BIG PICARD THEOREM FOR MODULI SPACES OF POLARIZED MANIFOLDS

YA DENG

Abstract. Consider a smooth projective family of complex polarized manifolds with semi-ample canonical sheaf over a quasi-projective manifold V. When the associated moduli map $V \to P_h$ from the base to coarse moduli space is quasi-finite, we prove that the generalized big Picard theorem holds for the base manifold V: for any projective compactification Y of V, any holomorphic map $f : \Delta - \{0\} \to V$ from the punctured unit disk to V extends to a holomorphic map of the unit disk Δ into Y. This result generalizes our previous work on the Brody hyperbolicity of V (*i.e.* there are no entire curves on V), as well as a more recent work by Lu-Sun-Zuo on the Borel hyperbolicity of V (*i.e.* any holomorphic map from a quasi-projective variety to V is algebraic). We also obtain generalized big Picard theorem for bases of log Calabi-Yau families.

Contents

0. Introduction

The classical big Picard theorem says that any holomorphic map from the punctured disk Δ^* into $\mathbb{P}^1$ which omits three points can be extended to a holomorphic map $\Delta \to \mathbb{P}^1$, where Δ denotes the unit disk. Therefore, we say the (generalized) big Picard theorem holds for a quasi-projective variety V if for some (thus any) projective compactification Y of V, any holomorphic map $f : \Delta^* \to V$ extends to a holomorphic map $\bar{f} : \Delta \to X$. This property is interesting for it implies the *Borel hyperbolicity*[1] of V: any holomorphic map from a quasi-projective variety to V is necessarily *algebraic*. A natural question is to find algebraic varieties satisfying the big Picard theorem. By the fundamental work of Kobayashi (see [Kob98, Theorem 6.3.6]), big Picard theorem holds for the quasi-projective manifold V which admits a projective compactification Y such that V is *hyperbolically embedded into* Y (see [Kob98, Chapter 3. §3] for the definition). This gives

Date: Wednesday 25th December, 2019.
2010 *Mathematics Subject Classification.* 32Q45, 32A22, 53C60.
Key words and phrases. big Picard theorem, logarithmic derivative lemma, Viehweg-Zuo Higgs bundles, negatively curved Finsler metric, moduli of polarized manifolds, log Calabi-Yau family.

[1]The notion of Borel hyperbolicity was first introduced by Javanpeykar-Kucharczyk in [JK18].

us an important criteria for varieties satisfying the big Picard theorem. By the work of Fujimoto [Fuj72] and Green [Gre77], complements of $2n+1$ general hyperplanes in $\mathbb{P}^n$ are hyperbolically embedded into $\mathbb{P}^n$. More recently Brotbek and the author [BD19] proved that the complement of a general hypersurface in $\mathbb{P}^n$ of high degree is also hyperbolically embedded into $\mathbb{P}^n$. By A. Borel [Bor72] and Kobayashi-Ochiai [KO71], the quotients of bounded symmetric domains by torsion free arithmetic groups are hyperbolically embedded into its Baily-Borel compactification. Hence these results provide examples of quasi-projective manifolds satisfying the big Picard theorem.

However, quasi-projective manifold V being hyperbolically embedded into some projective compactification Y is *minimal* in the sense that, for any birational modification $\mu: \tilde{Y} \to Y$ by blowing-up the boundary $Y \setminus V$, $\mu^{-1}(V) \simeq V$ is no more hyperbolically embedded into $\tilde{Y}$, while the big Picard theorem does not depends on its compactification. The first result in this paper is to establish a more flexible criteria for big Picard theorem. For our definition of Finsler metric (which is slightly different from the usual one in the literature), see Definition 1.6.

Theorem A (=Theorem 1.7)**.** *Let X be a projective manifold and let D be a simple normal crossing divisor on X. Let h be a (possibly degenerate) Finsler metric of $T_X(-\log D)$. Assume that $f: \Delta^* \to X \setminus D$ is a holomorphic map from the punctured unit disk Δ^* to $X \setminus D$ such that $|f'(t)|_h^2 \not\equiv 0$, and*

$$\sqrt{-1}\partial\overline{\partial} \log |f'(t)|_h^2 \geq f^*\omega$$

for some smooth Kähler metric ω on X. Then f extends to a holomorphic map $\bar{f}: \Delta \to X$ of the unit disk into X.

Theorem A is inspired by *fundamental vanishing theorem* for jet differentials vanishing on some ample divisor by Siu-Yeung [SY97] and Demailly [Dem97, §4], and its proof is mainly based on a *logarithmic derivative lemma* by Noguchi [Nog81].

The motivation of Theorem A is to study the hyperbolicity of moduli spaces of polarized manifolds with semi-ample canonical bundle. By the fundamental work of Viehweg-Zuo [VZ02,VZ03], and the recent development by Popa et al. [PS17,PTW18] and [Den18a, Den18b,Den19,LSZ19], a special Higgs bundle, the so-called Viehweg-Zuo Higgs bundle in Definition 2.1 below, turns out to be a powerful technique in studying hyperbolicity problems. For a quasi-projective manifold V equipped with a Viehweg-Zuo Higgs bundle, in [Den18a, Den18b] we prove that V can be equipped with a *generically positively definite* Finsler metric whose holomorphic sectional curvature is bounded from above by a negative constant. In particular, we prove that V is always *pseudo Kobayashi hyperbolic*. The second aim of this article is to give a new curvature estimate for that Finsler metric on V.

Theorem B (=Theorem 2.8)**.** *Let X be a projective manifold equipped with a smooth Kähler metric ω and let D be a simple normal crossing divisor on X. Assume that there is a Viehweg-Zuo Higgs bundle over (X, D). Then there are a positive constant δ and a Finsler metric h on $T_X(-\log D)$ which is positively definite on a dense Zariski open set V° of $V := X \setminus D$, such that for any holomorphic map $\gamma: C \to V$ from any open subset C of $\mathbb{C}$ to V with $\gamma(C) \cap V^\circ \neq \varnothing$, one has*

$$\sqrt{-1}\partial\overline{\partial} \log |\gamma'|_h^2 \geq \delta\gamma^*\omega.$$

In particular, by Theorem A, for any holomorphic map $f: \Delta^* \to X \setminus D$, with $f(\Delta^*) \cap V^\circ \neq \varnothing$, it extends to a holomorphic map $\bar{f}: \Delta \to X$.

By the work [VZ02, VZ03, PTW18], Viehweg-Zuo Higgs bundles exist on bases of maximally varying, smooth family of projective manifolds with semi-ample canonical bundle. Combining their results and Theorem B, we prove the big Picard theorem for moduli of polarized manifolds with semi-ample canonical bundle.

Theorem C. *Consider the moduli functor $\mathscr{P}_h$ of polarized manifolds with semi-ample canonical sheaf introduced by Viehweg [Vie95, §7.6], where h is the Hilbert polynomial associated to the polarization. Assume that for some quasi-projective manifold V over which there exists a smooth polarized family $(f_U : U \to V, \mathscr{L}) \in \mathscr{P}_h(V)$ such that the induced moduli map $\varphi_U : V \to P_h$ is quasi-finite. Let Y be an arbitrary projective compactification of V. Then any holomorphic map $\gamma : \Delta^* \to V$ from the punctured unit disk Δ^* to V extends to a holomorphic map from the unit disk Δ to Y.*

Under the same assumption as Theorem B, we have already in [Den18b] proved the Brody hyperbolicity of V: there exists no entire curves $\gamma : \mathbb{C} \to V$. Based on the infinitesimal Torelli-type theorem proven in [Den18b, Theorem C] (see Theorem 2.3 below), more recently, Lu-Sun-Zuo [LSZ19] proved the Borel hyperbolicity of V: any holomorphic map from a quasi-projective variety to V is algebraic. The use of Nevannlina theory in this article is inspired by their work, although our methods are different from theirs (see Remark 3.1).

Finally, let us mention that in [Den19], we construct Viehweg-Zuo Higgs bundles over bases of maximally varying, log smooth families of Calabi-Yau families (see Definition 3.2 for the definition of log smooth family). Applying Theorems A and B to this result, we also obtain big Picard theorem for these base manifolds.

Theorem D (=Theorem 3.4)**.** *Let $f^\circ : (X^\circ, D^\circ) \to Y^\circ$ be a log smooth family over a quasi-projective manifold Y°. Assume that each fiber $(X_y, D_y) := (f^\circ)^{-1}(y)$ of f° is a klt pair, and $K_{X_y} + D_y \equiv_{\mathbb{Q}} 0$. Assume that the logarithmic Kodaira-Spencer map*

$$T_{Y^\circ,y} \to H^1\big(X_y, T_{X_y}(-\log D_y)\big)$$

is injective for any $y \in Y^\circ$. Then for any projective compactification Y of the base Y°, any holomorphic map $\gamma : \Delta^* \to Y^\circ$ extends to a holomorphic map from Δ to Y.

0.1. **Acknowledgments.** I would like to thank Professors Jean-Pierre Demailly and Emmanuel Ullmo for their encouragements and supports. This work is supported by Institut des Hautes Études Scientifiques.

1. A differential geometric criteria for big Picard theorem

In the similar vein as the fundamental vanishing theorem for jet differentials vanishing on some ample divisor by Siu-Yeung [SY97] and Demailly [Dem97], in this section we will establish a differential geometric criteria for big Picard type theorem via the logarithmic derivative lemma by Noguchi [Nog81].

1.1. **Preliminary in Nevannlina theory.** Let $\mathbb{D}^* := \{t \in \mathbb{C} \mid |t| > 1\}$, and $\mathbb{D} := \mathbb{D}^* \cup \infty$. Then via the map $z \mapsto \frac{1}{z}$, $\mathbb{D}^*$ is isomorphic to the punctured unit disk Δ^* and $\mathbb{D}$ is isomorphic to the unit disk Δ. Therefore, for any holomorphic map f from the punctured disk Δ^* into a projective variety Y, f extends to the origin if and only if $f(\frac{1}{z}) : \mathbb{D}^* \to Y$ extends to the infinity.

Let (X, ω) be a compact Kähler manifold, and $\gamma : \mathbb{D}^* \to X$ be a holomorphic map. Fix any $r_0 > 1$. Write $\mathbb{D}_r := \{z \in \mathbb{C} \mid r_0 < |z| < r\}$. The *order function* is defined by

$$T_{\gamma,\omega}(r) := \int_{r_0}^{r} \frac{d\tau}{\tau} \int_{\mathbb{D}_\tau} \gamma^*\omega.$$

As is well-known, the asymptotic behavior of $T_{\gamma,\omega}(r)$ as $r \to \infty$ characterizes whether γ can be extended over the ∞ (see *e.g.* [Dem97, 2.11. Cas «local »] or [NW14, Remark 4.7.4.(ii)]).

Lemma 1.1. *$T_{\gamma,\omega}(r) = O(\log r)$ if and only if γ is extended holomorphically over ∞.* □

The following lemma is well-known to experts (see *e.g.* [Dem97, Lemme 1.6]).

Lemma 1.2. *Let X be a projective manifold equipped with a hermitian metric ω and let $u : X \to \mathbb{P}^1$ be a rational function. Then for any holomorphic map $\gamma : \mathbb{D}^* \to X$, one has*

$$T_{u\circ\gamma,\omega_{FS}}(r) \leq CT_{\gamma,\omega}(r) + O(1)$$

where ω_{FS} is the Fubini-Study metric for $\mathbb{P}^1$. □

The following logarithmic derivative lemma by Noguchi is crucial in the proof of Theorem A.

Lemma 1.3 ([Nog81, Lemma2.12], [Dem97, 3.4. Cas local]). *Let $u : \mathbb{D}^* \to \mathbb{P}^1$ be any meromorphic function. Then we have*

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+ |(\log u)'(re^{i\theta})|d\theta \leq C(\log^+ T_{u,\omega_{FS}}(r) + \log r) + O(1) \quad \|,$$

for some constant $C > 0$ which does not depend on r. Here the symbol $\|$ means that the inequality holds outside a Borel subset of $(r_0, +\infty)$ of finite Lebesgue measure. □

We need the lemma by E. Borel.

Lemma 1.4 ([NW14, Lemma 1.2.1]). *Let $\phi(r) \geq 0 (r \geq r_0 \geq 0)$ be a monotone increasing function. For every $\delta > 0$,*

$$\frac{d}{dr}\phi(r) \leq \phi(r)^{1+\delta} \quad \|.$$

□

We recall two useful formulas (the second one is the well-known Jensen formula).

Lemma 1.5. *Write* $\log^+ x := \max(\log x, 0)$.

$$\log^+(\sum_{i=1}^N x_i) \leq \sum_{i=1}^N \log^+ x_i + \log N \quad \textit{for } x_i \geq 0. \tag{1.1.1}$$

$$\frac{1}{\pi}\int_{r_0}^r \frac{d\tau}{\tau}\int_{\mathbb{D}_\tau} \sqrt{-1}\partial\bar{\partial}v = \frac{1}{2\pi}\int_0^{2\pi} v(re^{i\theta})d\theta - \frac{1}{2\pi}\int_0^{2\pi} v(r_0e^{i\theta})d\theta \tag{1.1.2}$$

*for all functions v so that $\sqrt{-1}\partial\bar{\partial}v$ exists as measures (*e.g.* v is the difference of two subharmonic functions).* □

1.2. A criteria for big Picard theorem.

Definition 1.6 (Finsler metric). Let E be a holomorphic vector bundle on a complex manifold X. A *Finsler metric* on E is a real non-negative *continuous* function $h : E \to [0, +\infty[$ such that

$$h(av) = |a|h(v)$$

for any $a \in \mathbb{C}$ and $v \in E$. The metric h is *degenerate* at a point $x \in X$ if $h(v) = 0$ for some nonzero $v \in E_x$, and the set of such degenerate points is denoted by Δ_h.

We shall mention that our definition is a bit different from that in [Kob98, Chapter 2, §3], which requires *convexity*, and the Finsler metric therein can be upper-semi continuous.

Let us now state and prove the main result in this section.

Theorem 1.7 (Criteria for big Picard theorem). *Let X be a projective manifold and let D be a simple normal crossing divisor on X. Let h be a (possibly degenerate) Finsler metric of $T_X(-\log D)$. Assume that $f : \mathbb{D}^* \to X \setminus D$ is a holomorphic map such that $|f'(t)|_h^2 \not\equiv 0$, and*

$$\frac{1}{\pi}\sqrt{-1}\partial\bar{\partial}\log|f'(t)|_h^2 \geq f^*\omega \tag{1.2.1}$$

for some smooth Kähler metric ω on X. Then f extends to a holomorphic map $\bar{f} : \mathbb{D} \to X$.

The proof is an application of logarithmic derivative lemma, which is inspired by [Dem97, §4] and [NW14, Lemma 4.7.1].

Proof of Theorem 1.7. We take a finite affine covering $\{U_\alpha\}_{\alpha\in I}$ of X and rational functions $(x_{\alpha 1},\ldots,x_{\alpha n})$ on X which are holomorphic on U_α so that

$$dx_{\alpha 1}\wedge\cdots\wedge dx_{\alpha n}\neq 0 \ \text{ on } U_\alpha$$
$$D\cap U_\alpha=(x_{\alpha,s(\alpha)+1}\cdots x_{\alpha n}=0)$$

Hence

$$(e_{\alpha 1},\ldots,e_{\alpha n}):=\Big(\frac{\partial}{\partial x_{\alpha 1}},\ldots,\frac{\partial}{\partial x_{\alpha s(\alpha)}},x_{\alpha,s(\alpha)+1}\frac{\partial}{\partial x_{\alpha,s(\alpha)+1}},\ldots,x_{\alpha n}\frac{\partial}{\partial x_{\alpha n}}\Big) \tag{1.2.2}$$

is a basis for $T_X(-\log D)|_{U_\alpha}$. Write

$$(f_{\alpha 1}(t),\ldots,f_{\alpha n}(t)):=(x_{\alpha 1}\circ f,\ldots,x_{\alpha n}\circ f).$$

With respect to the trivialization of $T_X(-\log D)$ induced by the basis (1.2.2), $f'(t)$ can be written as

$$f'(t)=f'_{\alpha 1}(t)e_{\alpha 1}+\cdots+f'_{\alpha s(\alpha)}(t)e_{\alpha s(\alpha)}+(\log f_{\alpha,s(\alpha)+1})'(t)e_{\alpha,s(\alpha)+1}+\cdots+(\log f_{\alpha n})'(t)e_{\alpha n}.$$

Let $\{\rho_\alpha\}_{\alpha\in I}$ be a partition of unity subordinated to $\{U_\alpha\}_{\alpha\in I}$. Since h is Finsler metric for $T_X(-\log D)$ which is continuous and locally bounded from above by Definition 1.6, and I is a finite set, there is a constant $C>0$ so that

$$\rho_\alpha\circ f\cdot|f'(t)|_h^2\leq C\Big(\sum_{j=1}^{s(\alpha)}\rho_\alpha\circ f\cdot|f'_{\alpha j}(t)|^2+\sum_{i=s(\alpha)+1}^{n}|(\log f_{\alpha i})'(t)|^2\Big)\quad\forall t\in\mathbb{D}^* \tag{1.2.3}$$

for any α. Hence

$$\begin{aligned}
T_{f,\omega}(r)&:=\int_{r_0}^{r}\frac{d\tau}{\tau}\int_{\mathbb{D}_\tau}f^*\omega\overset{(1.2.1)}{\leq}\int_{r_0}^{r}\frac{d\tau}{\tau}\int_{\mathbb{D}_\tau}\frac{1}{\pi}\sqrt{-1}\partial\bar\partial\log|f'|_h^2\\
&\overset{(1.1.2)}{\leq}\frac{1}{2\pi}\int_0^{2\pi}\log|f'(re^{i\theta})|_h d\theta-\frac{1}{2\pi}\int_0^{2\pi}\log|f'(r_0e^{i\theta})|_h d\theta\\
&\leq\frac{1}{2\pi}\int_0^{2\pi}\log^+|f'(re^{i\theta})|_h d\theta+O(1)=\frac{1}{2\pi}\int_0^{2\pi}\log^+\sum_\alpha|\rho_\alpha\circ f\cdot f'(re^{i\theta})|_h d\theta+O(1)\\
&\overset{(1.1.1)}{\leq}\sum_\alpha\frac{1}{2\pi}\int_0^{2\pi}\log^+|\rho_\alpha\circ f\cdot f'(re^{i\theta})|_h d\theta+O(1)\\
&\overset{(1.2.3)}{\leq}\sum_\alpha\sum_{i=s(\alpha)+1}^{n}\frac{1}{2\pi}\int_0^{2\pi}\log^+|(\log f_{\alpha i})'(re^{i\theta})|d\theta\\
&\quad+\sum_\alpha\sum_{j=1}^{s(\alpha)}\frac{1}{2\pi}\int_0^{2\pi}\log^+|\rho_\alpha\circ f\cdot f'_{\alpha j}(re^{i\theta})|d\theta+O(1)\\
&\overset{\text{Lemma 1.3}}{\leq}C_1\sum_\alpha\sum_{i=s(\alpha)+1}^{n}\big(\log^+T_{f_{\alpha i},\omega_{FS}}(r)+\log r\big)\\
&\quad+\sum_\alpha\sum_{j=1}^{s(\alpha)}\frac{1}{2\pi}\int_0^{2\pi}\log^+|\rho_\alpha\circ f\cdot f'_{\alpha j}(re^{i\theta})|d\theta+O(1)\quad\|
\end{aligned}$$

$$\overset{\text{Lemma 1.2}}{\leq}C_2(\log^+T_{f,\omega}(r)+\log r)+\sum_\alpha\sum_{j=1}^{s(\alpha)}\frac{1}{2\pi}\int_0^{2\pi}\log^+|\rho_\alpha\circ f\cdot f'_{\alpha j}(re^{i\theta})|d\theta+O(1)\quad\|. \tag{1.2.4}$$

where C_1 and C_2 are two positive constants which do not depend on r.

Claim 1.8. *For any $\alpha \in I$ and any $j \in \{1, \ldots, s(\alpha)\}$, one has*

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+ |\rho_\alpha \circ f \cdot f'_{\alpha j}(re^{i\theta})| d\theta \leq C_3(\log^+ T_{f,\omega}(r) + \log r) + O(1) \quad \| \tag{1.2.5}$$

for positive constant C_3 which does not depend on r.

Proof of Claim 1.8. The proof of the claim is borrowed from [NW14, eq.(4.7.2)]. Pick $C > 0$ so that $\rho_\alpha\sqrt{-1}dx_{\alpha j} \wedge d\bar{x}_{\alpha j} \leq C\omega$. Write $f^*\omega := \sqrt{-1}B(t)dt \wedge d\bar{t}$. Then

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} \log^+ |\rho_\alpha \circ f \cdot f'_{\alpha j}(re^{i\theta})| d\theta = \frac{1}{4\pi}\int_0^{2\pi} \log^+(|\rho_\alpha^2 \circ f| \cdot |f'_{\alpha j}(re^{i\theta})|^2) d\theta \\
&\leq \frac{1}{4\pi}\int_0^{2\pi} \log^+ B(re^{i\theta})d\theta + O(1) \leq \frac{1}{4\pi}\int_0^{2\pi} \log(1 + B(re^{i\theta}))d\theta + O(1) \\
&\leq \frac{1}{2}\log(1 + \frac{1}{2\pi}\int_0^{2\pi} B(re^{i\theta})d\theta) + O(1) = \frac{1}{2}\log(1 + \frac{1}{2\pi r}\frac{d}{dr}\int_{\mathbb{D}_r} rBdrd\theta) + O(1) \\
&= \frac{1}{2}\log(1 + \frac{1}{2\pi r}\frac{d}{dr}\int_{\mathbb{D}_r} f^*\omega) + O(1) \\
&\overset{\text{Lemma 1.4}}{\leq} \frac{1}{2}\log(1 + \frac{1}{2\pi r}(\int_{\mathbb{D}_r} f^*\omega)^{1+\delta}) + O(1) \quad \| \\
&= \frac{1}{2}\log(1 + \frac{r^\delta}{2\pi}(\frac{d}{dr}T_{f,\omega}(r))^{1+\delta}) + O(1) \quad \| \\
&\overset{\text{Lemma 1.4}}{\leq} \frac{1}{2}\log(1 + \frac{r^\delta}{2\pi}(T_{f,\omega}(r))^{(1+\delta)^2}) + O(1) \quad \| \\
&\leq 4\log^+ T_{f,\omega}(r) + \delta \log r + O(1) \quad \|.
\end{aligned}$$

Here we pick $0 < \delta < 1$ to apply Lemma 1.4. The claim is proved. □

Putting (1.2.5) to (1.2.4), one obtains

$$T_{f,\omega}(r) \leq C(\log^+ T_{f,\omega}(r) + \log r) + O(1) \quad \|$$

for some positive constant C. Hence $T_{f,\omega}(r) = O(\log r)$. We apply Lemma 1.1 to conclude that f extends to the ∞. □

2. The negatively curved metric via Viehweg-Zuo Higgs bundles

In [VZ02, VZ03], Viehweg-Zuo introduced a special type of Higgs bundles over the bases of smooth families of polarized manifolds with semi-ample canonical sheaves to study the hyperbolicity of such bases. It was later developed in [PTW18]. For any quasi-projective manifold V endowed with a Viehweg-Zuo Higgs bundle, we construct in [Den18a, Den18b] a generically positively definite Finsler metric over V whose holomorphic sectional curvature is bounded from above by a negative constant. In this section we will refine curvature estimate in [Den18a] to prove big Picard theorem for such quasi-projective manifold V.

2.1. Viehweg-Zuo Higgs bundles and their proper metrics.

The definition for Viehweg-Zuo Higgs bundles we present below follows from the formulation in [VZ02, VZ03] and [PTW18].

Definition 2.1 (Abstract Viehweg-Zuo Higgs bundles)**.** Let V be a quasi-projective manifold, and let $Y \supset V$ be a projective compactification of V with the boundary $D := Y \setminus V$ simple normal crossing. A *Viehweg-Zuo Higgs bundle* over (Y, D) (or say over V abusively) is a logarithmic Higgs bundle $(\tilde{\mathscr{E}}, \tilde{\theta})$ over Y consisting of the following data:

(i) a divisor S on Y so that $D+S$ is simple normal crossing,
(ii) a big and nef line bundle L over Y with $\mathbf{B}_+(L)\subset D\cup S$,
(iii) a Higgs bundle $(\mathscr{E},\theta):=\big(\bigoplus_{q=0}^n E^{n-q,q},\bigoplus_{q=0}^n \theta_{n-q,q}\big)$ induced by the lower canonical extension of a polarized VHS defined over $Y\setminus(D\cup S)$,
(iv) a sub-Higgs sheaf $(\mathscr{F},\eta)\subset(\tilde{\mathscr{E}},\tilde{\theta})$,

which satisfy the following properties.

(1) The Higgs bundle $(\tilde{\mathscr{E}},\tilde{\theta}):=(L^{-1}\otimes\mathscr{E},\mathbb{1}\otimes\theta)$. In particular, $\tilde{\theta}:\tilde{\mathscr{E}}\to\tilde{\mathscr{E}}\otimes\Omega_Y\big(\log(D+S)\big)$, and $\tilde{\theta}\wedge\tilde{\theta}=0$.
(2) The sub-Higgs sheaf $(\mathscr{F},\eta)$ has log poles only on the boundary D, that is, $\eta:\mathscr{F}\to\mathscr{F}\otimes\Omega_Y(\log D)$.
(3) Write $\tilde{\mathscr{E}}_k:=L^{-1}\otimes E^{n-k,k}$, and denote by $\mathscr{F}_k:=\tilde{\mathscr{E}}_k\cap\mathscr{F}$. Then the first stage $\mathscr{F}_0$ of $\mathscr{F}$ is an *effective line bundle*. In other words, there exists a non-trivial morphism $\mathcal{O}_Y\to\mathscr{F}_0$.

As shown in [VZ02], by iterating η for k-times, we obtain

$$\mathscr{F}_0\xrightarrow{\overbrace{\eta\circ\cdots\circ\eta}^{k\text{ times}}}\mathscr{F}_k\otimes\big(\Omega_Y(\log D)\big)^{\otimes k}.$$

Since $\eta\wedge\eta=0$, the above morphism factors through $\mathscr{F}_k\otimes\mathrm{Sym}^k\Omega_Y(\log D)$, and by (3) one thus obtains

$$\mathcal{O}_Y\to\mathscr{F}_0\to\mathscr{F}_k\otimes\mathrm{Sym}^k\Omega_Y(\log D)\to L^{-1}\otimes E^{n-k,k}\otimes\mathrm{Sym}^k\Omega_Y(\log D).$$

Equivalently, we have a morphism

$$\tau_k:\mathrm{Sym}^kT_Y(-\log D)\to L^{-1}\otimes E^{n-k,k}. \tag{2.1.1}$$

It was proven in [VZ02, Corollary 4.5] that τ_1 is always non-trivial. In [Den18b] we prove that $\tau_1:T_Y(-\log D)\to L^{-1}\otimes E^{n-1,1}$ in (2.1.1) is generically injective.

We will follow [PTW18] to give some "proper" metric on $\tilde{\mathscr{E}}=\oplus_{k=0}^n L^{-1}\otimes E^{n-k,k}$. Write the simple normal crossing divisor $D=D_1+\cdots+D_k$ and $S=S_1+\cdots+S_\ell$. Let $f_{D_i}\in H^0\big(Y,\mathcal{O}_Y(D_i)\big)$ and $f_{S_i}\in H^0\big(Y,\mathcal{O}_Y(S_i)\big)$ be the canonical section defining D_i and S_i. We fix smooth hermitian metrics g_{D_i} and g_{S_i} on $\mathcal{O}_Y(D_i)$ and $\mathcal{O}_Y(S_i)$. After rescaling g_{D_i} and g_{S_j}, we assume that $|f_{D_i}|_{g_{D_i}}<1$ and $|f_{S_j}|_{g_{S_j}}<1$ for $i=1,\ldots,k$ and $j=1,\ldots,\ell$. Set

$$r_D:=\prod_{i=1}^k(-\log|f_{D_i}|^2_{g_{D_i}}),\quad r_S:=\prod_{j=1}^\ell(-\log|f_{S_j}|^2_{g_{S_j}}).$$

Let g be a singular hermitian metric with analytic singularities of the big and nef line bundle L such that g is smooth on $Y\setminus\mathbf{B}_+(L)\supset Y\setminus D\cup S$, and the curvature current $\sqrt{-1}\Theta_g(L)\geqslant\omega$ for some smooth Kähler form ω on Y. For $\alpha\in\mathbb{N}$, define

$$h_L:=g\cdot(r_D\cdot r_S)^\alpha$$

The following proposition is a slight variant of [PTW18, Lemma 3.1, Corollary 3.4].

Proposition 2.2 ([PTW18]). *When $\alpha\gg 0$, after rescaling f_{D_i} and f_{S_i}, there exists a continuous, positively definite hermitian form ω_α on $T_Y(-\log D)$ such that*

(i) *over $V_0:=Y\setminus D\cup S$, the curvature form*

$$\sqrt{-1}\Theta_{h_L}(L)_{\restriction V_0}\geqslant r_D^{-2}\cdot\omega_{\alpha\restriction V_0},\quad \sqrt{-1}\Theta_{h_L}(L)\geq\omega$$

where ω is a smooth Kähler metric on Y.

(ii) *The singular hermitian metric $h:=h_L^{-1}\otimes h_{\rm hod}$ on $\tilde{\mathscr{E}}=L^{-1}\otimes\mathscr{E}$ is locally bounded on Y, and smooth outside $(D+S)$, where $h_{\rm hod}$ is the Hodge metric for the Hodge bundle $\mathscr{E}$. Moreover, h is degenerate on $D+S$.*

(iii) *The singular hermitian metric $r_D^2 h$ on $L^{-1}\otimes\mathscr{E}$ is also locally bounded on Y and is degenerate on $D+S$.* □

Hence by Definition 1.6, h and $r_D^2 h$ are both Finsler metrics on $\tilde{\mathscr{E}}$.

2.2. **Construction of negatively curved Finsler metric.** We adopt the same notations as § 2.1 throughout this subsection. Assume that the log manifold (Y,D) is endowed with a Viehweg-Zuo Higgs bundle. In [Den18a, §3.4] we construct Finsler metrics $F_1,\ldots,F_n$ on $T_Y(-\log D)$ as follows. By (2.1.1), for each $k=1,\ldots,n$, there exists

$$\tau_k : \mathrm{Sym}^k T_Y(-\log D)\to L^{-1}\otimes E^{n-k,k}. \tag{2.2.1}$$

Then it follows from Proposition 2.2.(ii) that the Finsler metric h on $L^{-1}\otimes E^{n-k,k}$ induces a Finsler metric F_k on $T_Y(-\log D)$ defined as follows: for any $e\in T_Y(-\log D)_y$,

$$F_k(e) := h\big(\tau_k(e^{\otimes k})\big)^{\frac{1}{k}} \tag{2.2.2}$$

Let $C\subset\mathbb{C}$ be any open set of $\mathbb{C}$. For any $\gamma: C\to V$, one has

$$d\gamma : T_C\to\gamma^*T_V\hookrightarrow\gamma^*T_Y(-\log D). \tag{2.2.3}$$

We denote by $\partial_t := \frac{\partial}{\partial t}$ the canonical vector fields in $C\subset\mathbb{C}$, $\bar\partial_t := \frac{\partial}{\partial\bar t}$ its conjugate. The Finsler metric F_k induces a continuous Hermitian pseudo-metric on C, defined by

$$\gamma^*F_k^2 = \sqrt{-1}G_k(t)dt\wedge d\bar t. \tag{2.2.4}$$

Hence $G_k(t) = |\tau_k\big(d\gamma(\partial_t)^{\otimes k}\big)|_h^{\frac{2}{k}}$, where τ_k is defined in (2.1.1). The reader might worried that all $G_k(t)$ will be identically equal to zero. In [Den18b, Theorem C], we prove that 'generically' this cannot happen.

Theorem 2.3 ([Den18b]). *There is a dense Zariski open set $V^\circ\subset V_0 = Y\setminus(D+S)$ of V° so that $\tau_1: T_Y(-\log D)|_{V^\circ}\to L^{-1}\otimes E^{n-1,1}|_{V^\circ}$ is injective.* □

We now fix any $\gamma: C\to V$ with $\gamma(C)\cap V^\circ\neq\varnothing$. By Proposition 2.2.(ii), the metric h for $L^{-1}\otimes\mathscr{E}$ is smooth and positively definite over V_0. It then follows from Theorem 2.3 that $G_1(t)\not\equiv 0$. Let $C^\circ\subset C$ be an (non-empty) open set whose complement $C\setminus C^\circ$ is a *discrete set* so that

- The image $\gamma(C^\circ)\subset V^\circ$.
- For every $k=1,\ldots,n$, either $G_k(t)\equiv 0$ on C° or $G_k(t)>0$ for any $t\in C^\circ$.
- $\gamma'(t)\neq 0$ for any $t\in C^\circ$.

By the definition of $G_k(t)$, if $G_k(t)\equiv 0$ for some $k>1$, then $\tau_k(\partial_t^{\otimes k})\equiv 0$ where τ_k is defined in (2.1.1). Note that one has $\tau_{k+1}(\partial_t^{\otimes(k+1)}) = \tilde\theta\big(\tau_k(\partial_t^{\otimes k})\big)(\partial_t)$, where $\tilde\theta: L^{-1}\otimes\mathscr{E}\to L^{-1}\otimes\mathscr{E}\otimes\Omega_Y(\log(D+S))$ is defined in Definition 2.1. We thus conclude that $G_{k+1}(t)\equiv 0$. Hence it exists $1\le m\le n$ so that the set $\{k\mid G_k(t)>0 \text{ over } C^\circ\} = \{1,\ldots,m\}$, and $G_\ell(t)\equiv 0$ for all $\ell = m+1,\ldots,n$. From now on, *all the computations* are made over C°.

In [Den18a] we proved the following curvature formula.

Theorem 2.4 ([Den18a, Proposition 3.12]). *For $k=1,\ldots,m$, over C° one has*

$$\frac{\partial^2\log G_1}{\partial t\partial\bar t}\ge\Theta_{L,h_L}(\partial_t,\bar\partial_t)-\frac{G_2^2}{G_1}\qquad \textit{if } k=1, \tag{2.2.5}$$

$$\frac{\partial^2\log G_k}{\partial t\partial\bar t}\ge\frac{1}{k}\Big(\Theta_{L,h_L}(\partial_t,\bar\partial_t)+\frac{G_k^k}{G_{k-1}^{k-1}}-\frac{G_{k+1}^{k+1}}{G_k^k}\Big)\qquad \textit{if } k>1. \tag{2.2.6}$$

Here we make the convention that $G_{n+1}\equiv 0$ and $\frac{0}{0}=0$. We also write ∂_t (resp. $\bar\partial_t$) for $d\gamma(\partial_t)$ (resp. $d\gamma(\bar\partial_t)$) abusively, where $d\gamma$ is defined in (2.2.3). □

Let us mention that in [Den18a, eq. (3.3.58)] we drop the term $\Theta_{L,h_L}(\partial_t, \bar{\partial}_t)$ in (2.2.6), though it can be easily seen from the proof of [Den18a, Lemma 3.9]. As we will see below, such a term is crucial in deriving the new curvature estimate.

By Theorem 2.4 we see that the curvature of F_k is not the desired type (1.2.1) for applying the criteria for big Picard theorem in Theorem 1.7. In [Den18a, §3.4], following ideas by [TY15, Sch17] we introduce a new Finsler metric F on $T_Y(-\log D)$ by taking convex sum in the following form

$$F := \sqrt{\sum_{k=1}^{n} k\alpha_k F_k^2}. \tag{2.2.7}$$

where $\alpha_1, \ldots, \alpha_n \in \mathbb{R}^+$ are some constants which will be fixed later. By Theorem 2.3, the set of degenerate points of F defined in Definition 1.6, denoted by Δ_F, is contained in a proper Zariski closed subset $Y \setminus V^\circ$. In [Den18a, Theorem 3.8] we prove that the holomorphic sectional curvature of F is bounded from above by a negative constant. Let us now prove a new curvature formula for F in this section.

For the above $\gamma : C \to V$ with $\gamma(C) \cap V^\circ \neq \varnothing$, we write

$$\gamma^* F^2 = \sqrt{-1}H(t)dt \wedge d\bar{t}.$$

Then

$$H(t) = \sum_{k=1}^{n} k\alpha_k G_k(t), \tag{2.2.8}$$

where G_k is defined in (2.2.4). Recall that for $k = 1, \ldots, m$, $G_k(t) > 0$ for $t \in C^\circ$.

We first recall a computational lemma by Schumacher.

Lemma 2.5 ([Sch17, Lemma 17]). *Let $\alpha_j > 0$ and G_j be positive real numbers for $j = 1, \ldots, n$. Then*

$$\begin{aligned} &\sum_{j=2}^{n} \Big(\alpha_j \frac{G_j^{j+1}}{G_{j-1}^{j-1}} - \alpha_{j-1} \frac{G_j^{j}}{G_{j-1}^{j-2}} \Big) \\ &\geqslant \frac{1}{2} \Big(- \frac{\alpha_1^3}{\alpha_2^2} G_1^2 + \frac{\alpha_{n-1}^{n-1}}{\alpha_n^{n-2}} G_n^2 + \sum_{j=2}^{n-1} \Big(\frac{\alpha_{j-1}^{j-1}}{\alpha_j^{j-2}} - \frac{\alpha_j^{j+2}}{\alpha_{j+1}^{j+1}} \Big) G_j^2 \Big) \end{aligned} \tag{2.2.9}$$

□

Now we are ready to compute the curvature of the Finsler metric F based on Theorem 2.4.

Theorem 2.6. *Fix a smooth Kähler metric ω on Y. There exist* universal *constants $0 < \alpha_1 < \ldots < \alpha_n$ and $\delta > 0$, such that for any $\gamma : C \to V$ with C an open set of $\mathbb{C}$ and $\gamma(C) \cap V^\circ \neq \varnothing$, one has*

$$\sqrt{-1}\partial\bar{\partial} \log |\gamma'(t)|_F^2 \geq \delta \gamma^* \omega \tag{2.2.10}$$

Proof. By Theorem 2.3 and the assumption that $\gamma(C) \cap V^\circ \neq \varnothing$, $G_1(t) \not\equiv 0$. We first recall a result in [Den18a, Lemma 3.11], and we write its proof here for it is crucial in what follows.

Claim 2.7. *There is a universal constant $c_0 > 0$ (*i.e.* it does not depend on γ) so that $\Theta_{L,h_L}(\partial_t, \bar{\partial}_t) \geq c_0 G_1(t)$ for all t.*

Proof of Claim 2.7. Indeed, by Proposition 2.2.(i), it suffice to prove that

$$\frac{|\partial_t|^2_{\gamma^*(r_D^{-2}\cdot\omega_\alpha)}}{|\tau_1(d\gamma(\partial_t))|_h^2} \geqslant c_0 \tag{2.2.11}$$

for some $c_0>0$, where ω_α is a positively definite Hermitian metric on $T_Y(-\log D)$. Note that

$$\frac{|\partial_t|^2_{\gamma^*(r_D^{-2}\cdot\omega_\alpha)}}{|\tau_1(d\gamma(\partial_t))|_h^2}=\frac{|\partial_t|^2_{\gamma^*(r_D^{-2}\cdot\omega_\alpha)}}{|\partial_t|^2_{\gamma^*\tau_1^*h}}=\frac{|\partial_t|^2_{\gamma^*(\omega_\alpha)}}{|\partial_t|^2_{\gamma^*\tau_1^*(r_D^2\cdot h)}},$$

where $\tau_1^*(r_D^2\cdot h)$ is a Finsler metric (indeed continuous pseudo hermitian metric) on $T_Y(-\log D)$ by Proposition 2.2.(iii). Since Y is compact, there exists a constant $c_0>0$ such that

$$\omega_\alpha\geqslant c_0\tau_1^*(r_D^2\cdot h).$$

Hence (2.2.11) holds for any $\gamma:C\to V$ with $\gamma(C)\cap V^\circ\neq\varnothing$. The claim is proved. □

By [Sch12, Lemma 8],

$$\sqrt{-1}\partial\bar\partial\log(\sum_{j=1}^n j\alpha_jG_j)\geqslant\frac{\sum_{j=1}^n j\alpha_jG_j\sqrt{-1}\partial\bar\partial\log G_j}{\sum_{i=1}^n j\alpha_jG_i}\tag{2.2.12}$$

Putting (2.2.5) and (2.2.6) to (2.2.12), and making the convention that $\frac{0}{0}=0$, we obtain

$$\begin{aligned}\frac{\partial^2\log H(t)}{\partial t\partial\bar t}&\geq\frac{1}{H}\Big(-\alpha_1G_2^2+\sum_{k=2}^n\alpha_k\big(\frac{G_k^{k+1}}{G_{k-1}^{k-1}}-\frac{G_{k+1}^{k+1}}{G_k^{k-1}}\big)\Big)+\frac{\sum_{k=1}^n\alpha_kG_k}{H}\Theta_{L,h_L}(\partial_t,\bar\partial_t)\\&=\frac{1}{H}\Big(\sum_{j=2}^n\Big(\alpha_j\frac{G_j^{j+1}}{G_{j-1}^{j-1}}-\alpha_{j-1}\frac{G_j^j}{G_{j-1}^{j-2}}\Big)\Big)+\frac{\sum_{k=1}^n\alpha_kG_k}{H}\Theta_{L,h_L}(\partial_t,\bar\partial_t)\\&\overset{(2.2.9)}{\geq}\frac{1}{H}\Big(-\frac12\frac{\alpha_1^3}{\alpha_2^2}G_1^2+\frac12\sum_{j=2}^{n-1}\Big(\frac{\alpha_{j-1}^{j-1}}{\alpha_j^{j-2}}-\frac{\alpha_j^{j+2}}{\alpha_{j+1}^{j+1}}\Big)G_j^2+\frac12\frac{\alpha_{n-1}^{n-1}}{\alpha_n^{n-2}}G_n^2\Big)\\&\quad+\frac{\sum_{k=1}^n\alpha_kG_k}{H}\Theta_{L,h_L}(\partial_t,\bar\partial_t)\\&\overset{\text{Claim 2.7}}{\geq}\frac{1}{H}\Big(\frac{\alpha_1}{2}(c_0-\frac{\alpha_1^2}{\alpha_2^2})G_1^2+\frac12\sum_{j=2}^{n-1}\Big(\frac{\alpha_{j-1}^{j-1}}{\alpha_j^{j-2}}-\frac{\alpha_j^{j+2}}{\alpha_{j+1}^{j+1}}\Big)G_j^2+\frac12\frac{\alpha_{n-1}^{n-1}}{\alpha_n^{n-2}}G_n^2\Big)\\&\quad+\frac1H(\frac12\alpha_1G_1+\sum_{k=2}^n\alpha_kG_k)\Theta_{L,h_L}(\partial_t,\bar\partial_t)\end{aligned}$$

One can take $\alpha_1=1$, and choose the further $\alpha_j>\alpha_{j-1}$ inductively so that

$$c_0-\frac{\alpha_1^2}{\alpha_2^2}>0,\quad\frac{\alpha_{j-1}^{j-1}}{\alpha_j^{j-2}}-\frac{\alpha_j^{j+2}}{\alpha_{j+1}^{j+1}}>0\quad\forall\, j=2,\ldots,n-1.\tag{2.2.13}$$

Hence

$$\frac{\partial^2\log H(t)}{\partial t\partial\bar t}\geq\frac1H(\frac12\alpha_1G_1+\sum_{k=2}^n\alpha_kG_k)\Theta_{L,h_L}(\partial_t,\bar\partial_t)\overset{(2.2.8)}{\geq}\frac1n\Theta_{L,h_L}(\partial_t,\bar\partial_t)$$

over C°. By Proposition 2.2.(i), this implies that

$$\sqrt{-1}\partial\bar\partial\log|\gamma'|_F^2=\sqrt{-1}\partial\bar\partial\log H(t)\geq\frac1n\gamma^*\sqrt{-1}\Theta_{L,h_L}\geq\delta\gamma^*\omega\tag{2.2.14}$$

over C° for some positive constant δ, which does not depend on γ. Since $|\gamma'(t)|_F^2$ is continuous and locally bounded from above over C, by the extension theorem of subharmonic function, (2.2.14) holds over the whole C. Since $c_0>0$ is a constant which does not depend on γ, so are $\alpha_1,\ldots,\alpha_n$ by (2.2.13). The theorem is thus proved. □

In summary of results in this subsection, we obtain the following theorem.

Theorem 2.8. *Let X be a projective manifold and let D be a simple normal crossing divisor on X. Assume that there is a Viehweg-Zuo Higgs bundle over (X, D). Then there are a Finsler metric h on $T_X(-\log D)$ which is positively definite on a dense Zariski open set V° of $V := X \setminus D$, and a smooth Kähler form on X such that for any holomorphic map $\gamma : C \to V$ from any open subset C of $\mathbb{C}$ with $\gamma(C) \cap V^\circ \neq \varnothing$, one has*

$$\sqrt{-1}\partial\bar{\partial}\log|\gamma'|_h^2 \geq \gamma^*\omega.$$

In particular, by Theorem A, for any holomorphic map $f : \Delta^* \to X \setminus D$, with $f(\Delta^*) \cap V^\circ \neq \varnothing$, it extends to a holomorphic map $\bar{f} : \Delta \to X$. □

3. Generalized Big Picard theorems

We will apply Theorems 1.7 and 2.8 to prove the big Picard theorem for moduli of polarized manifolds with semi-ample canonical sheaf, and for bases of log smooth families of Calabi-Yau pairs.

3.1. Big Picard theorem for moduli of polarized manifolds.

Proof of Theorem C. Let Z be the Zariski closure of $\gamma(\Delta^*)$ in Y. Take an embedded desingularization of singularities $\mu : \tilde{Y} \to Y$ so that the strict transform of Z, denoted by $\tilde{Z}$, is smooth. Write $\tilde{Z}^\circ := \tilde{Z} \cap \mu^{-1}(V)$, which is a dense Zariski open set of $\tilde{Z}$. We take the base change

$$\begin{array}{ccc} X^\circ = U \times_V \tilde{Z}^\circ & \longrightarrow & U \\ {\scriptstyle f_{X^\circ}}\downarrow & & \downarrow{\scriptstyle f_U} \\ \tilde{Z}^\circ & \xrightarrow{\quad\iota\quad} & V \end{array}$$

Then polarized family $(f_{X^\circ} : X^\circ \to \tilde{Z}^\circ, \iota^*\mathscr{L}) \in \mathscr{P}_h(\tilde{Z}^\circ)$. We denote by $\varphi_{X^\circ} : \tilde{Z}^\circ \to P_h$ the moduli map associated to f_{X°. Then $\varphi_{X^\circ} = \varphi_U \circ \iota$, which is generically finite. Hence f_{X° is of maximal variation. By [VZ02,PTW18], after passing to a birational modification $\nu : W \to \tilde{Z}$, there exists a Viehweg-Zuo Higgs bundle on $W^\circ := \nu^{-1}(\tilde{Z}^\circ)$. By Theorem 2.8, there is a dense Zariski open set $W' \subset W^\circ$ so that any holomorphic map $\Delta^* \to W^\circ$ extends to $\Delta \to W$ provided that its image is not contained in $W \setminus W'$. Since $\gamma : \Delta^* \to Z$ is Zariski dense, it thus does not lie on the discriminant locus of the birational morphism $\mu|_{\tilde{Z}} \circ \nu : W \to Z$, and thus $\tilde{\gamma} = (\mu|_{\tilde{Z}} \circ \nu)^{-1} \circ \gamma : \Delta^* \to W$ exists with its image contained in W°. Moreover, $\tilde{\gamma} : \Delta^* \to W$ is also Zariski dense, and thus $\tilde{\gamma}(\Delta^*) \cap W' \neq \varnothing$. By Theorem 1.7, $\tilde{\gamma} : \Delta^* \to W$ extends to a holormorphic map $\bar{\tilde{\gamma}} : \Delta \to W$. The holomorphic map $\mu \circ \nu \circ \bar{\tilde{\gamma}} : \Delta \to Y$ is the desired extension of $\gamma : \Delta^* \to V$. The theorem is proved. □

Remark 3.1. Based on the fundamental work [VZ02,VZ03,PTW18], in [Den18b] we prove that the base V in Theorem C is both Brody hyperbolic and pseudo Kobayashi hyperbolic. In [LSZ19], Lu-Sun-Zuo combine the original approach in [VZ03] with our Torelli type result Theorem 2.3 to construct negatively curved pseudo hermitian metric on any algebraic curve in V, so that they can apply the celebrated work of Griffiths-King [GK73] to prove the Borel hyperbolicity of V.

3.2. Big Picard theorem for bases of log Calabi-Yau families.

In [Den19], we prove that for maximally varying, log smooth family of Calabi-Yau pairs, its base can be equipped with a Viehweg-Zuo Higgs bundle. Let us start with the following definition.

Definition 3.2 (log smooth family)**.** Let X° and Y° be quasi-projective manifolds, and let $D^\circ = \sum_{i=1}^m a_i D_i^\circ$ be a Kawamata log terminal (klt for short) $\mathbb{Q}$-divisor on X° with simple normal crossing support. The morphism $f^\circ : (X^\circ, D^\circ) \to Y^\circ$ is a *log smooth family* if $f^\circ : X^\circ \to Y^\circ$ is a smooth projective morphism with connected fibers, and D° is *relatively normal crossing over* Y°, namely each stratum $D_{i_1}^\circ \cap \cdots \cap D_{i_k}^\circ$ of D° is dominant onto and smooth over Y° under f°.

Let us recall the main result in [Den19].

Theorem 3.3 ([Den19, Theorem A]). *Let $f^\circ : (X^\circ, D^\circ) \to Y^\circ$ be a log smooth family over a quasi-projective manifold Y°. Assume that each fiber $(X_y, D_y) := (f^\circ)^{-1}(y)$ is a klt pair, and $K_{X_y} + D_y \equiv_{\mathbb{Q}} 0$. Assume that the logarithmic Kodaira-Spencer map $\rho : T_{Y^\circ} \to R^1 f^\circ_* \big(T_{X^\circ/Y^\circ}(-\log D^\circ)\big)$ is generically injective. Then after replacing Y° by a birational model, there exists a Viehweg-Zuo Higgs bundle over Y°.* □

By Theorem 3.3, one can perform the same proof as that of Theorem C to conclude the following result.

Theorem 3.4. *In the setting of Theorem 3.3, assume additionally that the logarithmic Kodaira-Spencer map*

$$T_{Y^\circ,y} \to H^1\big(X_y, T_{X_y}(-\log D_y)\big)$$

is injective for any $y \in Y^\circ$. Then for any projective compactification Y of the base Y°, any holomorphic map $\gamma : \Delta^* \to Y^\circ$ extends into the origin. □

References

[BD19] Damian Brotbek and Ya Deng, *Kobayashi hyperbolicity of the complements of general hypersurfaces of high degree*, Geometric and Functional Analysis (2019), https://doi.org/10.1007/s00039-019-00496-2. ↑ 2

[Bor72] Armand Borel, *Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem*, J. Differential Geometry **6** (1972), 543–560, http://projecteuclid.org/euclid.jdg/1214430642. ↑ 2

[Dem97] Jean-Pierre Demailly, *Variétés hyperboliques et équations différentielles algébriques*, Gaz. Math. (1997), no. 73, 3–23. ↑ 2, 3, 4, 5

[Den18a] Ya Deng, *Kobayashi hyperbolicity of moduli spaces of minimal projective manifolds of general type (with the appendix by Dan Abramovich)*, arXiv:1806.01666 (2018). ↑ 2, 6, 8, 9

[Den18b] ______, *Pseudo Kobayashi hyperbolicity of base spaces of families of minimal projective manifolds with maximal variation*, arXiv:1809.05891 (2018). ↑ 2, 3, 6, 7, 8, 11

[Den19] ______, *Hyperbolicity of bases of log Calabi-Yau families*, hal-02266744 (2019). ↑ 2, 3, 11, 12

[Fuj72] Hirotaka Fujimoto, *On holomorphic maps into a taut complex space*, Nagoya Math. J. **46** (1972), 49–61, http://projecteuclid.org/euclid.nmj/1118798592. ↑ 2

[GK73] Phillip Griffiths and James King, *Nevanlinna theory and holomorphic mappings between algebraic varieties*, Acta Math. **130** (1973), 145–220, https://doi.org/10.1007/BF02392265. ↑ 11

[Gre77] Mark L. Green, *The hyperbolicity of the complement of $2n + 1$ hyperplanes in general position in P_n and related results*, Proc. Amer. Math. Soc. **66** (1977), no. 1, 109–113, https://doi.org/10.2307/2041540. ↑ 2

[JK18] Ariyan Javanpeykar and Robert A. Kucharczyk, *Algebraicity of analytic maps to a hyperbolic variety*, arXiv e-prints (2018), arXiv:1806.09338. ↑ 1

[KO71] Shoshichi Kobayashi and Takushiro Ochiai, *Satake compactification and the great Picard theorem*, J. Math. Soc. Japan **23** (1971), 340–350, https://doi.org/10.2969/jmsj/02320340. ↑ 2

[Kob98] Shoshichi Kobayashi, *Hyperbolic complex spaces*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 318, Springer-Verlag, Berlin, 1998, https://doi.org/10.1007/978-3-662-03582-5. ↑ 1, 4

[LSZ19] Steven Lu, Ruiran Sun, and Kang Zuo, *Nevanlinna Theory on Moduli Space and the big Picard Theorem*, arXiv e-prints (2019), arXiv:1911.02973. ↑ 2, 3, 11

[Nog81] Junjiro Noguchi, *Lemma on logarithmic derivatives and holomorphic curves in algebraic varieties*, Nagoya Math. J. **83** (1981), 213–233, http://projecteuclid.org/euclid.nmj/1118786486. ↑ 2, 3, 4

[NW14] Junjiro Noguchi and Jörg Winkelmann, *Nevanlinna theory in several complex variables and Diophantine approximation*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 350, Springer, Tokyo, 2014, https://doi.org/10.1007/978-4-431-54571-2. ↑ 3, 4, 5, 6

[PS17] Mihnea Popa and Christian Schnell, *Viehweg's hyperbolicity conjecture for families with maximal variation*, Invent. Math. **208** (2017), no. 3, 677–713, https://doi.org/10.1007/s00222-016-0698-9. ↑ 2

[PTW18] Mihnea Popa, Behrouz Taji, and Lei Wu, *Brody hyperbolicity of base spaces of certain families of varieties*, arXiv:1801.05898 (2018). ↑ 2, 6, 7, 11

[Sch12] Georg Schumacher, *Positivity of relative canonical bundles and applications*, Invent. Math. **190** (2012), no. 1, 1–56, https://doi.org/10.1007/s00222-012-0374-7. ↑ 10

[Sch17] Georg Schumacher, *Moduli of canonically polarized manifolds, higher order Kodaira-Spencer maps, and an analogy to Calabi-Yau manifolds*, arXiv e-prints (2017), arXiv:1702.07628. ↑ 9

[SY97] Yum-Tong Siu and Sai-Kee Yeung, *Defects for ample divisors of abelian varieties, Schwarz lemma, and hyperbolic hypersurfaces of low degrees*, Amer. J. Math. **119** (1997), no. 5, 1139–1172, http://muse.jhu.edu/journals/american_journal_of_mathematics/v119/119.5siu.pdf. ↑ 2, 3

[TY15] Wing-Keung To and Sai-Kee Yeung, *Finsler metrics and Kobayashi hyperbolicity of the moduli spaces of canonically polarized manifolds*, Ann. of Math. (2) **181** (2015), no. 2, 547–586, https://doi.org/10.4007/annals.2015.181.2.3. ↑ 9

[Vie95] Eckart Viehweg, *Quasi-projective moduli for polarized manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 30, Springer-Verlag, Berlin, 1995, https://doi.org/10.1007/978-3-642-79745-3. ↑ 3

[VZ02] Eckart Viehweg and Kang Zuo, *Base spaces of non-isotrivial families of smooth minimal models*, Complex geometry (Göttingen, 2000), Springer, Berlin, 2002, pp. 279–328. ↑ 2, 6, 7, 11

[VZ03] ______, *On the Brody hyperbolicity of moduli spaces for canonically polarized manifolds*, Duke Math. J. **118** (2003), no. 1, 103–150, https://doi.org/10.1215/S0012-7094-03-11815-3. ↑ 2, 6, 11

Institut des Hautes Études Scientifiques, Université Paris-Saclay, 35 route de Chartres, 91440, Bures-sur-Yvette, France

Email address: deng@ihes.fr

URL: https://www.ihes.fr/~deng