(X, x) \xrightarrow{\cong} \mathrm{Hol}(Y, y)$$

defined by $\tilde{\gamma} \mapsto \mu \circ \tilde{\gamma}$. Indeed, observe that $\mu^{-1} : Y \dashrightarrow X$ is a meromorphic map, hence so is $\mu^{-1} \circ \gamma$ for any $\gamma \in \mathrm{Hol}(Y, y)$. Since $\dim \mathbb{D} = 1$, the map $\mu^{-1} \circ \gamma$ is moreover holomorphic. It follows from (2.1.2) that

$$\kappa_X(\xi) = \kappa_Y(\mu_*(\xi))$$

for any $\xi \in \mathcal{T}_{X,x}$. Hence

$$\mu^* \kappa_Y|_{\mu^{-1}(U)} = \kappa_X|_{\mu^{-1}(U)} \geq F|_{\mu^{-1}(U)}.$$

Let $G : \mathcal{T}_U \rightarrow [0, +\infty[$ be the Finsler metric on U such that $\mu^* G = F|_{\mu^{-1}(U)}$. Then G is continuous and positive definite over $U \setminus \mu(\Delta_F)$, and

$$\kappa_Y|_U \geq G.$$

Therefore, for any $y \in Y \setminus \mu(\Delta_F \cup \mathrm{Ex}(\mu))$, one has $d_Y(y, z) > 0$ for any $z \neq y$, which proves the second statement. ■

The above criterion can be refined further to show the Kobayashi hyperbolicity of the complex manifold.

Lemma 2.5 (Bimeromorphic criterion for Kobayashi hyperbolicity). *Let X be a complex manifold. Assume that for each point $p \in X$, there is a bimeromorphic morphism $\mu : \tilde{X} \rightarrow X$ with \tilde{X} equipped with a negatively curved Finsler metric F such that $p \notin \mu(\Delta_F \cup \mathrm{Ex}(\mu))$. Then X is Kobayashi hyperbolic.*

Proof. It suffices to show that $d_X(p, q) > 0$ for any distinct $p, q \in X$. We take the bimeromorphic morphism $\mu : \tilde{X} \rightarrow X$ in the lemma with respect to p . By Lemma 2.4, \tilde{X} is Kobayashi hyperbolic modulo $\mu(\Delta_F \cup \mathrm{Ex}(\mu))$, which shows that $d_X(p, q) > 0$ for any $q \neq p$. The lemma follows. ■

2.2. Curvature formula

Let $(\tilde{\mathcal{E}}, \tilde{\theta})$ be the VZ Higgs bundle on a quasi-projective manifold V defined in §1.1. In the next two subsections, we will construct a negatively curved Finsler metric on V via $(\tilde{\mathcal{E}}, \tilde{\theta})$. Our main result is the following.

Theorem 2.6 (Existence of negatively curved Finsler metrics). *Same notations as Definition 1.1. Assume that τ_1 is injective over a non-empty Zariski open set $V_1 \subseteq Y \setminus (D \cup S)$. Then there exists a Finsler metric F (see (2.3.6) below) on $\mathcal{T}_Y(-\log D)$ such that*

- (i) F is positive definite over V_1 ;
- (ii) when F is seen as a Finsler metric on $V = Y \setminus D$, it is negatively curved in the sense of Definition 2.3(ii).

Let us first construct the desired Finsler metric F , and then prove the curvature property. By (1.1.1), for each $k = 1, \dots, n$, there exists

$$\tau_k : \text{Sym}^k \mathcal{T}_Y(-\log D) \rightarrow \mathcal{L}^{-1} \otimes E^{n-k,k}. \quad (2.2.1)$$

Then it follows from Proposition 1.3(ii) that the Finsler metric h_g^α on $\mathcal{L}^{-1} \otimes E^{n-k,k}$ induces a Finsler metric F_k on $\mathcal{T}_Y(-\log D)$ defined as follows: for any $e \in \mathcal{T}_Y(-\log D)_y$,

$$F_k(e) := (\tau_k^* h_g^\alpha)^{1/k}(e) = h_g^\alpha(\tau_k(e^{\otimes k}))^{1/k}. \quad (2.2.2)$$

For any $\gamma : \mathbb{D} \rightarrow V$, one has

$$d\gamma : \mathcal{T}_{\mathbb{D}} \rightarrow \gamma^* \mathcal{T}_V \hookrightarrow \gamma^* \mathcal{T}_Y(-\log D)$$

and thus the Finsler metric F_k induces a continuous hermitian pseudo-metric on \mathbb{D} , denoted by

$$\gamma^* F_k^2 := \sqrt{-1} G_k(t) dt \wedge d\bar{t}. \quad (2.2.3)$$

In general, $G_k(t)$ may be identically equal to zero for all k . However, if we further assume that $\gamma(\mathbb{D}) \cap V_1 \neq \emptyset$, by the assumption in Theorem 2.6 that the restriction of τ_1 to V_1 is injective, one has $G_1(t) \neq 0$. Denote by $\partial_t := \frac{\partial}{\partial t}$ the canonical vector field in \mathbb{D} , and $\bar{\partial}_t := \frac{\partial}{\partial \bar{t}}$ its conjugate. Set $C := \gamma^{-1}(V_1)$, and note that $\mathbb{D} \setminus C$ is a discrete set in \mathbb{D} .

Lemma 2.7. *Assume that $G_k(t) \neq 0$ for some $k > 1$. Then the Gaussian curvature K_k of the continuous pseudo-hermitian metric $\gamma^* F_k^2$ on C satisfies*

$$K_k := -\frac{\partial^2 \log G_k}{\partial t \partial \bar{t}} / G_k \leq \frac{1}{k} \left(-\left(\frac{G_k}{G_{k-1}} \right)^{k-1} + \left(\frac{G_{k+1}}{G_k} \right)^{k+1} \right) \quad (2.2.4)$$

over $C \subset \mathbb{D}$.

Proof. For $i = 1, \dots, n$, let us write $e_i := \tau_i(d\gamma(\partial_t)^{\otimes i})$, which can be seen as a section of $\gamma^*(\mathcal{L}^{-1} \otimes E^{n-i,i})$. Then by (2.2.2) one observes that

$$G_i(t) = \|e_i\|_{h_g^\alpha}^{2/i}. \quad (2.2.5)$$

Let $\mathcal{R}_k = \Theta_{h_g^\alpha}(\mathcal{L}^{-1} \otimes E^{n-k,k})$ be the curvature form of $\mathcal{L}^{-1} \otimes E^{n-k,k}$ on the set $V_0 := Y \setminus (D \cup S)$ induced by the metric $h_g^\alpha = g_\alpha^{-1} \cdot h$ defined in Proposition 1.3(ii), and let D' be the $(1, 0)$ -part of the Chern connection D of $(\mathcal{L}^{-1} \otimes E^{n-k,k}, h_g^\alpha)$. Then for $k = 1, \dots, n$, one has

$$\begin{aligned} -\sqrt{-1} \partial \bar{\partial} \log G_k &= \frac{1}{k} \left(\frac{\{\sqrt{-1} \mathcal{R}_k(e_k), e_k\}_{h_g^\alpha}}{\|e_k\|_{h_g^\alpha}^2} + \frac{\sqrt{-1} \{D' e_k, e_k\}_{h_g^\alpha} \wedge \{e_k, D' e_k\}_{h_g^\alpha}}{\|e_k\|_{h_g^\alpha}^4} \right. \\ &\quad \left. - \frac{\sqrt{-1} \{D' e_k, D' e_k\}_{h_g^\alpha}}{\|e_k\|_{h_g^\alpha}^2} \right) \\ &\leq \frac{1}{k} \frac{\{\sqrt{-1} \mathcal{R}_k(e_k), e_k\}_{h_g^\alpha}}{\|e_k\|_{h_g^\alpha}^2} \end{aligned}$$

thanks to the Lagrange inequality

$$\sqrt{-1} \|e_k\|_{h_g^\alpha}^2 \cdot \langle D'e_k, D'e_k \rangle_{h_g^\alpha} \geq \sqrt{-1} \langle D'e_k, e_k \rangle_{h_g^\alpha} \wedge \langle e_k, D'e_k \rangle_{h_g^\alpha}.$$

Hence

$$-\frac{\partial^2 \log G_k}{\partial t \partial \bar{t}} \leq \frac{1}{k} \cdot \frac{\langle \mathcal{R}_k(e_k)(\partial_t, \bar{\partial}_t), e_k \rangle_{h_g^\alpha}}{\|e_k\|_{h_g^\alpha}^2}. \quad (2.2.6)$$

Recall that for the logarithmic Higgs bundle $(\bigoplus_{k=0}^n E^{n-k,k}, \bigoplus_{k=0}^n \theta_{n-k,k})$, the curvature Θ_k on $E_{\downarrow V_0}^{n-k,k}$ induced by the Hodge metric h is given by

$$\Theta_k = -\theta_{n-k,k}^* \wedge \theta_{n-k,k} - \theta_{n-k+1,k-1} \wedge \theta_{n-k+1,k-1}^*,$$

where we recall that $\theta_{n-k,k} : E^{n-k,k} \rightarrow E^{n-k-1,k+1} \otimes \Omega_Y(\log(D+S))$. Set $\tilde{\theta}_{n-k,k} := \mathbb{1} \otimes \theta_{n-k,k} : \mathcal{L}^{-1} \otimes E^{n-k,k} \rightarrow \mathcal{L}^{-1} \otimes E^{n-k-1,k+1} \otimes \Omega_Y(\log(D+S))$, and one has

$$\begin{array}{ccccc} & \xrightarrow{\tilde{\theta}_{n-k+1,k-1}(\partial_t)} & & \xrightarrow{\tilde{\theta}_{n-k,k}(\partial_t)} & \\ \mathcal{L}^{-1} \otimes E^{n-k+1,k-1} & & \mathcal{L}^{-1} \otimes E^{n-k,k} & & \mathcal{L}^{-1} \otimes E^{n-k-1,k+1} \\ & \xleftarrow{\tilde{\theta}_{n-k+1,k-1}^*(\bar{\partial}_t)} & & \xleftarrow{\tilde{\theta}_{n-k,k}^*(\bar{\partial}_t)} & \end{array}$$

where $\tilde{\theta}_{n-k,k}^*$ is the adjoint of $\tilde{\theta}_{n-k,k}$ with respect to the metric h_g^α over $Y \setminus (D \cup S)$. Here we also write ∂_t (resp. $\bar{\partial}_t$) for $d\gamma(\partial_t)$ (resp. $d\gamma(\bar{\partial}_t)$) abusively. Then over V_0 , we have

$$\begin{aligned} \mathcal{R}_k &= -\Theta_{\mathcal{L}, g_\alpha} \otimes \mathbb{1} + \mathbb{1} \otimes \Theta_k \\ &= -\Theta_{\mathcal{L}, g_\alpha} \otimes \mathbb{1} - \tilde{\theta}_{n-k,k}^* \wedge \tilde{\theta}_{n-k,k} - \tilde{\theta}_{n-k+1,k-1} \wedge \tilde{\theta}_{n-k+1,k-1}^*. \end{aligned} \quad (2.2.7)$$

By the definition of τ_k in (1.1.1), for any $k = 2, \dots, n$ one has

$$e_k = \tilde{\theta}_{n-k+1,k-1}(\partial_t)(e_{k-1}), \quad (2.2.8)$$

and we can derive the following curvature formula:

$$\begin{aligned} &\langle \mathcal{R}_k(e_k)(\partial_t, \bar{\partial}_t), e_k \rangle_{h_g^\alpha} \\ &= -\Theta_{\mathcal{L}, g_\alpha}(\partial_t, \bar{\partial}_t) \|e_k\|_{h_g^\alpha}^2 \\ &\quad + \langle \tilde{\theta}_{n-k,k}^*(\bar{\partial}_t) \circ \tilde{\theta}_{n-k,k}(\partial_t)(e_k) - \tilde{\theta}_{n-k+1,k-1}(\partial_t) \circ \tilde{\theta}_{n-k+1,k-1}^*(\bar{\partial}_t)(e_k), e_k \rangle_{h_g^\alpha} \\ &\leq \langle \tilde{\theta}_{n-k,k}^*(\bar{\partial}_t) \circ \tilde{\theta}_{n-k,k}(\partial_t)(e_k), e_k \rangle_{h_g^\alpha} \\ &\quad - \langle \tilde{\theta}_{n-k+1,k-1}(\partial_t) \circ \tilde{\theta}_{n-k+1,k-1}^*(\bar{\partial}_t)(e_k), e_k \rangle_{h_g^\alpha} \\ &\stackrel{(2.2.8)}{=} \|e_{k+1}\|_{h_g^\alpha}^2 - \|\tilde{\theta}_{n-k+1,k-1}^*(\bar{\partial}_t)(e_k)\|_{h_g^\alpha}^2 \end{aligned}$$

$$\begin{aligned}
&\leq \|e_{k+1}\|_{h_g^\alpha}^2 - \frac{|\langle \tilde{\theta}_{n-k+1, k-1}^*(\bar{\partial}_t)(e_k), e_{k-1} \rangle_{h_g^\alpha}|^2}{\|e_{k-1}\|_{h_g^\alpha}^2} \quad (\text{Cauchy-Schwarz inequality}) \\
&= \|e_{k+1}\|_{h_g^\alpha}^2 - \frac{|\langle e_k, \tilde{\theta}_{n-k+1, k-1}(\partial_t)(e_{k-1}) \rangle_{h_g^\alpha}|^2}{\|e_{k-1}\|_{h_g^\alpha}^2} \\
&\stackrel{(2.2.8)}{=} \|e_{k+1}\|_{h_g^\alpha}^2 - \frac{\|e_k\|_{h_g^\alpha}^4}{\|e_{k-1}\|_{h_g^\alpha}^2} \stackrel{(2.2.5)}{=} G_{k+1}^{k+1} - \frac{G_k^{2k}}{G_{k-1}^{k-1}}
\end{aligned}$$

Putting this into (2.2.6), we obtain (2.2.4). \blacksquare

Remark 2.8. For the final stage $E^{0,n}$ of the Higgs bundle $(\bigoplus_{q=0}^n E^{n-q,q}, \bigoplus_{q=0}^n \theta_{n-q,q})$. We make the convention that $G_{n+1} \equiv 0$. Then the Gaussian curvature for \tilde{G}_n in (2.2.6) is always semi-negative, which is similar to the Griffiths curvature formula for Hodge bundles in [31].

When $k = 1$, by (2.2.6) one has

$$\begin{aligned}
-\frac{\partial^2 \log G_1}{\partial t \partial \bar{t}} / G_1 &\leq \frac{\langle \mathcal{R}_1(e_1)(\partial_t, \bar{\partial}_t), e_1 \rangle_{h_g^\alpha}}{\|e_1\|_{h_g^\alpha}^4} \\
&\stackrel{(2.2.7)}{=} \frac{-\Theta_{\mathcal{L}, g_\alpha}(\partial_t, \bar{\partial}_t)}{\|e_1\|_{h_g^\alpha}^2} \\
&\quad + \frac{\langle \tilde{\theta}_{n-1,1}^*(\bar{\partial}_t) \circ \tilde{\theta}_{n-1,1}(\partial_t)(e_1) - \tilde{\theta}_{n,0}(\partial_t) \circ \tilde{\theta}_{n,0}^*(\bar{\partial}_t)(e_1), e_1 \rangle_{h_g^\alpha}}{\|e_1\|_{h_g^\alpha}^4} \\
&\stackrel{(2.2.8)}{\leq} \frac{-\Theta_{\mathcal{L}, g_\alpha}(\partial_t, \bar{\partial}_t) \|e_1\|_{h_g^\alpha}^2 + \|e_2\|_{h_g^\alpha}^2}{\|e_1\|_{h_g^\alpha}^4} \\
&= \frac{-\Theta_{\mathcal{L}, g_\alpha}(\partial_t, \bar{\partial}_t)}{\|e_1\|_{h_g^\alpha}^2} + \left(\frac{G_2}{G_1} \right)^2.
\end{aligned}$$

We need the following lemma to control the negative term in the above inequality.

Lemma 2.9. *When $\alpha \gg 0$, there exists a universal constant $c > 0$ such that for any $\gamma : \mathbb{D} \rightarrow V$ with $\gamma(\mathbb{D}) \cap V_0 \neq \emptyset$, one has*

$$\frac{\Theta_{\mathcal{L}, g_\alpha}(\partial_t, \bar{\partial}_t)}{\|e_1\|_{h_g^\alpha}^2} \geq c.$$

In particular,

$$-\frac{\partial^2 \log G_1}{\partial t \partial \bar{t}} / G_1 \leq -c + \left(\frac{G_2}{G_1} \right)^2.$$

Proof. By Proposition 1.3(ii), it suffices to prove that

$$\frac{\gamma^*(r_D^{-2} \cdot \omega_\alpha)(\partial_t, \bar{\partial}_t)}{\|e_1\|_{h_g^\alpha}^2} \geq c. \quad (2.2.9)$$

Note that

$$\frac{\gamma^*(r_D^{-2} \cdot \omega_\alpha)(\partial_t, \bar{\partial}_t)}{\|e_1\|_{h_g^\alpha}^2} = \frac{\gamma^*(\omega_\alpha)(\partial_t, \bar{\partial}_t)}{\gamma^*(r_D^2) \cdot \|e_1\|_{h_g^\alpha}^2} = \frac{\gamma^*\omega_\alpha(\partial_t, \bar{\partial}_t)}{\gamma^*\tau_1^*(r_D^2 \cdot h_g^\alpha)(\partial_t, \bar{\partial}_t)},$$

where $\tau_1^*(r_D^2 \cdot h_g^\alpha)$ is the Finsler metric on $\mathcal{T}_Y(-\log D)$ defined by (2.1.1). By Proposition 1.3(iii), ω_α is a positive definite hermitian metric on $\mathcal{T}_Y(-\log D)$. Since Y is compact, there exists a *uniform constant* $c > 0$ such that

$$\omega_\alpha \geq c\tau_1^*(r_D^2 \cdot h_g^\alpha).$$

We have thus obtained the desired inequality (2.2.9). \blacksquare

In summary, we have the following curvature estimate for the Finsler metrics F_1, \dots, F_n defined in (2.2.2), which is similar to [54, Lemma 9] for the Weil–Petersson metric.

Proposition 2.10. *Let $\gamma : \mathbb{D} \rightarrow V$ be such that $\gamma(\mathbb{D}) \cap V_1 \neq \emptyset$. Assume that $G_k \neq 0$ for $k = 1, \dots, q$, and $G_{q+1} \equiv 0$ (thus $G_j \equiv 0$ for all $j > q + 1$). Then $q \geq 1$, and over $C := \gamma^{-1}(V_1)$, which is the complement of a discrete set in \mathbb{D} , one has*

$$-\frac{\partial^2 \log G_1}{\partial t \partial \bar{t}} / G_1 \leq -c + \left(\frac{G_2}{G_1} \right)^2, \quad (2.2.10)$$

$$-\frac{\partial^2 \log G_k}{\partial t \partial \bar{t}} / G_k \leq \frac{1}{k} \left(-\left(\frac{G_k}{G_{k-1}} \right)^{k-1} + \left(\frac{G_{k+1}}{G_k} \right)^{k+1} \right) \quad \forall 1 < k \leq q. \quad (2.2.11)$$

Here the constant $c > 0$ does not depend on the choice of γ .

2.3. Construction of the Finsler metric

By Proposition 2.10, we observe that none of the Finsler metrics F_1, \dots, F_n defined in (2.2.2) is negatively curved. Following the similar strategies in [7, 54, 58], we construct a new Finsler metric F (see (2.3.6) below) by defining a convex sum of all F_1, \dots, F_n , to cancel the positive terms in (2.2.10) and (2.2.11) by negative terms in the next stage. By Remark 2.8, we observe that the last highest order term is always semi-negative. We mainly follow the computations in [54], and try to make this subsection as self-contained as possible. Let us first recall the following basic inequalities by Schumacher.

Lemma 2.11 ([53, Lemma 8]). *Let V be a complex manifold, and let G_1, \dots, G_n be non-negative \mathcal{C}^2 functions on V . Then*

$$\sqrt{-1} \partial \bar{\partial} \log \left(\sum_{i=1}^n G_i \right) \geq \frac{\sum_{j=1}^n G_j \sqrt{-1} \partial \bar{\partial} \log G_j}{\sum_{i=1}^n G_i} \quad (2.3.1)$$

Lemma 2.12 ([54, Lemma 17]). *Let $\alpha_j > 0$ for $j = 1, \dots, n$. Then for all $x_j \geq 0$,*

$$\begin{aligned} & \sum_{j=2}^n (\alpha_j x_j^{j+1} - \alpha_{j-1} x_j^j) x_{j-1}^2 \cdots x_1^2 \\ & \geq \frac{1}{2} \left(-\frac{\alpha_1^3}{\alpha_2^2} x_1^2 + \frac{\alpha_{n-1}^{n-1}}{\alpha_n^{n-2}} x_n^2 \cdots x_1^2 + \sum_{j=2}^{n-1} \left(\frac{\alpha_{j-1}^{j-1}}{\alpha_j^{j-2}} - \frac{\alpha_j^{j+2}}{\alpha_{j+1}^{j+1}} \right) x_j^2 \cdots x_1^2 \right) \end{aligned} \quad (2.3.2)$$

Set $x_j := G_j/G_{j-1}$ for $j = 2, \dots, n$ and $x_1 := G_1$ where $G_j \geq 0$ for $j = 1, \dots, n$. Put these into (2.3.2) to obtain

$$\begin{aligned} & \sum_{j=2}^n \left(\alpha_j \frac{G_j^{j+1}}{G_{j-1}^{j-1}} - \alpha_{j-1} \frac{G_j^j}{G_{j-1}^{j-2}} \right) \\ & \geq \frac{1}{2} \left(-\frac{\alpha_1^3}{\alpha_2^2} G_1^2 + \frac{\alpha_{n-1}^{n-1}}{\alpha_n^{n-2}} G_n^2 + \sum_{j=2}^{n-1} \left(\frac{\alpha_{j-1}^{j-1}}{\alpha_j^{j-2}} - \frac{\alpha_j^{j+2}}{\alpha_{j+1}^{j+1}} \right) G_j^2 \right). \end{aligned} \quad (2.3.3)$$

The following technical lemma is crucial in constructing our negatively curved Finsler metric F .

Lemma 2.13 ([54, Lemma 10]). *Let F_1, \dots, F_n be Finsler metrics on a complex space X , with holomorphic sectional curvatures K_1, \dots, K_n . Then for the holomorphic sectional curvature of the Finsler metric $F := (F_1^2 + \dots + F_n^2)^{1/2}$, one has*

$$K_F \leq \frac{\sum_{j=1}^n K_j F_j^4}{F^4}. \quad (2.3.4)$$

Proof. For any holomorphic map $\gamma : \mathbb{D} \rightarrow X$, we denote by G_1, \dots, G_n the semi-positive functions on \mathbb{D} such that

$$\gamma^* F_i^2 = \sqrt{-1} G_i dt \wedge d\bar{t}$$

for $i = 1, \dots, n$. Then

$$\gamma^* F^2 = \sqrt{-1} \left(\sum_{i=1}^n G_i \right) dt \wedge d\bar{t},$$

and it follows from (2.1.3) that the Gaussian curvature of $\gamma^* F^2$ satisfies

$$\begin{aligned} K_{\gamma^* F^2} &= -\frac{1}{\sum_{i=1}^n G_i} \frac{\partial^2 \log(\sum_{i=1}^n G_i)}{\partial t \partial \bar{t}} \\ &\stackrel{(2.3.1)}{\leq} -\frac{1}{(\sum_{i=1}^n G_i)^2} \sum_{j=1}^n G_j \frac{\partial^2 \log G_j}{\partial t \partial \bar{t}} \leq \frac{\sum_{j=1}^n K_j G_j^2}{(\sum_{i=1}^n G_i)^2}. \end{aligned}$$

The lemma follows from Definition 2.3(i). ■

For any $\gamma : \mathbb{D} \rightarrow V$ with $C := \gamma^{-1}(V_1) \neq \emptyset$, we define a hermitian pseudo-metric $\sigma := \sqrt{-1} H(t)dt \wedge d\bar{t}$ on \mathbb{D} by taking a convex sum in the following form:

$$H(t) := \sum_{k=1}^n k\alpha_k G_k(t),$$

where G_k is defined in (2.2.3), and $\alpha_1, \dots, \alpha_n \in \mathbb{R}^+$ are some *universal constants* which will be fixed later. Following the similar estimate in [54, Proposition 11], one can choose those constants so that the Gaussian curvature K_σ of σ is uniformly bounded.

Proposition 2.14. *There exist universal constants $0 < \alpha_1 \leq \dots \leq \alpha_n$ and $K > 0$ (independent of $\gamma : \mathbb{D} \rightarrow V$) such that*

$$K_\sigma \leq -K \quad \text{on } C.$$

Proof. It follows from (2.3.4) that

$$K_\sigma \leq \frac{1}{H^2} \sum_{j=1}^n j\alpha_j K_j G_j^2 \quad \text{with} \quad K_j := -\frac{\partial^2 \log G_j}{\partial t \partial \bar{t}} / G_j.$$

By Proposition 2.10, one has

$$\begin{aligned} K_\sigma &\leq \frac{\alpha_1 G_1^2}{H^2} \left(-c + \left(\frac{G_2}{G_1} \right)^2 \right) + \frac{1}{H^2} \sum_{j=2}^n \alpha_j G_j^2 \left(-\left(\frac{G_j}{G_{j-1}} \right)^{j-1} + \left(\frac{G_{j+1}}{G_j} \right)^{j+1} \right) \\ &\leq \frac{1}{H^2} \left(-c\alpha_1 G_1^2 - \sum_{j=2}^n \left(\alpha_j \frac{G_j^{j+1}}{G_{j-1}^{j-1}} - \alpha_{j-1} \frac{G_j^j}{G_{j-1}^{j-2}} \right) \right) \\ &\stackrel{(2.3.3)}{\leq} \frac{1}{H^2} \left(\left(-c + \frac{1}{2} \frac{\alpha_1^2}{\alpha_2^2} \right) \alpha_1 G_1^2 + \frac{1}{2} \sum_{j=2}^{n-1} \left(\frac{\alpha_j^{j+2}}{\alpha_{j+1}^{j+1}} - \frac{\alpha_{j-1}^{j-1}}{\alpha_j^{j-2}} \right) G_j^2 - \frac{1}{2} \frac{\alpha_{n-1}^{n-1}}{\alpha_n^{n-2}} G_n^2 \right) \\ &=: -\frac{1}{H^2} \sum_{j=1}^n \beta_j G_j^2. \end{aligned}$$

One can take $\alpha_1 = 1$, and choose the further $\alpha_j > \alpha_{j-1}$ inductively such that $\min_j \beta_j > 0$. Set $\beta_0 := \min_j \frac{\beta_j}{(j\alpha_j)^2}$. Then

$$\begin{aligned} K_\sigma &\leq -\frac{1}{H^2} \beta_0 \sum_{j=1}^n (j\alpha_j G_j)^2 \leq -\frac{\beta_0}{nH^2} \left(\sum_{j=1}^n j\alpha_j G_j \right)^2 \\ &= -\frac{\beta_0}{n} =: -K. \end{aligned}$$

Note that $\alpha_1, \dots, \alpha_n$ and K are universal. The lemma is thus proved. \blacksquare

It follows from Proposition 2.14 and (2.1.3) that

$$\frac{\partial^2 \log H(t)}{\partial t \partial \bar{t}} \geq KH(t) \geq 0 \quad (2.3.5)$$

over the Zariski dense open set $C \subseteq \mathbb{D}$, and in particular $\log H(t)$ is a subharmonic function over C . Since $H(t) \in [0, +\infty[$ is continuous (in particular locally bounded from above) over \mathbb{D} , $\log H(t)$ is a subharmonic function over \mathbb{D} , and the estimate (2.3.5) holds over the whole \mathbb{D} .

In summary, we construct a *negatively curved Finsler metric* F on $Y \setminus D$, defined by

$$F := \left(\sum_{k=1}^n k \alpha_k F_k^2 \right)^{1/2}, \quad (2.3.6)$$

where F_k is defined in (2.2.2), such that $\gamma^* F^2 = \sqrt{-1} H(t) dt \wedge d\bar{t}$ for any $\gamma : \mathbb{D} \rightarrow V$. Since we assume that τ_1 is injective over V_0 , the Finsler metric F_1 is positive definite on V_0 , and *a fortiori* so is F . Therefore, the proof of Theorem 2.6 is finished.

2.4. Proof of Theorem B

By Theorem 1.2, there is a VZ Higgs bundle over some birational model \tilde{V} of V . By Theorem D and Theorem 2.6, we can associate to this VZ Higgs bundle a negatively curved Finsler metric which is positive definite over some Zariski dense open set of \tilde{V} . The theorem follows directly from the bimeromorphic criterion for pseudo Kobayashi hyperbolicity in Lemma 2.4.

Remark 2.15. Let me mention that Sun and Zuo also have a similar idea of constructing a Finsler metric over the base using Viehweg–Zuo Higgs bundles combined with To–Yeung’s method [58].

3. Kobayashi hyperbolicity of the base

In this section we will prove Theorem C. We first refine Viehweg–Zuo’s result on the positivity of direct images. We then apply this result to take a different branch covering in the construction of VZ Higgs bundles to prove the Kobayashi hyperbolicity of the base in Theorem C.

3.1. Preliminaries on positivity of direct images

We first recall a *pluricanonical extension theorem* due to Cao [17, Theorem 2.10]. Its proof is a combination of the Ohsawa–Takegoshi–Manivel L^2 -extension theorem with the semi-positivity of the m -relative Bergman metric studied by Berndtsson–Păun [5, 6] and Păun–Takayama [45].

Theorem 3.1 (Pluricanonical L^2 -extension). *Let $f : X \rightarrow Y$ be an algebraic fiber space such that the Kodaira dimension of the general fiber is non-negative. Assume that f is smooth over a dense Zariski open subset of $Y_0 \subset Y$ such that both $B := Y \setminus Y_0$ and f^*B are normal crossing. Let L be any pseudo-effective line bundle L on X equipped with a positively curved singular metric h_L with algebraic singularities such that*

- (i) *there exists some regular value $z \in Y$ of f such that for some $m \in \mathbb{N}$, all the sections in $H^0(X_z, (mK_X + L)|_{X_z})$ extend locally near z ;*
- (ii) $H^0(X_z, (mK_{X_z} + L|_{X_z}) \otimes \mathcal{I}(h_{L|_{X_z}}^{1/m})) \neq \emptyset$.

Then for any regular value y of f such that

- (i) *all sections in $H^0(X_y, mK_{X_y} + L|_{X_y})$ extend locally near y ;*
- (ii) *the metric $h_{L|_{X_y}}$ is not identically equal to $+\infty$,*

the restriction map

$$H^0(X, mK_{X/Y} - m\Delta_f + L + f^*A_Y) \twoheadrightarrow H^0(X_y, (mK_{X_y} + L|_{X_y}) \otimes \mathcal{I}(h_{L|_{X_y}}^{1/m}))$$

is surjective. Here A_Y is a universal ample line bundle on Y which does not depend on L , f or m , and

$$\Delta_f := \sum_j (a_j - 1)V_j. \quad (3.1.1)$$

*where the sum is taken over all prime divisors V_j of f^*B with multiplicity a_j and with image $f(V_j)$ being a divisor in Y .*

We will apply a technical result of [18, Claim 3.5] to prove Theorem 3.7(i). Let us first recall some definitions of singularities of divisors from [61, Chapter 5.3] in a slightly different language.

Definition 3.2. Let X be a smooth projective variety, and let \mathcal{L} be a line bundle such that $H^0(X, \mathcal{L}) \neq \emptyset$. One defines

$$e(\mathcal{L}) = \sup \{1/c(D) \mid D \in |\mathcal{L}| \text{ is an effective divisor}\} \quad (3.1.2)$$

where

$$c(D) := \sup \{c > 0 \mid (X, c \cdot D) \text{ is a klt divisor}\}$$

is the *log-canonical threshold* of D .

Viehweg showed that one can control the lower bound of $e(\mathcal{L})$.

Lemma 3.3 ([61, Corollary 5.11]). *Let X be a smooth projective variety equipped with a very ample line bundle \mathcal{H} , and let \mathcal{L} be a line bundle such that $H^0(X, \mathcal{L}) \neq \emptyset$.*

- (i) *There is a uniform estimate*

$$e(\mathcal{L}) \leq c_1(\mathcal{H})^{\dim X - 1} \cdot c_1(\mathcal{L}) + 1. \quad (3.1.3)$$

- (ii) *Let $Z := X \times \cdots \times X$ (r factors) and $\mathcal{M} := \bigotimes_{i=1}^r \text{pr}_i^* \mathcal{L}$. Then $e(\mathcal{M}) = e(\mathcal{L})$.*

Let us recall the following result by Cao–Păun [18].

Lemma 3.4 (Cao–Păun). *Let $f : X \rightarrow Y$ be an algebraic fiber space such that the Kodaira dimension of the general fiber is non-negative. Assume that f is smooth over a dense Zariski open subset of $Y_0 \subset Y$ such that both $B := Y \setminus Y_0$ and f^*B are normal crossings. Then there exists a positive integer $C \geq 2$ such that for any $m \geq m_0$ and $a \in \mathbb{N}$, any $y \in Y_0$ and any section*

$$\sigma \in H^0(X_y, amCK_{X_y}),$$

there exists a section

$$\Sigma \in H^0(X, f^*A_Y - af^* \det f_*(mK_{X/Y}) + amr_mCK_{X/Y} + a(P_m + F_m)) \quad (3.1.4)$$

whose restriction to the fiber X_y is equal to $\sigma^{\otimes r_m}$. Here F_m and P_m are effective divisors on X (independent of a) such that F_m is f -exceptional with $f(F_m) \subset \text{Supp}(B)$, $\text{Supp}(P_m) \subset \text{Supp}(\Delta_f)$, $r_m := \text{rank } f_*(mK_{X/Y})$, and A_Y is the universal ample line bundle on Y defined in Theorem 3.1.

We recall the definition of a Kollár family of varieties with semi-log-canonical singularities (slc family for short).

Definition 3.5 (slc family). An *slc family* is a flat proper morphism $f : X \rightarrow B$ such that

- (i) each fiber $X_b := f^{-1}(b)$ is a projective variety with slc singularities;
- (ii) $\omega_{X/B}^{[m]}$ is flat;
- (iii) the family $f : X \rightarrow B$ satisfies the *Kollár condition*, which means that, for any $m \in \mathbb{N}$, the reflexive power $\omega_{X/B}^{[m]}$ commutes with arbitrary base change.

To make Definition 3.5(iii) precise, for every base change $\tau : B' \rightarrow B$, given the induced morphism $\rho : X' = X \times_B B' \rightarrow X$ we note that the natural homomorphism $\rho^* \omega_{X/B}^{[m]} \rightarrow \omega_{X'/B'}^{[m]}$ is an isomorphism.

Let us collect the basic properties of slc families, well-known to experts.

Lemma 3.6. *Let $g : Z \rightarrow W$ be a surjective morphism between quasi-projective manifolds with connected fibers, which is birational to an slc family $g' : Z' \rightarrow W$ whose generic fiber has at most Gorenstein canonical singularities.*

- (i) *The total space Z' is normal and has only canonical singularities at worst.*
- (ii) *If $\nu : W' \rightarrow W$ is a dominant morphism with W' smooth quasi-projective, then $Z' \times_W W' \rightarrow W'$ is still an slc family whose generic fiber has at most Gorenstein canonical singularities, and is birational to $(Z \times_W W')^\sim \rightarrow W'$.*
- (iii) *Denote by Z'^r the r -fold fiber product $Z' \times_W \cdots \times_W Z'$. Then $g'^r : Z'^r \rightarrow W$ is also an slc family whose generic fiber has at most Gorenstein canonical singularities. Moreover, Z'^r is birational to the main component $(Z^r)^\sim$ of Z^r dominating W .*
- (iv) *Let $Z^{(r)}$ be a desingularization of $(Z^r)^\sim$. Then*

$$(g^{(r)})_*(\ell K_{Z^{(r)}/W}) \simeq (g'^r)_*(\ell K_{Z'^r/W})$$

is reflexive for every sufficiently divisible $\ell > 0$.

3.2. Positivity of direct images

This section is devoted to proving Theorem 3.7 on positivity of direct images, which refines results by Viehweg–Zuo [28, Proposition 3.4] and [63, Proposition 4.3]. It will be crucially used to prove Theorem C.

Theorem 3.7. *Let $f_0 : X_0 \rightarrow Y_0$ be a smooth family of projective manifolds of general type. Assume that for any $y \in Y_0$, the set of $z \in Y_0$ with $X_z \xrightarrow{\text{bir}} X_y$ is finite.*

- (i) *For any smooth projective compactification $f : X \rightarrow Y$ of $f_0 : X_0 \rightarrow Y_0$ and any sufficiently ample line bundle A_Y over Y , $f_*(\ell K_{X/Y})^{**} \otimes A_Y^{-1}$ is globally generated over Y_0 for any $\ell \gg 0$. In particular, $f_*(\ell K_{X/Y})$ is ample with respect to Y_0 .*
- (ii) *In the setting of (i), $\det f_*(\ell K_{X/Y}) \otimes A_Y^{-r_\ell}$ is also globally generated over Y_0 for any $\ell \gg 0$, where $r_\ell = \text{rank } f_*(\ell K_{X/Y})$. In particular, $\mathbf{B}_+(\det f_*(\ell K_{X/Y})) \subset Y \setminus Y_0$.*
- (iii) *For some $r \gg 0$, there exists an algebraic fiber space $f : X \rightarrow Y$ compactifying $X_0^r \rightarrow Y_0$ such that $f_*(\ell K_{X/Y}) \otimes A_Y^{-\ell}$ is globally generated over Y_0 for ℓ large and divisible enough. Here X_0^r denotes the r -fold fiber product of $X_0 \rightarrow Y_0$, and A_Y is some sufficiently ample line bundle over Y .*

Proof. Let us first show that to prove (i) and (ii), one can assume that both $B := Y \setminus Y_0$ and f^*B are normal crossing.

For an arbitrary smooth projective compactification $f' : X' \rightarrow Y'$ of $f_0 : X_0 \rightarrow Y_0$, we take a log resolution $v : Y \rightarrow Y'$ with centers supported on $Y' \setminus Y_0$ such that $B := v^{-1}(Y' \setminus Y_0)$ is a simple normal crossing divisor. Define X to be a strong desingularization of the main component $(X' \times_{Y'} Y)^\sim$ dominant over Y

$$\begin{array}{ccccc}
 X & \longrightarrow & X' \times_{Y'} Y & \longrightarrow & X' \\
 & \searrow f & \downarrow & & \downarrow f' \\
 & & Y & \xrightarrow{v} & Y'
 \end{array} \tag{3.2.1}$$

so that f^*B is normal crossing. By [60, Lemma 2.5.a], there is the inclusion

$$v_* f_*(m K_{X/Y}) \hookrightarrow f'_*(m K_{X'/Y'}), \tag{3.2.2}$$

which is an isomorphism over Y_0 for each $m \in \mathbb{N}$. Hence for any ample line bundle A over Y' , once $f'_*(m K_{X'/Y'})^{**} \otimes (v^*A)^{-1}$ is globally generated over $v^{-1}(Y_0) \simeq Y_0$ for some $m \geq 0$, $f_*(m K_{X/Y})^{**} \otimes A^{-1}$ will also be globally generated over Y_0 . As we will see, (ii) is a direct consequence of (i). This proves the above statement.

(i) Let us fix a sufficiently ample line bundle A_Y on Y . Assume that both $B := Y \setminus Y_0$ and f^*B are normal crossing. It follows from [60, Theorem 5.2] that one can take some $b \gg a \gg 0$, $\mu \gg m \gg 0$ and $s \gg 0$ such that $\mathcal{L} := \det f_*(\mu m K_{X/Y})^{\otimes a} \otimes \det f_*(m K_{X/Y})^{\otimes b}$ is ample over Y_0 . In other words, $\mathbf{B}_+(\mathcal{L}) \subset \text{Supp}(B)$. By the definition of augmented base locus, one can even find $a, b \gg 0$ such that there exists a singular hermitian metric h_1 on $\mathcal{L} - 4A_Y$ which is smooth over Y_0 , and the curvature current satisfies $\sqrt{-1} \Theta_{h_1, \mathcal{L}}(\mathcal{L}) \geq \omega$ for some Kähler form ω in Y . Denote $r_1 := \text{rank } f_*(\mu m K_{X/Y})$

and $r_2 := \text{rank } f_*(mK_{X/Y})$. It follows from Lemma 3.4 that for any sections

$$\sigma_1 \in H^0(X_y, a\mu mCK_{X_y}), \quad \sigma_2 \in H^0(X_y, bmCK_{X_y}),$$

there exist effective divisors Σ_1 and Σ_2 such that

$$\begin{aligned} \Sigma_1 + af^* \det f_*(m\mu K_{X/Y}) - f^* A_Y &\stackrel{\text{linear}}{\sim} am\mu r_1 CK_{X/Y} + P_1 + F_1, \\ \Sigma_2 + bf^* \det f_*(mK_{X/Y}) - f^* A_Y &\stackrel{\text{linear}}{\sim} bmr_2 CK_{X/Y} + P_2 + F_2, \end{aligned}$$

and

$$\Sigma_1|_{X_y} = \sigma_1^{\otimes r_1}, \quad \Sigma_2|_{X_y} = \sigma_2^{\otimes r_2}.$$

Here F_i is f -exceptional with $f(F_i) \subset \text{Supp}(B)$, $\text{Supp}(P_i) \subset \text{Supp}(\Delta_f)$ for $i = 1, 2$.

Write $N := am\mu r_1 C + bmr_2 C$, $P := P_1 + P_2$ and $F := F_1 + F_2$. Fix any $y \in Y_0$. Then the effective divisor $\Sigma_1 + \Sigma_2$ induces a singular hermitian metric h_2 on the line bundle $L_2 := NK_{X/Y} - f^* \mathcal{L} + 2f^* A_Y + P + F$ such that $h_2|_{X_y}$ is not identically equal to $+\infty$, and the singular hermitian metric $h := f^* h_1 \cdot h_2$ over $L_0 := L_2 + f^* \mathcal{L} - 4f^* A_Y = NK_{X/Y} - 2f^* A_Y + P + F$ is not identically $+\infty$ either. In particular, for ℓ sufficiently large, the multiplier ideal sheaf $\mathcal{I}(h|_{X_y}^{1/\ell})$ equals \mathcal{O}_{X_y} . By Siu's invariance of plurigenera, all global sections in $H^0(X_y, (\ell K_X + L_0)|_{X_y}) \simeq H^0(X_y, (\ell + N)K_{X_y})$ extend locally, and we can thus apply Theorem 3.1 to obtain the desired surjectivity

$$H^0(X, \ell K_{X/Y} + L_0 - \ell \Delta_f + f^* A_Y) \twoheadrightarrow H^0(X_y, (\ell + N)K_{X_y}). \quad (3.2.3)$$

Recall that $\text{Supp}(P) \subset \text{Supp}(\Delta_f)$. Then $\ell f^* B \geq P$ for $\ell \gg 0$, and one has the inclusion of sheaves

$$\ell K_{X/Y} + L_0 - \ell \Delta_f + f^* A_Y \hookrightarrow (N + \ell)K_{X/Y} - f^* A_Y + F,$$

which is an isomorphism over X_0 . By (3.2.3) this implies that the direct image sheaves $f_*(\ell K_{X/Y} - f^* A_Y + F)$ are globally generated over some Zariski open set $U_y \subset Y_0$ containing y for $\ell \gg 0$. Since y is an arbitrary point in Y_0 , the direct image $f_*(\ell K_{X/Y} + F) \otimes A_Y^{-1}$ is globally generated over Y_0 for $\ell \gg 0$ by noetherianity. Recall that F is f -exceptional with $f(F) \subset \text{Supp}(B)$. Then there is an injection

$$f_*(\ell K_{X/Y} + F) \otimes A_Y^{-1} \hookrightarrow f_*(\ell K_{X/Y})^{**} \otimes A_Y^{-1}$$

which is an isomorphism over Y_0 . Hence $f_*(\ell K_{X/Y})^{**} \otimes A_Y^{-1}$ is also globally generated over Y_0 . Hence $f_*(\ell K_{X/Y})$ is ample with respect to Y_0 for $\ell \gg 0$. The first claim follows.

(ii) The trick proving the second claim has already appeared in [23] in proving a conjecture by Demailly–Peternell–Schneider. We first recall that $f_*(\ell K_{X/Y})$ is locally free outside a codimension 2 analytic subset of Y . By the proof of Theorem 3.7(i), for ℓ sufficiently large and divisible, $f_*(\ell K_{X/Y} + F) \otimes A_Y^{-1}$ is locally free and generated by global sections over Y_0 , where F is some f -exceptional effective divisor. Therefore, its determinant $\det f_*(\ell K_{X/Y} + F) \otimes A_Y^{-r_\ell}$ is also globally generated over Y_0 , where $r_\ell := \text{rank } f_*(\ell K_{X/Y})$. Since F is f -exceptional and effective, one has

$$\det f_*(\ell K_{X/Y} + F) \otimes A_Y^{-r_\ell} = \det f_*(\ell K_{X/Y}) \otimes A_Y^{-r_\ell},$$

and therefore $\det f_*(\ell K_{X/Y}) \otimes A_Y^{-r\ell}$ is also globally generated over Y_0 . By the very definition of the augmented base locus $\mathbf{B}_+(\bullet)$ we conclude that

$$\mathbf{B}_+(\det f_*(\ell K_{X/Y})) \subset \text{Supp}(B).$$

The second claim is proved.

(iii) We combine the ideas in [63, Proposition 4.1] as well as the pluricanonical extension techniques of Theorem 3.1. By Corollary A.2, there exists a smooth projective compactification Y of Y_0 with $B := Y \setminus Y_0$ simple normal crossing, a non-singular finite covering $\psi : W \rightarrow Y$, and an slc family $g' : Z' \rightarrow W$ which extends the family $X_0 \times_{Y_0} W$. By Lemma 3.6(iii), for any $r \in \mathbb{Z}_{>0}$, the r -fold fiber product $g'' : Z'' \rightarrow W$ is still an slc family, which compactifies the smooth family $X_0^r \times_{Y_0} W \rightarrow W_0$, where $W_0 := \psi^{-1}(Y_0)$. Note that Z'' has canonical singularities.

Take a smooth projective compactification $f : X \rightarrow Y$ of $X_0^r \rightarrow Y_0$ such that f^*B is normal crossing. Let $Z \rightarrow Z''$ be a strong desingularization of Z'' , which also resolves this birational map $Z'' \dashrightarrow (X \times_Y W)^{\sim}$. Then $g : Z \rightarrow W$ is smooth over $W_0 := \psi^{-1}(Y_0)$.

$$\begin{array}{ccccccc} & & & & & & \searrow \\ & & & & & & \text{---} \\ Z & \longrightarrow & Z'' & \dashrightarrow & X & \longleftarrow & (X \times_Y W)^{\sim} \\ & \searrow & \downarrow & & \downarrow & & \downarrow \\ & & W & \xrightarrow{\psi} & Y & \xleftarrow{\psi} & W \\ & \downarrow & & & & & \\ W & \xlongequal{\quad} & W & & & & \end{array}$$

Let \tilde{Z} be a strong desingularization of Z' , which is thus smooth over $W_0 := \psi^{-1}(Y_0)$. For the new family $\tilde{g} : \tilde{Z} \rightarrow W$, we denote $\tilde{Z}_0 := \tilde{g}^{-1}(W_0)$. Then $\tilde{Z}_0 \rightarrow W_0$ is also a smooth family, and any fiber of Z_w with $w \in W_0$ is a projective manifold of general type. By our assumption in the theorem, for any $w \in W_0$, the set of $w' \in W_0$ with $\tilde{Z}_{w'} \stackrel{\text{bir}}{\sim} \tilde{Z}_w$ is finite as $\psi : W \rightarrow Y$ is a finite morphism. We can thus apply Theorem 3.7(i, ii) to our new family $\tilde{g} : \tilde{Z} \rightarrow W$.

From now on, we will always assume that $\ell \gg 0$ is sufficiently divisible so that $\ell K_{Z'}$ is Cartier. Let A_Y be a sufficiently ample line bundle over Y , so that $A_W := \psi^*A_Y$ is also *sufficiently ample*. Since Z' has canonical singularity, $\tilde{g}_*(\ell K_{\tilde{Z}/W}) = g'_*(\ell K_{Z'/W})$. It follows from Theorem 3.7(ii) that, for any $\ell \gg 0$, the line bundle

$$\det \tilde{g}_*(\ell K_{\tilde{Z}/W}) \otimes A_W^{-r} = \det g'_*(\ell K_{Z'/W}) \otimes A_W^{-r} \quad (3.2.4)$$

is globally generated over W_0 , where $r := \text{rank } g'_*(\ell K_{Z'/W})$ depends on ℓ . Then there exists a positively curved singular hermitian metric h_{\det} on the line bundle $\det g'_*(\ell K_{Z'/W}) \otimes A_W^{-r}$ such that h_{\det} is smooth over W_0 .

By the base change properties of slc families (see [8, Proposition 2.12] and [40, Lemma 2.6]), one has

$$\omega_{Z''/W}^{[\ell]} \simeq \bigotimes_{i=1}^r \text{pr}_i^* \omega_{Z'/W}^{[\ell]}, \quad g''^*(\ell K_{Z''/W}) \simeq \bigotimes^r g'^*(\ell K_{Z'/W}),$$

where $\text{pr}_i : Z'^r \rightarrow Z'$ is the i -th directional projection map. Hence $\ell K_{Z'^r}$ is Cartier as well, and we have

$$\bigotimes^r g'_*(\ell K_{Z'/W}) \simeq g_*^r(\ell K_{Z'^r/W}) = g_*(\ell K_{Z/W}).$$

By Lemma 3.6(iv), $g_*(\ell K_{Z/W})$ is reflexive, and we thus have

$$\det g'_*(\ell K_{Z'/W}) \rightarrow \bigotimes^r g'_*(\ell K_{Z'/W}) \simeq g_*(\ell K_{Z/W}),$$

which induces a natural effective divisor

$$\Gamma \in |\ell K_{Z/W} - g^* \det g'_*(\ell K_{Z'/W})|$$

such that $\Gamma|_{Z_w} \neq 0$ for any (smooth) fiber Z_w with $w \in W_0$. By Lemma 3.3 for any $w \in W_0$ the log-canonical threshold

$$\begin{aligned} c(\Gamma|_{Z_w}) &\geq \frac{1}{e(\ell K_{Z_w})} = \frac{1}{e(\bigotimes_{i=1}^r \text{pr}_i^*(K_{Z'_w}^{\otimes \ell}))} = \frac{1}{e(\ell K_{Z'_w})} \\ &\geq \frac{1}{\ell \cdot c_1(\mathcal{A})^{d-1} \cdot c_1(K_{Z'_w}) + 1} \geq \frac{2}{(C-1)\ell} \end{aligned} \quad (3.2.5)$$

for some $C \in \mathbb{N}$ which does not depend on ℓ or $w \in W_0$. Denote by h the singular hermitian metric on

$$\ell K_{Z/W} - g^* \det g'_*(\ell K_{Z'/W})$$

induced by Γ . By the definition of log-canonical threshold, the multiplier ideal sheaf $\mathcal{J}(h|_{Z_w}^{\frac{1}{(C-1)\ell}})$ equals \mathcal{O}_{Z_w} for any fiber Z_w with $w \in W_0$. Let us define a positively curved singular metric $h_{\mathcal{F}}$ for the line bundle $\mathcal{F} := \ell K_{Z/W} - r g^* A_W$ by setting $h_{\mathcal{F}} := h \cdot g^* h_{\det}$. Then $\mathcal{J}(h_{\mathcal{F}}|_{Z_w}^{\frac{1}{(C-1)\ell}}) = \mathcal{O}_{Z_w}$ for any $w \in W_0$.

For any $n \in \mathbb{N}^*$, applying Theorem 3.1 to $n\mathcal{F}$ we obtain the surjectivity

$$H^0(Z, (C-1)n\ell K_{Z/W} + n\mathcal{F} + g^* A_W) \twoheadrightarrow H^0(Z_w, Cn\ell K_{Z_w}) \quad (3.2.6)$$

for all $w \in W_0$. In other words,

$$g_*(C\ell n K_{Z/W}) \otimes A_W^{-(nr-1)}$$

is globally generated over W_0 for any $\ell \gg 0$ and any $n \geq 1$.

Since K_{X_Y} is big, one thus has

$$r = r_\ell \sim \ell^d \quad \text{as } \ell \rightarrow +\infty$$

where $d := \dim Z_w \geq 2$ (if the fibers of f are curves, one can take a fiber product to replace the original family). Recall that C is a constant which does not depend on ℓ . One can thus *a priori* take an $\ell \gg 0$ such that $r \gg C\ell$. In conclusion, for sufficiently large and divisible m ,

$$g_*(m K_{Z/W}) \otimes A_W^{-2m} = g_*(m K_{Z/W}) \otimes \psi^* A_Y^{-2m}$$

is globally generated over W_0 . Therefore, we have a morphism

$$\bigoplus_{i=1}^N \psi^* A_Y^m \rightarrow g_*(mK_{Z/W}) \otimes \psi^* A_Y^{-m}, \quad (3.2.7)$$

which is surjective over W_0 . On the other hand, by [60, Lemma 2.5.b], one has the inclusion

$$g_*(mK_{Z/W}) \hookrightarrow \psi^* f_*(mK_{X/Y}),$$

which is an isomorphism over W_0 . Now, (3.2.7) induces a morphism

$$\bigoplus_{i=1}^N \psi_* \mathcal{O}_W \otimes A_Y^m \rightarrow \psi_* g_*(mK_{Z/W}) \otimes A_Y^{-m} \rightarrow \psi_* \psi^*(f_*(mK_{X/Y})) \otimes A_Y^{-m}, \quad (3.2.8)$$

which is surjective over Y_0 . Note that even if $f_*(mK_{X/Y})$ is merely a coherent sheaf, the projection formula $\psi_* \psi^*(f_*(mK_{X/Y})) = f_*(mK_{X/Y}) \otimes \psi_* \mathcal{O}_W$ still holds since ψ is finite (see [4, Lemma 5.7]). The trace map

$$\psi_* \mathcal{O}_W \rightarrow \mathcal{O}_Y$$

splits the natural inclusion $\mathcal{O}_Y \rightarrow \psi_* \mathcal{O}_W$, and is thus surjective. Hence (3.2.8) gives rise to a morphism

$$\bigoplus_{i=1}^N \psi_* \mathcal{O}_W \otimes A_Y^m \rightarrow \psi_* g_*(mK_{Z/W}) \otimes A_Y^{-m} \xrightarrow{\Phi} f_*(mK_{X/Y}) \otimes A_Y^{-m}, \quad (3.2.9)$$

which is surjective over Y_0 . By taking m sufficiently large, we may assume that $\psi_* \mathcal{O}_W \otimes A_Y^m$ is generated by its global sections. Then $f_*(mK_{X/Y}) \otimes A_Y^{-m}$ is globally generated over Y_0 , completing the proof. \blacksquare

Let us prove the following Bertini-type result, which will be used in the proof of Theorem 3.11.

Lemma 3.8 (A Bertini-type result). *Let $f : X \rightarrow Y$ be the projective family in Theorem 3.7(iii). Then for any given smooth fiber X_y with $y \in Y_0$, there exists $H \in |\ell K_{X/Y} - \ell f^* A_Y|$ such that $H|_{X_y}$ is smooth. In particular, there is a Zariski open neighborhood $V_0 \supset y$ such that H is smooth over V_0 .*

Proof. By Siu's invariance of plurigenera and Grauert–Grothendieck's ‘‘cohomology and base change’’, we know that $f_*(\ell K_{X/Y}) \otimes A_Y^{-\ell}$ is locally free on Y_0 , and the natural map

$$(f_*(\ell K_{X/Y}) \otimes A_Y^{-\ell})_y \rightarrow H^0(X_y, \ell K_{X_y})$$

is an isomorphism for any $y \in Y_0$. Since K_{X_y} is assumed to be semi-ample, one can take $\ell \gg 0$ such that $|\ell K_{X_y}|$ is base point free. By the Bertini theorem, one can take a

section $s \in H^0(X_y, \ell K_{X_y})$ whose zero locus is a smooth hypersurface on X_y . By Theorem 3.7(iii), one has the surjection

$$H^0(Y, f_*(\ell K_{X/Y}) \otimes A_Y^{-\ell}) \twoheadrightarrow (f_*(\ell K_{X/Y}) \otimes A_Y^{-\ell})_y \xrightarrow{\cong} H^0(X_y, \ell K_{X_y}).$$

Hence there is

$$\sigma \in H^0(X, \ell K_{X/Y} - \ell f^* A_Y) = H^0(Y, f_*(\ell K_{X/Y}) \otimes A_Y^{-\ell})$$

which extends the section s . In other words, for the zero divisor $H = (\sigma = 0)$, its restriction to X_y is smooth. Hence there is a Zariski open neighborhood $V_0 \ni y$ such that H is smooth over V_0 . The lemma is proved. \blacksquare

Remark 3.9. We do not know how to find a hypersurface $H \in |\ell K_{X/Y} - \ell f^* A_Y|$ whose discriminant locus in Y is normal crossing. We have to blow up the base, $v : Y' \rightarrow Y$, to achieve this. As $f : X \rightarrow Y$ is not flat in general, for the new family

$$\begin{array}{ccccccc} X' & \longrightarrow & (X \times_Y Y')^\sim & \longrightarrow & X \times_Y Y' & \longrightarrow & X \\ \downarrow f' & & \downarrow & & \downarrow & & \downarrow f \\ Y' & \xlongequal{\quad} & Y' & \xlongequal{\quad} & Y' & \xrightarrow{\quad v \quad} & Y \end{array}$$

where X' is a desingularization of $(X \times_Y Y')^\sim$, in general

$$v^* f_*(\ell K_{X/Y}) \not\subset f'_*(\ell K_{X'/Y'}).$$

In other words, although the discriminant locus of $v^* H \rightarrow Y'$ is a simple normal crossing divisor in Y' , $v^* H$ might not lie in $|\ell K_{X'/Y'} - \ell f'^* v^* A_Y|$. We will overcome this problem in Theorem 3.11 at the cost of appearance of some f' -exceptional divisors.

Since the \mathbb{Q} -mild reduction in Corollary A.2 holds for any smooth surjective projective morphism with connected fibers and smooth base, it follows from our proof of Theorem 3.7(iii) and Kawamata's theorem [34] that one still has the *generic* global generation as follows.

Theorem 3.10. *Let $f_U : U \rightarrow V$ be a smooth projective morphism between quasi-projective varieties with connected fibers. Assume that the general fiber F of f_U has semi-ample canonical bundle, and f_U is of maximal variation. Then there is an integer $r \gg 0$ and a smooth projective compactification $f : X \rightarrow Y$ of $U^r \rightarrow V$ such that $f_*(m K_{X/Y}) \otimes \mathcal{A}^{-m}$ is globally generated over some Zariski open subset of V . Here $U^r \rightarrow V$ is the r -fold fiber product of $U \rightarrow V$, and \mathcal{A} is some ample line bundle on Y . \blacksquare*

3.3. Sufficiently many “moving” hypersurfaces

As we have seen in §1.2 on the construction of VZ Higgs bundles, one has to apply the branch cover trick to construct a negatively twisted Hodge bundle on the compactification of the base, which is well-defined outside a simple normal crossing divisor. This means that the hypersurface $H \in |\ell K_{X/Y} - \ell f^* A_Y|$ in constructing the cyclic cover is smooth

over the complement of an SNC divisor of the base. As discussed in Lemma 3.8, in general we cannot perform a simple blow-up of the base to achieve this. In this subsection we will overcome this difficulty by applying the methods of [48, Proposition 4.4]. It will be our basic setup in constructing refined VZ Higgs bundles in §3.4.

Theorem 3.11. *Let $X_0 \rightarrow Y_0$ be a smooth family of minimal projective manifolds of general type over a quasi-projective manifold Y_0 . Suppose that for any $y \in Y_0$, the set of $z \in Y_0$ with $X_z \stackrel{\text{bir}}{\sim} X_y$ is finite. Let $Y \supset Y_0$ be the smooth compactification in Corollary A.2. Fix any $y_0 \in Y_0$ and some sufficiently ample line bundle A_Y on Y . Then there exist a birational morphism $v : Y' \rightarrow Y$ and a new algebraic fiber space $f' : X' \rightarrow Y'$ which is smooth over $v^{-1}(Y_0)$ and such that for any sufficiently large and divisible ℓ , one can find a hypersurface*

$$H \in |\ell K_{X'/Y'} - \ell(v \circ f')^* A_Y + \ell E| \quad (3.3.1)$$

with the following properties:

- the divisor $D := v^{-1}(Y \setminus Y_0)$ is simple normal crossing;
- there exists a reduced divisor S in Y' such that $D + S$ is simple normal crossing and $H \rightarrow Y'$ is smooth over $Y' \setminus (D \cup S)$;
- the exceptional locus $\text{Ex}(v)$ is contained in $\text{Supp}(D + S)$, and $y_0 \notin v(D \cup S)$;
- the divisor E is effective and f' -exceptional with $f'(E) \subset \text{Supp}(D + S)$.

Moreover, when $X_0 \rightarrow Y_0$ is effectively parametrized over some open set containing y_0 , so is the new family $X' \rightarrow Y'$.

Proof. The proof is a continuation of that of Theorem 3.7(iii), and we adopt the notations therein. By (3.2.9) and the isomorphism

$$H^0(Z, \ell K_{Z/W} - \ell g^* A_W) \simeq H^0(Z'^r, \ell K_{Z'^r/W} - \ell (g'^r)^* A_W),$$

the morphism $\Phi : \psi_* g_* (\ell K_{Z/W}) \otimes A_Y^{-\ell} \rightarrow f_* (\ell K_{X/Y}) \otimes A_Y^{-\ell}$ in (3.2.9) gives rise to a natural map

$$\Upsilon : H^0(Z'^r, \ell K_{Z'^r/W} - \ell (g'^r)^* A_W) \rightarrow H^0(Y, f_* (\ell K_{X/Y}) \otimes A_Y^{-\ell}) \quad (3.3.2)$$

whose image I generates $f_* (\ell K_{X/Y}) \otimes A_Y^{-\ell}$ over Y_0 . Note that Υ is functorial in the sense that it does not depend on the choice of the birational model $Z \rightarrow Z'^r$. By the base point free theorem, for any $y \in Y_0$, K_{X_y} is semi-ample, and we can assume that $\ell \gg 0$ is sufficiently large and divisible so that $\ell K_{X/Y}$ is relatively semi-ample over Y_0 . Hence we can take a section

$$\sigma \in H^0(Z'^r, \ell K_{Z'^r/W} - \ell (g'^r)^* A_W) \quad (3.3.3)$$

such that the zero divisor of

$$\Upsilon(\sigma) \in H^0(X, \ell K_{X/Y} - \ell f^* A_Y) = H^0(Y, f_* (\ell K_{X/Y}) \otimes A_Y^{-\ell}),$$

denoted by $H_1 \in |\ell K_{X^{(r)}/Y} - \ell(f^{(r)})^* A_Y|$, is *transverse* to the fiber X_{y_0} . Denote by T the *discriminant locus* of $H_1 \rightarrow Y$, and set $B := Y \setminus Y_0$. Then $y_0 \notin T \cup B$. Take a log-resolution $\nu : Y' \rightarrow Y$ with centers in $T \cup B$ so that both $D := \nu^{-1}(B)$ and $D + S := \nu^{-1}(T \cup B)$ are simple normal crossing. Let X' be a strong desingularization of $(X \times_Y Y')^\sim$, and write $f' : X' \rightarrow Y'$, which is smooth over $Y'_0 := \nu^{-1}(Y_0)$. Set $X'_0 := f'^{-1}(Y'_0)$. It suffices to show that there exists a hypersurface H in (3.3.1) with $H_{\downarrow(\nu \circ f')^{-1}(V)} = H_{1\downarrow(f^{(r)})^{-1}(V)}$, where $V := Y \setminus (S' \cup B) \subset Y_0$. Since the birational morphism ν is isomorphic at y_0 , we can write y_0 as $\nu^{-1}(y_0)$ abusively.

Now we use similar arguments to [48, Proposition 4.4] to prove the existence of H (the authors of [48] apply their methods to *mild morphisms*). Let W' be a strong desingularization of $W \times_Y Y'$ which is finite at $y_0 \in Y'$. Write $W'_0 := \nu^{-1}(W_0)$. By Lemma 3.6(ii), the new family $Z'' := Z'' \times_{W'} W' \rightarrow W'$ is still an slc family, which compactifies the smooth family $X'_0 \times_{Y'_0} W' \rightarrow W'_0$. Let M' be a desingularization of Z'' such that it resolves the rational maps to X' as well as Z .

$$\begin{array}{ccccc}
 & & X & \longleftarrow & Z & \longrightarrow & Z'' \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & \mu & & \eta & & \mu' \\
 & & \downarrow & & \downarrow & & \downarrow \\
 X' & \longleftarrow & M' & \longrightarrow & Z'' & & \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & Y & \longleftarrow & W & \xlongequal{\quad} & W \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & \psi & & \psi' & & \psi' \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & Y' & \longleftarrow & W' & \xlongequal{\quad} & W' \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & \nu & & \nu' & & \nu' \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & Y' & \longleftarrow & W' & \xlongequal{\quad} & W'
 \end{array}$$

By the properties of slc families, $\mu'^* \omega_{Z''/W}^{[\ell]} = \omega_{Z''/W'}^{[\ell]}$, which induces a natural map

$$\mu^* : H^0(Z'', \ell K_{Z''/W} - \ell(g^{(r)})^* A_W) \rightarrow H^0(Z'', \ell K_{Z''/W'} - \ell(\nu' \circ g'')^* A_W). \quad (3.3.4)$$

Since both Z'' and Z' have canonical singularities, one has the natural morphisms

$$g_*(\ell K_{Z/W}) \simeq (g^{(r)})_*(\ell K_{Z''/W}), \quad h'_*(\ell K_{M'/W'}) = g''_*(\ell K_{Z''/W'}).$$

We can leave out a subvariety of codimension at least 2 in Y' supported on $D + S$ (which thus avoids y_0 by our construction) so that $\psi' : W' \rightarrow Y'$ becomes a *flat finite* morphism. As discussed at the beginning of the proof, there is also a natural map

$$\Upsilon' : H^0(Z'', \ell K_{Z''/W'} - \ell(\nu' \circ g'')^* A_W) \rightarrow H^0(X', \ell K_{X'/Y'} - \ell(\nu \circ f')^* A_Y) \quad (3.3.5)$$

as in (3.3.2), obtained by factorizing through M' .

Note that for $V := Y \setminus (T \cup B)$, $\nu : \nu^{-1}(V) \xrightarrow{\simeq} V$ is also an isomorphism, and thus the restriction of $X \rightarrow Y$ to V is isomorphic to the restriction of $X' \rightarrow Y'$ to $\nu^{-1}(V)$.

Hence by our construction, the restriction of $Z'' \rightarrow W$ to $\psi^{-1}(V)$ is isomorphic to the restriction of $Z'' \rightarrow W'$ to $(v \circ \psi')^{-1}(V) = (v' \circ \psi)^{-1}(V)$. In particular, under the above isomorphism, for the section $\sigma \in H^0(Z'', \ell K_{Z''/W} - \ell(g'')^* A_W)$ in (3.3.3) with $\Upsilon(\sigma)$ defining H_1 , one has

$$\Upsilon(\sigma)|_{f^{-1}(V)} \simeq \Upsilon'(\mu^* \sigma)|_{(v \circ f')^{-1}(V)}$$

where μ^* and Υ' are defined in (3.3.4) and (3.3.5). Denote by \tilde{H} the zero divisor defined by

$$\Upsilon'(\mu^* \sigma) \in H^0(X', \ell K_{X'/Y'} - \ell(v \circ f')^* A_Y).$$

Recall that H_1 is smooth over V , then \tilde{H} is also smooth over $v^{-1}(V)$.

Note that $\Upsilon'(\mu^* \sigma) \in H^0(Y', f'_*(\ell K_{X'/Y'}) \otimes v^* A_Y^\ell)$ is only defined over a big open subset of Y' containing $v^{-1}(V)$. Hence it extends to a global section

$$s \in H^0(X', \ell K_{X'/Y'} - \ell(v \circ f')^* A_Y + \ell E),$$

where E is an f' -exceptional effective divisor with $f'(E) \subset \text{Supp}(D + S)$. Denote by H the hypersurface in X' defined by s . Hence $H|_{(v \circ f')^{-1}(V)} = \tilde{H}|_{(v \circ f')^{-1}(V)}$, which is smooth over $v^{-1}(V) = Y' \setminus (D \cup S) \simeq V \ni y_0$. Note that the property of effective parametrization is invariant under fiber product. The theorem follows. \blacksquare

3.4. Kobayashi hyperbolicity of the moduli spaces

In this subsection, for an effectively parametrized smooth family of minimal projective manifolds of general type, we refine the Viehweg–Zuo Higgs bundles of Theorem 1.2 so that we can apply Theorem 2.6 and the bimeromorphic criterion for Kobayashi hyperbolicity in Lemma 2.5 to prove Theorem C.

Theorem 3.12. *Let $U \rightarrow V$ be an effectively parametrized smooth family of minimal projective manifolds of general type over the quasi-projective manifold V . Then for any given point $y \in V$, there exists a smooth projective compactification Y for a birational model $v : \tilde{V} \rightarrow V$, and a VZ Higgs bundle $(\tilde{\mathcal{E}}, \tilde{\theta}) \supset (\mathcal{F}, \eta)$ over Y with the following properties:*

- (i) *there is a Zariski open set V_0 of V containing y such that $v : v^{-1}(V_0) \rightarrow V_0$ is an isomorphism;*
- (ii) *both $D := Y \setminus \tilde{V}$ and $D + S := Y \setminus v^{-1}(V_0)$ are simple normal crossing divisors in Y ;*
- (iii) *the Higgs bundle $(\tilde{E}, \tilde{\theta})$ has log poles supported on $D \cup S$, that is, $\tilde{\theta} : \tilde{E} \rightarrow \tilde{E} \otimes (\log(D + S))$;*
- (iv) *the morphism*

$$\tau_1 : \mathcal{T}_Y(-\log D) \rightarrow \mathcal{L}^{-1} \otimes E^{n-1,1} \tag{3.4.1}$$

induced by the Higgs subsheaf (\mathcal{F}, η) is injective over V_0 .

Proof. The proof is a continuation of that of Theorem 1.2, and we will adopt the same notations.

We first prove that for any $y \in V$, the set of $z \in V$ with $X_z \stackrel{\text{bir}}{\sim} X_y$ is finite. Take a polarization \mathcal{H} for $U \rightarrow V$ with Hilbert polynomial h . Denote by $\mathcal{P}_h(V)$ the set of such pairs $(U \rightarrow V, \mathcal{H})$, up to isomorphism and up to fiberwise numerical equivalence for \mathcal{H} . By [61, Section 7.6], there exists a coarse quasi-projective moduli scheme P_h for \mathcal{P}_h , and thus the family induces a morphism $V \rightarrow P_h$. By the assumption that the family $U \rightarrow V$ is effectively parametrized, the induced morphism $V \rightarrow P_h$ is quasi-finite, which in turn shows that the set of $z \in V$ with X_z isomorphic to X_y is finite. Note that a projective manifold of general type has finitely many minimal models. Hence the set of $z \in V'$ with $X_z \stackrel{\text{bir}}{\sim} X_y$ is finite as well.

Now we will choose the hypersurface in (1.2.1) carefully so that the cyclic cover construction in Theorem 1.2 can provide the desired refined VZ Higgs bundle. Let $Y' \supset V$ be the smooth compactification in Corollary A.2. By Theorem 3.11, for any given point $y \in V$ and any sufficiently ample line bundle \mathcal{A} on Y' , there exists a birational morphism $v : Y \rightarrow Y'$ and a new algebraic fiber space $f : X \rightarrow Y$ such that one can find a hypersurface

$$H \in |\ell\Omega_{X/Y}^n(\log \Delta) - \ell(v \circ f)^*\mathcal{A} + \ell E|, \quad n := \dim X - \dim Y, \quad (3.4.2)$$

such that

- the inverse image $D := v^{-1}(Y' \setminus V)$ is a simple normal crossing divisor;
- there exists a reduced divisor S such that $D + S$ is simple normal crossing, and $H \rightarrow Y$ is smooth over $V_0 := Y \setminus (D \cup S)$;
- the restriction $v : v^{-1}(V_0) \rightarrow V_0$ is an isomorphism;
- the given point y is contained in V_0 ;
- the divisor E is effective and f -exceptional with $f(E) \subset \text{Supp}(D + S)$;
- for any $z \in V := v^{-1}(V')$, the canonical bundle of the fiber $X_z := f^{-1}(z)$ is big and nef;
- the restricted family $f^{-1}(V_0) \rightarrow V_0$ is smooth and effectively parametrized.

Here we set $\Delta := f^*D$ and $\Sigma := f^*S$. Write $\mathcal{L} := v^*\mathcal{A}$. Now we take the cyclic cover with respect to H in (3.4.2) instead of that in (1.2.1), and perform the same construction of VZ Higgs bundle $(\tilde{\mathcal{E}}, \tilde{\theta}) \supset (\mathcal{F}, \tau)$ as in Theorem 1.2. Theorem 3.12(i–iii) can be seen directly from the properties of H and the cyclic construction.

Theorem 3.12(iv) has already appeared in [48, Proposition 2.11] implicitly, and we give a proof here for the sake of completeness. Recall that both Z and H are smooth over V_0 . Denote

$$\begin{aligned} H_0 &:= H \cap f^{-1}(V_0), \\ f_0 : X_0 = f^{-1}(V_0) &\rightarrow V_0, \quad g_0 : Z_0 = g^{-1}(V_0) \rightarrow V_0. \end{aligned}$$

We have

$$\begin{aligned}
F_{\downarrow V_0}^{n,0} &= f_*(\Omega_{X/Y}^n(\log \Delta) \otimes \mathcal{L}^{-1})_{\downarrow V_0} = \mathcal{O}_{V_0}, \\
E_{\downarrow V_0}^{n-1,1} &= R^1(g_0)_*(\Omega_{Z_0/V_0}^{n-1}) \\
&= R^1(f_0)_* \left(\Omega_{X_0/V_0}^{n-1} \oplus \bigoplus_{i=1}^{\ell-1} \Omega_{X_0/V_0}^{n-1}(\log H_0) \otimes (K_{X_0/V_0} \otimes f_0^* \mathcal{L}^{-1})^{-i} \right) \quad (3.4.3) \\
F_{\downarrow V_0}^{n-1,1} &= R^1 f_*(\Omega_{X/Y}^{n-1}(\log \Delta) \otimes \mathcal{L}^{-1})_{\downarrow V_0} \\
&= R^1(f_0)_*(\Omega_{X_0/V_0}^{n-1} \otimes K_{X_0/V_0}^{-1}) \simeq R^1(f_0)_*(\mathcal{T}_{X_0/V_0}).
\end{aligned}$$

Hence $\tau_1 \downarrow_{V_0}$ factors through

$$\begin{aligned}
\tau_1 \downarrow_{V_0} : \mathcal{T}_{V_0} &\xrightarrow{\rho} R^1(f_0)_*(\mathcal{T}_{X_0/V_0}) \xrightarrow{\simeq} R^1(f_0)_*(\Omega_{X_0/V_0}^{n-1} \otimes K_{X_0/V_0}^{-1}) \\
&\rightarrow R^1(f_0)_*(\Omega_{X_0/V_0}^{n-1}(\log H_0) \otimes K_{X_0/V_0}^{-1}) \rightarrow R^1(g_0)_*(\Omega_{Z_0/V_0}^{n-1}) \otimes \mathcal{L}^{-1},
\end{aligned}$$

where ρ is the Kodaira–Spencer map. Although the intermediate objects in the above factorization might not be locally free, the \mathbb{C} -linear map induced by the sheaf morphism $\tau_1 \downarrow_{V_0}$ at $z \in V_0$,

$$\tau_{1,z} : \mathcal{T}_{Y,z} \rightarrow (\mathcal{L}^{-1} \otimes E^{n-1,1})_z,$$

coincides with the following composition of \mathbb{C} -linear maps between finite-dimensional complex vector spaces:

$$\begin{aligned}
\tau_{1,z} : \mathcal{T}_{Y,z} &\xrightarrow{\rho_z} H^1(X_z, \mathcal{T}_{X_z}) \xrightarrow{\simeq} H^1(X_z, \Omega_{X_z}^{n-1} \otimes K_{X_z}^{-1}) \\
&\xrightarrow{j_z} H^1(X_z, \Omega_{X_z}^{n-1}(\log H_z) \otimes K_{X_z}^{-1}) \rightarrow H^1(Z_z, \Omega_{Z_z}^{n-1}). \quad (3.4.4)
\end{aligned}$$

To prove Theorem 3.12(iv), it then suffices to prove that each linear map in (3.4.4) is injective for any $z \in V_0$.

By the effective parametrization assumption, ρ_z is injective. The map j_z in (3.4.4) is the same as the H^1 -cohomology map of the short exact sequence

$$0 \rightarrow K_{X_z}^{-1} \otimes \Omega_{X_z}^{n-1} \rightarrow K_{X_z}^{-1} \otimes \Omega_{X_z}^{n-1}(\log H_z) \rightarrow K_{X_z \downarrow H_z}^{-1} \otimes \Omega_{H_z}^{n-2} \rightarrow 0.$$

Observe that $K_{X_z \downarrow H_z}$ is big. Indeed, this follows from

$$\text{vol}(K_{X_z \downarrow H_z}) = c_1(K_{X_z \downarrow H_z})^{n-1} = c_1(K_{X_z})^{n-1} \cdot H_z = \ell c_1(K_{X_z})^n = \ell \text{vol}(K_{X_z}) > 0.$$

Hence j_z is injective by the Bogomolov–Sommese vanishing theorem

$$H^0(H_z, K_{X_z \downarrow H_z}^{-1} \otimes \Omega_{H_z}^{d-2}) = 0,$$

as observed in [48]. Since $\psi_z : Z_z \rightarrow X_z$ is the cyclic cover obtained by taking the ℓ -th roots out of the smooth hypersurface $H_z \in |\ell K_{X_z}|$, the morphism ψ is finite. It follows from the degeneration of the Leray spectral sequence that

$$\begin{aligned}
H^1(Z_z, \Omega_{Z_z}^{n-1}) &\simeq H^1(X_z, (\psi_z)_* \Omega_{Z_z}^{n-1}) \\
&= H^1(X_z, \Omega_{X_z}^{n-1}) \oplus \bigoplus_{i=1}^{\ell-1} H^1(X_z, \Omega_{X_z}^{n-1}(\log H_z) \otimes K_{X_z}^{-i}). \quad (3.4.5)
\end{aligned}$$

The last map in (3.4.4) is therefore injective, because the cohomology group $H^1(X_z, \Omega_{X_z}^{n-1}(\log H_z) \otimes K_{X_z}^{-1})$ is a direct summand of $H^1(Z_z, \Omega_{Z_z}^{n-1})$ by (3.4.5). As a consequence, the composition $\tau_{1,z}$ in (3.4.4) is injective at each point $z \in V_0$. Theorem 3.12(iv) is thus proved. ■

Let us explain how Lemma 2.5 and Theorems 2.6 and 3.12 imply Theorem C.

Proof of Theorem C. We first take a smooth compactification $Y \supset V$ as in Corollary A.2. By Theorem 3.12, for any given point $y \in V$, there exists a birational morphism $\nu : Y' \rightarrow Y$ which is isomorphic at y , so that $D := Y' \setminus \nu^{-1}(V)$ is a simple normal crossing divisor, and there exists a VZ Higgs bundle $(\tilde{\mathcal{E}}, \tilde{\theta})$ whose log pole $D + S$ avoids $y' := \nu^{-1}(y)$. Moreover, by Theorem 3.12(iv), τ_1 is injective at y' . Applying Theorem 2.6, we can associate to $(\tilde{\mathcal{E}}, \tilde{\theta})$ a Finsler metric F on $\mathcal{T}_{Y'}(-D)$ which is positive definite at y' . Moreover, if we think of F as a Finsler metric on $\nu^{-1}(V)$, it is negatively curved in the sense of Definition 2.3(ii). Hence the base V satisfies the conditions in Lemma 2.5, and we conclude that V is Kobayashi hyperbolic. ■

Dan Abramovich

Appendix A. \mathbb{Q} -mild reductions

Let us work over the complex number field \mathbb{C} .

The main result in this appendix is the following:

Theorem A.1. *Let $f_0 : S_0 \rightarrow T_0$ be a projective family of smooth varieties with T_0 quasi-projective.*

- (i) *There are compactifications $S_0 \subset \mathcal{S}$ and $T_0 \subset \mathcal{T}$, with \mathcal{S} and \mathcal{T} Deligne–Mumford stacks with projective coarse moduli spaces, and a projective morphism $f : \mathcal{S} \rightarrow \mathcal{T}$ extending f_0 which is a Kollár family of slc varieties.*
- (ii) *Given a finite subset $Z \subset T_0$, there is a projective variety W and a finite surjective lci morphism $\rho : W \rightarrow \mathcal{T}$, unramified over Z , such that $\rho^{-1}\mathcal{T}^{sm} = W^{sm}$.*

Here the notion of Kollár family refers to the condition that the sheaf $\omega_{\mathcal{S}/\mathcal{T}}^{[m]}$ is flat and its formation commutes with arbitrary base change for each m . We refer the readers to [1, Definition 5.2.1] for further details.

Note that the pull-back family $\mathcal{S} \times_{\mathcal{T}} W \rightarrow W$ is a Kollár family of slc varieties compactifying the pull-back $S_0 \times_{T_0} W_0 \rightarrow W_0$ of the original family to $W_0 := W \times_{\mathcal{T}} T_0$.

This is applied in the present paper, where some mild regularity assumption on T_0 and W is required:

Corollary A.2 (\mathbb{Q} -mild reduction). *Assume further T_0 is smooth. For any given finite subset $Z \subset T_0$, there exist*

- (i) *a compactification $T_0 \subset \mathcal{T}$ with \mathcal{T} a regular projective scheme,*
- (ii) *a simple normal crossings divisor $D \subset \mathcal{T}$ containing $\mathcal{T} \setminus T_0$ and disjoint from Z ,*
- (iii) *a finite morphism $W \rightarrow \mathcal{T}$ unramified outside D , and*
- (iv) *a Kollár family $S_W \rightarrow W$ of slc varieties extending the given family $S_0 \times_{\mathcal{T}} W$.*

The significance of these extended families is due to their \mathbb{Q} -mildness property. Recall from [2] that a family $S \rightarrow T$ is \mathbb{Q} -mild if whenever $T_1 \rightarrow T$ is a dominant morphism with T_1 having at most Gorenstein canonical singularities, the total space $S_1 = T_1 \times_S T$ has canonical singularities. It was shown by Kollár–Shepherd-Barron [39, Theorem 5.1] and Karu [32, Theorem 2.5] that Kollár families of slc varieties whose generic fiber has at most Gorenstein canonical singularities are \mathbb{Q} -mild.

The main result is proved using moduli of Alexeev stable maps.

Let V be a projective variety. A morphism $\phi : U \rightarrow V$ is a *stable map* if U is slc and K_U is ϕ -ample. More generally, given $\pi : U \rightarrow T$, a morphism $\phi : U \rightarrow V$ is a *stable map over T* or a *family of stable maps parametrized by T* if π is a Kollár family of slc varieties and $K_{U/T}$ is $\phi \times \pi$ -ample. Note that this condition is very flexible and does not require the fibers to be of general type, although key applications in Theorems 3.7(iii) and 3.11 require some positivity of the fibers.

Theorem A.3 ([30, Theorem 1.5]). *The stable maps form an algebraic stack $M(V)$ locally of finite type over \mathbb{C} , each of whose connected components is a proper global quotient stack with projective coarse moduli space.*

The existence of an algebraic stack satisfying the valuative criterion for properness was known to Alexeev, and can also be deduced directly from the results of [1], which presents it as a global quotient stack. The work [30] shows that the stack has bounded, hence proper components, admitting projective coarse moduli spaces. An algebraic approach to these statements is provided in [32, Corollary 1.2].

Proof of Theorem A.1. (i) Let $T_0 \subset T$ and $S_0 \subset S$ be projective compactifications with $\pi : S \rightarrow T$ extending f_0 . The family $S_0 \rightarrow T_0$ with the injective morphism $\phi : S_0 \rightarrow S$ is a family of stable maps into S , providing a morphism $T_0 \rightarrow M(S)$ which is in fact injective. Let \mathcal{T} be the closure of T_0 . Since $M(S)$ is proper, \mathcal{T} is proper. Let \mathcal{S} be the pull-back of the universal family along $\mathcal{T} \rightarrow M(S/T)$. Then $\mathcal{S} \supset S_0$ is a compactification as needed.

(ii) The existence of W follows from the main result of [41]. ■

Proof of Corollary A.2. Consider the coarse moduli space \mathcal{T} of the stack \mathcal{T} provided by the first part of the main result. This might be singular, but by Hironaka’s theorem we may replace it by a resolution of singularities such that $D_\infty := \mathcal{T} \setminus T_0$ is a simple normal crossings divisor. Thus condition (i) is satisfied.

For each component $D_i \subset D_\infty$ denote by m_i the ramification index of $\mathcal{T} \rightarrow \mathcal{T}$. In particular any covering $W \rightarrow \mathcal{T}$ whose ramification indices over D_i are divisible by m_i lifts along the generic point of D_i to \mathcal{T} .

Choosing a Kawamata covering package [2] disjoint from Z we obtain a simple normal crossings divisor D as required by (ii), and a finite covering $W \rightarrow \mathcal{T}$ as required by (iii), such that $W \rightarrow \mathcal{T}$ factors through \mathcal{T} at every generic point of D_i .

By the Purity Lemma [3, Lemma 2.4.1] the morphism $W \rightarrow \mathcal{T}$ extends over all of W , hence we obtain a family $S_W \rightarrow W$ as required by (iv). ■

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