BIG PICARD THEOREM AND ALGEBRAIC HYPERBOLICITY FOR VARIETIES ADMITTING A VARIATION OF HODGE STRUCTURES

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Abstract. In the paper we study various hyperbolicity properties for the quasi-compact Kähler manifold U which admits a complex polarized variation of Hodge structures so that each fiber of the period map is zero dimensional. In the first part we prove that U is algebraically hyperbolic, and that the generalized big Picard theorem holds for U. In the second part, we prove that there is a finite unramified cover $\tilde{U}$ of U from a quasi-projective manifold $\tilde{U}$ so that any projective compactification X of $\tilde{U}$ is Picard hyperbolic modulo the boundary $X - \tilde{U}$, and any irreducible subvariety of X not contained in $X - \tilde{U}$ is of general type. This result coarsely incorporates previous works by Nadel, Rousseau, Brunebarbe and Cadorel on the hyperbolicity of compactifications of quotients of bounded symmetric domains by torsion free lattices.

Contents

0. Introduction

0.1. **Background.** The classical big Picard theorem says that any holomorphic map from the punctured disk Δ^* into $\mathbb{P}^1$ which omits three points can be extended to a holomorphic map $\Delta \to \mathbb{P}^1$, where Δ denotes the unit disk. Therefore, we introduce a new notion of hyperbolicity which generalizes the big Picard theorem. We say a complex manifold U is *quasi-compact Kähler* if it is a Zariski open set of a compact Kähler manifold Y. Such Y will be called a *smooth Kähler compactification* of U.

Definition 0.1 (Picard hyperbolicity)**.** Let U be a quasi-compact Kähler manifold. U is called *Picard hyperbolic* if there is a smooth Kähler compactification Y of U such that any holomorphic map $f : \Delta^* \to U$ extends to a holomorphic map $\bar{f} : \Delta \to Y$.

We will prove in Lemma 4.3 that this definition does not depend on the compactification of U. Picard hyperbolic varieties first attracted the author's interest because of the recent interesting work [JK18] by Javanpeykar-Kucharczyk on the *algebraicity of analytic maps*. In [JK18, Definition 1.1], they introduce a new notion of hyperbolicity:

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a quasi-projective variety U is *Borel hyperbolic* if any holomorphic map from a quasi-projective variety to U is necessarily *algebraic*. In [JK18, Corollary 3.11] they prove that a Picard hyperbolic variety is Borel hyperbolic. We refer the readers to [JK18, §1] for their motivation on the Borel hyperbolicity. Picard hyperbolic varieties fascinate us further when we realize in Proposition 4.2 that a more general extension theorem is also valid for them: any holomorphic map from $\Delta^p\times(\Delta^*)^q$ to the manifold U in Definition 0.1 extends to a *meromorphic map* from Δ^{p+q} to Y.

By A. Borel [Bor72] and Kobayashi-Ochiai [KO71], it has long been known to us that the quotients of bounded symmetric domains by torsion free arithmetic lattice are hyperbolically embedded into their Baily-Borel compactification, and thus they are Picard hyperbolic (see [Kob98, Theorem 6.1.3]). A *transcendental* analogue of bounded symmetric domains is the rich theory of period domain, which was first introduced by Griffiths [Gri68a] and was later systematically studied by him in the seminal work [Gri68b, Gri70a, Gri70b]. In [JK18, §1.1] Javanpeykar-Kucharczyk conjectured that an algebraic variety U which admits an integral variation of Hodge structures ($\mathbb{Z}$-VHS for short) with quasi-finite period map is Borel hyperbolic. Their conjecture was recently proved in a recent remarkable work of Bakker-Brunebarbe-Tsimerman [BBT18]. The proof is based on the tame geometry for quotient ${}^{\mathcal{D}}\!/_{\Gamma}$ of period domains $\mathcal{D}$ by arithmetic groups Γ containing the monodromy group of the $\mathbb{Z}$-VHS. However, when one studies Picard hyperbolicity (or Borel hyperbolicity) for varieties admitting the more general *complex polarized variation of Hodge structures*, there are several problems which seem difficult to be tackled if one uses o-minimal geometry in [BBT18]:

- the monodromy group Γ might not act on the period domain $\mathcal{D}$ discretely, and thus the quotient of the period domain by the monodromy group ${}^{\mathcal{D}}\!/_{\Gamma}$ is even non-Haudorff.
- The local monodromies at infinity might not be quasi-unipotent, though it is always the case for $\mathbb{Z}$-VHS by the theorem of Borel.

Therefore, the great differences between $\mathbb{Z}$-VHS and $\mathbb{C}$-PVHS require new ideas if one would like to extend the theorem by Bakker-Brunebarbe-Tsimerman to the context of $\mathbb{C}$-PVHS. This is one of the main goals in this paper. In the first part, we prove the Picard hyperbolicity of quasi-compact Kähler manifolds admitting a *complex polarized variation of Hodge structures* using techniques in complex analytic geometry and Hodge theory.

0.2. Big Picard theorem and algebraic hyperbolicity.

Theorem A. *Let U be a quasi-compact Kähler manifold. Assume that there is a complex polarized variation of Hodge structures ($\mathbb{C}$-PVHS for short) over U so that each fiber of the period map $U_{\rm uni}\to\mathcal{D}$ is zero dimensional, where $U_{\rm uni}$ is the universal cover of U and $\mathcal{D}$ is the period domain associated to the above $\mathbb{C}$-PVHS. Then U is both algebraically hyperbolic, and Picard hyperbolic. In particular, U is Borel hyperbolic.*

Note that we make no assumptions on the local monodromies of the $\mathbb{C}$-PVHS at infinity (which can be non quasi-unipotent), or on its global monodromy group (thus it might act non-discretely on U). Let us mention that when the $\mathbb{C}$-PVHS over U in Theorem A is moreover a $\mathbb{Z}$-PVHS, Borel hyperbolicity of U in Theorem A has been proven in [BBT18, Corollary 7.1], and algebraic hyperbolicity of U is implicitly shown by Javanpeykar-Litt in [JL19, Theorem 4.2] if local monodromies of the $\mathbb{Z}$-VHS at infinity are unipotent (see Remark 4.5). Our proof of Theorem A is based on complex analytic and Hodge theoretic methods, and it does not use the delicate o-minimal geometry in [PS08, PS09, BKT18, BBT18]. Indeed, we are not sure that the tame geometry for period domains of $\mathbb{Z}$-PVHS in [BKT18, BBT18] can be applied to prove Theorem A.

We can even generalize Theorem A to higher dimensional domain spaces.

Corollary 0.2 (=Theorem A+Proposition 4.2)**.** *Let U be the quasi-compact Kähler manifold in Theorem A, and let Y be a smooth Kähler compactification of U. Then any holomorphic map $f : \Delta^p \times (\Delta^*)^q \to U$ extends to a meromorphic map $\overline{f} : \Delta^{p+q} \dashrightarrow Y$. In particular, if W is a Zariski open set of a compact complex manifold X, then any holomorphic map $g : W \to U$ extends to a meromorphic map $\overline{g} : X \dashrightarrow Y$.*

0.3. **Hyperbolicity for the compactification after finite unramified cover.** The second main result of this paper is on the hyperbolicity for the compactification of some finite unramified cover of the quasi-compact Kähler manifold U in Theorem A. Let us first introduce several definitions of hyperbolicity. We refer the readers to the recent survey by Javenpeykar [Jav20, §8] for more conjectural relations among them.

Definition 0.3 (Notions of hyperbolicity)**.** Let (X, ω) be a compact Kähler manifold and let $Z \subsetneq X$ be a closed subset of X.

(1) X is *Kobayashi hyperbolic modulo Z* if the Kobayashi pseudo-distance $d_X(x, y) > 0$ for distinct points $x, y \in X$ not both contained in Z.
(2) X is *Picard hyperbolic modulo Z* if any holomorphic map $\gamma : \Delta^* \to X$ not contained in Z extends across the origin.
(3) X is *Brody hyperbolic modulo Z* if any entire curve $\gamma : \mathbb{C} \to X$ is contained in Z.
(4) X is *algebraically hyperbolic modulo Z* if there is $\varepsilon > 0$ so that for any compact irreducible curve $C \subset X$ not contained in Z, one has
$$2g(\tilde{C}) - 2 \geq \varepsilon \deg_\omega C,$$
where $g(\tilde{C})$ is the genus of the normalization $\tilde{C}$ of C, and $\deg_\omega C := \int_C \omega$.

It is easy to show that if X is Kobayashi hyperbolic or Picard hyperbolic modulo Z, then X is Brody hyperbolic modulo Z.

The second main result of the paper is the following theorem.

Theorem B. *Let U be a quasi-compact Kähler manifold. Assume that there is a $\mathbb{C}$-PVHS over U so that each fiber of the period map is zero dimensional. Then there is a quasi-projective manifold $\tilde{U}$ and a finite unramified cover $\tilde{U} \to U$ such that for any projective compactification X of $\tilde{U}$,*

(i) *any irreducible Zariski closed subvariety of X is of general type if it is not contained in $\tilde{D} := X - \tilde{U}$.*
(ii) *The variety X is Picard hyperbolic modulo $\tilde{D}$.*
(iii) *The variety X is Brody hyperbolic modulo $\tilde{D}$.*
(iv) *The variety X is algebraically hyperbolic modulo $\tilde{D}$.*

By the work of Deligne, the quotient of any bounded symmetric domain by a torsion free lattice always admits a $\mathbb{C}$-PVHS whose period map is *immersive everywhere*. Theorem B then yields

Corollary C. *Let U be the quotient of a bounded symmetric domain by a torsion free lattice. Then there is a finite unramified cover $\tilde{U} \to U$ from a quasi-projective manifold $\tilde{U}$ so that any projective compactification X of $\tilde{U}$ is Picard and algebraically hyperbolic modulo $X - \tilde{U}$.*

Let us stress here that, Nadel [Nad89] and Rousseau [Rou16] proved that the variety X in Corollary C is Brody and Kobayashi hyperbolic modulo $X - \tilde{U}$; and Brunebarbe [Bru20a] and Cadorel [Cad16,Cad18] proved that any Zariski closed subvariety not contained in $X - \tilde{U}$ is of general type. Theorem B thus incorporates their results, but in the cost of loss of effectivity for the level structures (see Remark 5.4) due to the generality of our result in Theorem B.

0.4. **Main strategy.**

0.4.1. *Negatively curved Finsler metric.* Let Y be a compact Kähler manifold and let D be a simple normal crossing divisor on Y. Assume that there is a $\mathbb{C}$-PVHS over $U := Y - D$. Note that one also has a strictly positive line bundle on U if the period map is immersive at one point, which was constructed by Griffiths in [Gri70a] half century ago! Based on the work by Simpson and Mochizuki, in Proposition 2.5 we can extend this Griffiths line bundle over Y to obtain a more refined positivity result[1]. We then construct a special system of log Hodge bundles $(E,\theta) = (\oplus_{p+q=m}E^{p,q}, \oplus_{p+q=m}\theta_{p,q})$ on the log pair (Y,D) so that some higher stage E^{p_0,q_0} contains a big line bundle which admits enough *local positivity along D*. Inspired by our previous work [Den18] on the proof of Viehweg-Zuo's conjecture on Brody hyperbolicity of moduli of polarized manifolds, in Theorem 2.9 we show that (E,θ) still enjoys a 'partially' infinitesimal Torelli property. These results enable us construct a negatively curved, and generically positively definite Finsler metric on $T_Y(-\log D)$, in a similar vein as [Den18].

Theorem 0.4 (=Theorem 2.6+Theorem 3.6)**.** *Let Y be a compact Kähler manifold and let D be a simple normal crossing divisor on Y. Assume that there is a $\mathbb{C}$-PVHS over $U := Y - D$, whose period map is immersive at one point. Then there are a* Finsler metric h *(see Definition 3.1) on $T_Y(-\log D)$ which is positively definite on a dense Zariski open set U° of $Y - D$, and a smooth Kähler form ω on Y such that for any holomorphic map $\gamma : C \to U$ from an open set $C \subset \mathbb{C}$ to U, one has*

$$\mathrm{dd^c} \log |\gamma'(t)|_h^2 \geq \gamma^*\omega \tag{0.4.1}$$

when $\gamma(C) \cap U^\circ \neq \varnothing$.

Let us mention that, though we only construct (possibly degenerate) Finsler metric over $T_Y(-\log D)$, it follows from (0.4.1) that we know exactly the behavior of its curvature near the boundary D since ω is a smooth Kähler form over Y. The proof of Theorem A is then based on Theorem 0.4 and some criteria for big Picard theorem established in [DLSZ19] (see Theorem 4.4). Let us also mention that the Finsler metric constructed in Theorem 0.4 is also crucially used in the proof of Theorem B.

0.4.2. *On the hyperbolicity of the compactification.* The proof of Theorem B is based on Theorem 5.1, whose proof is technically involved. It is worthwhile to mention that our proof is quite different from those in [Nad89, Rou16, Bru20a, Cad18]. All these proofs relied heavily on the special property of quotients of bounded symmetric domains by torsion free lattices. They all applied the toroidal compactifications by Mumford to find the desired finite unramified cover $\tilde{U} \to U$ when U is a quotient of bounded symmetric domain by a torsion free lattice. We construct the cover $\tilde{U} \to U$ in Theorem B in a subtle way using the residual finiteness of the global monodromy group. We refer the readers to the beginning of § 5 for the general strategies.

0.5. **Some new developments.** Shortly after this paper appeared on arXiv, Brunebarbe-Brotbek [BB20, Theorem 1.5] proved the Borel hyperbolicity of U in Theorem A under the additional assumption that the local monodromy of the $\mathbb{C}$-PVHS at infinity is unipotent. Moreover, they also obtained a weaker result than Theorem B.(ii) in [BB20, Theorem 1.7], in which they showed that for a quasi-projective manifold U admitting a $\mathbb{Z}$-PVHS with quasi-finite period map, there is a finite étale cover $\tilde{U} \to U$ so that the projective compactification X of $\tilde{U}$ is *Borel hyperbolic* modulo $X - \tilde{U}$. Our proofs are indeed quite different: Brotbek-Brunebarbe's proof is based on their Second Main theorem

[1] If the local monodromy around D is unipotent, this is well-known.

using the *Griffiths-Schmid metric*, which coincides with the curvature form of the Griffiths line bundle. This keen observation is crucial in their proof and thus their method is more “canonical" than us. However, the methods developed in this paper is more flexible to extend main results in a more general setting: in a forthcoming paper, we proved that Theorems A and B in this paper still hold for varieties admitting *generalized variation of Hodge structures*.

Let us also mention that a similar result of Theorem B.(i) is also obtained by Brunebarbe in [Bru20b, Theorem 1.1] when the underlying local system of the ℂ-PVHS is defined over ℤ.

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Notations and Conventions

- A *log pair* (Y, D) consists of a (possibly non-compact) complex manifold and a simple normal crossing divisor D. (Y, D) will be called *compact Kähler log pair* (resp. *projective log pair*) if Y is a compact Kähler (resp. projective) manifold.
- A complex manifold is called *quasi-compact Kähler* if it is a Zariski open set of a compact Kähler manifold.
- A *log morphism* $f : (X, \tilde{D}) \to (Y, D)$ between log pairs is a morphism $f : X \to Y$ with $\tilde{D} \subset f^{-1}(D)$.
- For a big line bundle L on a projective manifold, $\mathbf{B}_+(L)$ denotes its *augmented base locus* (see [Laz04, Definition 10.3.2]).
- ℂ-PVHS stands for *complex polarized variation of Hodge structures*.

1. Preliminary on Hodge theory

1.1. **System of Hodge bundles.** Following Simpson [Sim88], a complex polarized variation of Hodge structures (ℂ-PVHS) is equivalent to a system of Hodge bundles. Let us recall the definition in this subsection.

Definition 1.1 (Higgs bundle)**.** A Higgs bundle on a complex manifold Y is a pair (E, θ) consisting of a holomorphic vector bundle E on Y and an $\mathcal{O}_Y$-linear map

$$\theta : E \to E \otimes \Omega_Y^1$$

so that $\theta \wedge \theta = 0$. Such θ is called *Higgs field*.

Definition 1.2 (Harmonic bundle)**.** A *harmonic bundle* (E, θ, h) consists of a Higgs bundle (E, θ) and a hermitian metric h for E so that

$$D := \partial_h + \bar{\partial}_E + \theta + \theta_h^*$$

is flat. Here $\partial_h + \bar{\partial}_E$ is the Chern connection, and $\theta_h^* \in C^\infty(Y, \mathrm{End}(E) \otimes \Omega_Y^{0,1})$ is the adjoint of θ with respect to h.

Definition 1.3 (System of Hodge bundles)**.** A *system of Hodge bundles* of weight m is a harmonic bundle (E, θ, h) satisfying the following:

- $E = \oplus_{p+q=m} E^{p,q}$ is a direct sum of holomorphic vector bundles $E^{p,q}$.
- $$\theta|_{E^{p,q}} : E^{p,q} \to E^{p-1,q+1} \otimes \Omega_Y^1$$
- The splitting $E = \oplus_{p+q=m} E^{p,q}$ is orthogonal with respect to h.

We denote by $h_{p,q} = h|_{E^{p,q}}$, and $\theta_{p,q} = \theta|_{E^{p,q}}$. This harmonic metric will be called *Hodge metric*.

Throughout this paper, we make the convention that $0 \leq p, q \leq m$ for the decomposition $E = \oplus_{p+q=m} E^{p,q}$. This can always be achieved if we make a Tate twist (k, k) to increase the weight by $2k$ when $k \in \mathbb{Z}_{>0}$ is large enough.

1.2. **Filtered bundles and parabolic Higgs bundle.** In this section, we recall the notions of filtered bundles and parabolic Higgs bundles in [Sim88, Moc07]. Let $(Y, D = \sum_{i=1}^{c} D_i)$ be log pair.

Definition 1.4. A filtered bundle $(E, \mathcal{P}_{\boldsymbol{a}} E)$ on (Y, D) is a locally free sheaf E on $Y - D$, together with an $\mathbb{R}^c$-indexed filtration $\mathcal{P}_{\boldsymbol{a}} E$ by locally free sheaves on Y such that

(1) $\boldsymbol{a} \in \mathbb{R}^c$ and $\mathcal{P}_{\boldsymbol{a}} E|_U = E$.
(2) $\mathcal{P}_{\boldsymbol{a}} E \subset \mathcal{P}_{\boldsymbol{b}} E$ for $\mathbf{a} \leq \mathbf{b}$ (*i.e.* $a_i \leq b_i$ for all i).
(3) $\mathcal{P}_{\boldsymbol{a}} E \otimes \mathcal{O}_Y(D_i) = \mathcal{P}_{\mathrm{a}+1_i} E$ with $1_i = (0, \ldots, 1, \ldots, 0)$ with 1 in the i-th component.
(4) $\mathcal{P}_{\boldsymbol{a}+\boldsymbol{\epsilon}} E = \mathcal{P}_{\boldsymbol{a}} E$ for any vector $\boldsymbol{\epsilon} = (\epsilon, \ldots, \epsilon)$ with $0 < \epsilon \ll 1$.
(5) Write $\mathcal{P}_{<\boldsymbol{a}} E = \cup_{\boldsymbol{b}<\boldsymbol{a}} \mathcal{P}_{\boldsymbol{b}} E$. The set of *weights* $\boldsymbol{a}$ such that $\mathcal{P}_{\boldsymbol{a}} E / \mathcal{P}_{<\boldsymbol{a}} E \neq 0$ is discrete in $\mathbb{R}^c$.

A weight is normalized if it lies in $(-1, 0]^c$. Denote $\mathcal{P}_{\mathbf{0}} E$ by $^\diamond E$, where $\mathbf{0} = (0, \ldots, 0)$. Note that the set of weight of $(E, \mathcal{P}_{\boldsymbol{a}} E)$ is uniquely determined by the filtration for weights lying in $(-1, 0]^c$.

Definition 1.5. A *parabolic Higgs bundle* on (Y, D) is a filtered bundle bundle $(E, \mathcal{P}_{\boldsymbol{a}} E)$ together with $\mathcal{O}_Y$ linear map

$$\theta : {}^\diamond E \longrightarrow \Omega_Y^1(\log D) \otimes {}^\diamond E$$

such that

$$\theta \wedge \theta = 0$$

and

$$\theta(\mathcal{P}_{\boldsymbol{a}} E) \subseteq \Omega_Y^1(\log D) \otimes \mathcal{P}_{\boldsymbol{a}} E,$$

for $\boldsymbol{a} \in [-1, 0)^c$.

A natural class of filtered bundles comes from prolongations of systems of Hodge bundles, which will be discussed in § 1.4.

1.3. **Admissible coordinate.**

Definition 1.6 (Admissible coordinate)**.** Let $(Y, D = \sum_{i=1}^{c} D_i)$ be log pair. Let p be a point of Y, and assume that $\{D_j\}_{j=1,\ldots,\ell}$ be components of D containing p. An *admissible coordinate* around p is the tuple $(\mathcal{U}; z_1, \ldots, z_n; \varphi)$ (or simply $(\mathcal{U}; z_1, \ldots, z_n)$ if no confusion arises) where

- $\mathcal{U}$ is an open subset of Y containing p.
- there is a holomorphic isomorphism $\varphi : \mathcal{U} \to \Delta^n$ so that $\varphi(D_j) = (z_j = 0)$ for any $j = 1, \ldots, \ell$.

We shall write $\mathcal{U}^* := \mathcal{U} - D$.

Recall that the complete Poincaré metric ω_P on $(\Delta^*)^\ell \times \Delta^{n-\ell}$ is described as

$$\omega_P = \sum_{j=1}^{\ell} \frac{\sqrt{-1} dz_j \wedge d\bar{z}_j}{|z_j|^2 (\log |z_j|^2)^2} + \sum_{k=\ell+1}^{n} \sqrt{-1} \frac{dz_k \wedge d\bar{z}_k}{(1 - |z_k|^2)^2} \tag{1.3.1}$$

Note that $\omega_P = dd^c\varphi$ with

$$\varphi := -\log\Big(\prod_{j=1}^{\ell}(-\log|z_j|^2)\cdot\big(\prod_{k=\ell+1}^{n}(1-|z_k|)^2\big)\Big). \tag{1.3.2}$$

Remark 1.7 (Global Kähler metric with Poincaré growth). Let (Y,ω) be a compact Kähler manifold and let $D=\sum_{i=1}^{\ell}D_i$ be a simple normal crossing divisor on Y. Let σ_i be the section $H^0(Y,\mathcal{O}_Y(D_i))$ defining D_i, and we pick any smooth metric h_i for the line bundle $\mathcal{O}_Y(D_i)$. One can prove that when $\varepsilon>0$ is small enough, the closed $(1,,1)$-current

$$T := \omega - dd^c\log(-\prod_{i=1}^{\ell}\log|\varepsilon\cdot\sigma_i|^2_{h_i}), \tag{1.3.3}$$

is a Kähler current (*i.e.* $T\geq\delta\omega$ for some $\delta>0$), and on any admissible coordinate $(\mathcal{U};z_1,\ldots,z_n)$, $T|_{\mathcal{U}-D}\sim\omega_P$.

1.4. **Prolongation of systems of Hodge bundles.** Let $(Y,D=\sum_{i=1}^{\ell}D_i)$ be log pair. Let (E,h) be a hermitian bundle on $Y-D$. For any $\mathbf{a}=(a_1,\ldots,a_\ell)\in\mathbb{R}^\ell$, we can prolong E over Y by $\mathcal{P}^h_{\boldsymbol{a}}E$ as follows:

$$\mathcal{P}^h_{\boldsymbol{a}}E(\mathcal{U})=\{\sigma\in\Gamma(\mathcal{U}-D,E|_{\mathcal{U}-D})\mid|\sigma|_h\lesssim\frac{1}{\prod_{i=1}^{\ell}|z_i|^{a_i+\varepsilon}}\ \forall\varepsilon>0\}, \tag{1.4.1}$$

where $(\mathcal{U};z_1,\ldots,z_n)$ is any admissible coordinate. We still use the notation ${}^\diamond E$ in the case $\boldsymbol{a}=(0,\ldots,0)$. In general, $\mathcal{P}^h_{\boldsymbol{a}}E$ is not coherent. However, by the deep work of Simpson [Sim88, Theorem 3] and Mochizuki [Moc07], this is the case for systems of Hodge bundles.

Theorem 1.8 (Simpson, Mochizuki). *If $(E=\oplus_{p+q=m}E^{p,q},\theta,h)$ is a system of Hodge bundles on $Y-D$, then $(E,\mathcal{P}^h_{\boldsymbol{a}}E,\theta)$ is a parabolic Higgs bundles on (Y,D).* □

In this case, we write $\mathcal{P}_{\boldsymbol{a}}E$ for $\mathcal{P}^h_{\boldsymbol{a}}E$ to lighten the notation, and denote by

$$\theta:\mathcal{P}_{\boldsymbol{a}}E\to\mathcal{P}_{\boldsymbol{a}}E\otimes\Omega^1_Y(\log D)$$

the prolonged Higgs field by abuse of notation. From Theorem 1.8 one can easily deduce the following.

Lemma 1.9. *Let $(E=\oplus_{p+q=m}E^{p,q},\theta,h)$ be as above. Then*

(i) $\mathcal{P}_{\boldsymbol{a}}E=\oplus_{p+q=m}\mathcal{P}_{\boldsymbol{a}}E^{p,q}$. *Here $\mathcal{P}_{\boldsymbol{a}}E^{p,q}$ is the prolongation of $(E^{p,q},h_{p,q})$.*

(ii)

$$\theta|_{\mathcal{P}_{\boldsymbol{a}}E^{p,q}}:\mathcal{P}_{\boldsymbol{a}}E^{p,q}\to\mathcal{P}_{\boldsymbol{a}}E^{p-1,q+1}\otimes\Omega^1_Y(\log D).$$

□

Remark 1.10. If $(E=\oplus_{p+q=m}E^{p,q},\theta,h)$ is a system of Hodge bundles, $\mathcal{P}_{\boldsymbol{a}}E$ coincides with the Deligne extension with real part of the eigenvalue in $[\boldsymbol{a},\boldsymbol{a}+\mathbf{1})$.

Definition 1.11. Let (Y,D) be a log pair. Let $(E=\oplus_{p+q=m}E^{p,q},\theta,h)$ be a system of Hodge bundle defined over $Y-D$. The prolongation $({}^\diamond E=\oplus_{p+q=m}{}^\diamond E^{p,q},\theta)$ is called *canonical extension* of $(E=\oplus_{p+q=m}E^{p,q},\theta,h)$.

Lemma 1.9 inspires us to introduce the definition of systems of log Hodge bundles.

Definition 1.12 (System of log Hodge bundles). Let (Y,D) be a log pair. A *system of log Hodge bundles* of weight m over (Y,D) consists of $(E=\oplus_{p+q=m}E^{p,q},\theta=\oplus_{p+q=m}\theta_{p,q})$ where

- $E=\oplus_{p+q=m}E^{p,q}$ is a direct sum of holomorphic vector bundles $E^{p,q}$ on Y.

- θ is a direct sum of

$$\theta_{p,q} : E^{p,q} \longrightarrow E^{p-1,q+1} \otimes \Omega^1_Y(\log D)$$

with

$$\theta \wedge \theta = 0.$$

2. Construction of special system of log Hodge bundles

In this section, we first study the refined positivity for the Griffiths line bundle associated to a system of Hodge bundles. This positivity is well-known when the corresponding $\mathbb{C}$-PVHS has unipotent monodromies near the boundary. We then construct a special system of log Hodge bundles (see Theorem 2.6) over the log pair (Y, D) in Theorem 0.4. Such system of Hodge bundles will be used to construct a negatively curved Finsler metric in § 3.

2.1. **Refined positivity for Griffiths line bundle.** For a system of Hodge bundles $(E = \oplus_{p+q=m} E^{p,q}, \theta, h)$ over a complex manifold U, in [Gri70a] Griffiths constructed a line bundle $\mathfrak{L}$ on U, which can be endowed with a natural metric with semi-positive curvature. Precisely, one has

$$\mathfrak{L} := (\det E^{m,0})^{\otimes m} \otimes (\det E^{m-1,1})^{\otimes(m-1)} \otimes \cdots \otimes \det E^{1,m-1}. \tag{2.1.1}$$

Here $\theta^*_{p,q}$ the adjoint of $\theta_{p,q}$ with respect to $h_{p,q}$. The Hodge metric h then induces a metric $h_{\mathfrak{L}}$ on $\mathfrak{L}$ whose curvature is

$$\sqrt{-1}\Theta_{h_{\mathfrak{L}}}(\mathfrak{L}) = -\mathrm{tr}\Big(\sum_{q=0}^{m-1} \theta^*_{m-q,q} \wedge \theta_{m-q,q}\Big) \geq 0. \tag{2.1.2}$$

One can see that $\sqrt{-1}\Theta_{h_{\mathfrak{L}}}(\mathfrak{L}) > 0$ at the point y where $\theta : T_{Y,y} \longrightarrow \mathrm{End}(E_y)$ is injective. Note that θ is the differential of the period map. This means that $\sqrt{-1}\Theta_{h_{\mathfrak{L}}}(\mathfrak{L})$ is strictly positive at the point where the period map is immersive.

Assume now $U = Y - D$, where (Y, D) is a compact Kähler log pair. Let T be the Kähler current on Y defined in Remark 1.7. Then $\omega_U := T|_U$ is a complete Kähler metric with Poincaré type near D. We recall the following theorem by Simpson [Sim88, Lemma 10.1] and Mochizuki [Moc07].

Theorem 2.1 (Simpson, Mochizuki)**.** *Let $(E = \oplus_{p+q=m} E^{p,q}, \theta, h)$ be a system of Hodge bundles on $U = Y - D$. Then*

$$|\theta|_{h,\omega_U} \leq C \tag{2.1.3}$$

for some constant $C > 0$. □

Lemma 2.2. *Same notations as above. Then*

$$\sqrt{-1}\Theta_{h_{\mathfrak{L}}}(\mathfrak{L}) \lesssim \omega_U.$$

Proof. By Theorem 2.1 one has $|\theta_{p,q}|_{h,\omega_U} \leq C$. Then $|\theta^*_{p,q}|_{h,\omega_U} \leq C$. Hence

$$|\theta^*_{p,q} \wedge \theta_{p,q}|_{h,\omega_U} \leq C^2.$$

It follows from (2.1.2) that

$$|\sqrt{-1}\Theta_{h_{\mathfrak{L}}}(\mathfrak{L})| \leq C'$$

for some constant $C' > 0$. The lemma follows directly from the above inequality. □

By Lemma 2.2, the mass of $\sqrt{-1}\Theta_{h_{\mathfrak{L}}}(\mathfrak{L})$ is bounded near D, and one can thus apply the Skoda extension theorem so that the trivial extension of $\sqrt{-1}\Theta_{h_{\mathfrak{L}}}(\mathfrak{L})$ over Y is a positive closed $(1,1)$-current, which we denote by S. S is therefore less singular than the current T defined in Remark 1.7, which we denote by $S \lesssim T$.

Let us consider the prolongation $\mathcal{P}_{\mathbf{1}}\mathfrak{L}$ of $(\mathfrak{L}, h_{\mathfrak{L}})$ defined in (1.4.1), where $\mathbf{1} = (1, \ldots, 1)$. Then $h_{\mathfrak{L}}$ can be seen as the *singular hermitian metric* for $\mathcal{P}_{\mathbf{1}}\mathfrak{L}$.

Lemma 2.3. *The curvature $\sqrt{-1}\Theta_{h_{\mathfrak{L}}}(\mathcal{P}_{\mathbf{1}}\mathfrak{L})$ is a closed positive $(1,1)$-current. In particular, $\mathcal{P}_{\mathbf{1}}\mathfrak{L}$ is a pseudo effective line bundle on Y.*

Proof. Pick any $p \in Y$. We take an admissible coordinate $(W; z_1, \ldots, z_n)$ around p as Definition 1.6.

Since S is a closed positive current on Y, then over W there is a plurisubarmonic function ψ so that $S = \mathrm{dd}^{\mathrm{c}}\psi$. Note that $S \lesssim T$. One thus has $\varphi \lesssim \psi$ where φ is defined in (1.3.2). For the metric

$$\tilde{h}_{\mathfrak{L}} := h_{\mathfrak{L}} \cdot e^{\psi},$$

one has $\Theta_{\tilde{h}_{\mathfrak{L}}}(\mathfrak{L}) = 0$ over W. Let ∇ be the corresponding Chern connection, which is flat. It corresponds to a unitary representation $\mathbb{Z}^{\ell} \simeq \pi_1(W - D) \to U(1)$. Let $T_i \in U(1)$ be the monodromy corresponding to the loop γ_i. Choose a multi-valued flat section σ with respect to ∇ and thus $|\sigma|_{\tilde{h}_{\mathfrak{L}}} \equiv 1$. Choose $e^{2\pi\sqrt{-1}t_i} = z_i$. One has

$$T_i \cdot \sigma(t) = \sigma(t + \mathbf{1}_i).$$

Write $T_i = e^{2\pi\sqrt{-1}b_i}$ for some $0 < b_i \leq 1$. Define

$$\sigma(t) := e(t)e^{-2\pi\sqrt{-1}\sum_{i=1}^{\ell} b_i t_i},$$

which is indeed single valued, and descends to $\sigma(z)$. Note that

$$\nabla(\sigma(z)) = -b_i d\log z_i \cdot \sigma(z).$$

Hence $\sigma(z)$ is a holomorphic section trivializing $\mathfrak{L}|_{W-D}$. Moreover, by

$$|\sigma(t)|_{\tilde{h}_{\mathfrak{L}}} = |e^{-2\pi\sqrt{-1}\sum_{i=1}^{\ell} b_i t_i}|,$$

one has $|\sigma(z)|_{\tilde{h}_{\mathfrak{L}}} = \prod_{i=1}^{\ell} |z_i|^{-b_i}$, and thus

$$|\sigma(z)|_{h_{\mathfrak{L}}} = \prod_{i=1}^{\ell} |z_i|^{-2b_i} \cdot e^{-\psi}. \tag{2.1.4}$$

Since $\varphi \lesssim \psi$, one has

$$1 \lesssim e^{-\psi} \lesssim \Big(\prod_{j=1}^{\ell}(-\log|z_j|^2)\Big)^N$$

for some $N > 0$. Therefore,

$$\prod_{i=1}^{\ell} \frac{1}{|z_i|^{b_i-\varepsilon}} \lesssim |\sigma(z)|_{h_{\mathfrak{L}}} \lesssim \prod_{i=1}^{\ell} \frac{1}{|z_i|^{b_i+\varepsilon}} \tag{2.1.5}$$

for any $\varepsilon > 0$. Since $0 < b_i \leq 1$, one has $\sigma \in \mathcal{P}_{\mathbf{1}}\mathfrak{L}|_W$ by (1.4.1). Let us show that σ is a generator of $\mathcal{P}_{\mathbf{1}}\mathfrak{L}|_W$.

For any section $s \in \mathcal{P}_{\mathbf{1}}\mathfrak{L}(W)$, there is a holomorphic function $f \in \mathcal{O}(W^*)$ so that $s = f \cdot \sigma$, where $W^* := W - D$. By (1.4.1) again,

$$|f| \cdot |\sigma|_{h_{\mathfrak{L}}} = |s|_{h_{\mathfrak{L}}} \lesssim \frac{1}{\prod_{i=1}^{\ell} |z_i|^{1+\varepsilon}}$$

for all $\varepsilon > 0$. By (2.1.5) one has

$$|f| \lesssim \frac{1}{\prod_{i=1}^{\ell} |z_i|^{1-b_i+\varepsilon}}$$

for all $\varepsilon > 0$. Pick $\varepsilon \ll 1$ with $1 - b_i + \varepsilon < 1$ for all i. The above inequality shows that f extends to a holomorphic function over U. Hence σ is a generator of $\mathcal{P}_1\mathfrak{L}|_W$.

By (2.1.4), one has

$$\sqrt{-1}\Theta_{h_{\mathfrak{L}}}(\mathcal{P}_1\mathfrak{L}) = \mathrm{dd^c} \log |\sigma|_{h_{\mathfrak{L}}} = S + \sum_{i=1}^{\ell} b_i [D_i], \tag{2.1.6}$$

where $[D_i]$ is the current of integration associated to D_i. This finishes the proof of the theorem. □

The following lemma is thus a consequence of the above proof.

Lemma 2.4. *For any $N \in \mathbb{Z}_{>0}$, let $\mathcal{P}_1(\mathfrak{L}^{\otimes N})$ be the prolongation of $(\mathfrak{L}^{\otimes N}, h_{\mathfrak{L}}^{\otimes N})$ defined in* (1.4.1)*. Then*

$$\mathcal{P}_1(\mathfrak{L}^{\otimes N}) = (\mathcal{P}_1\mathfrak{L})^{\otimes N} \otimes \mathcal{O}(-\sum_{i=1}^{\ell}(\lceil Nb_i \rceil - 1)D_i).$$

Proof. We use the same notations as those in the above proof. Consider the section σ^N which a generator of $(\mathcal{P}_1\mathfrak{L})^{\otimes N}|_W$. For any section $s \in \mathcal{P}_1(\mathfrak{L}^{\otimes N})(W)$, there is a holomorphic function $f \in \mathcal{O}(W^*)$ so that $s = f \cdot \sigma^N$, where $W^* := W - D$. By (1.4.1) again, one has

$$|f| \cdot |\sigma^N|_{h_{\mathfrak{L}}^{\otimes N}} = |s|_{h_{\mathfrak{L}}^{\otimes N}} \lesssim \frac{1}{\prod_{i=1}^{\ell} |z_i|^{1+\varepsilon}}$$

for all $\varepsilon > 0$. By (2.1.5) one has

$$|f| \lesssim \frac{1}{\prod_{i=1}^{\ell} |z_i|^{1-Nb_i+\varepsilon}}$$

for all $\varepsilon > 0$. This shows that $f \in \mathcal{O}_Y(-\sum_{i=1}^{\ell}(\lceil Nb_i \rceil - 1)D_i)$.

On the other hand, if $g \in \mathcal{O}_Y(-\sum_{i=1}^{\ell}(\lceil Nb_i \rceil - 1)D_i)$, then by (2.1.5), one has

$$|g \cdot \sigma^N| \lesssim \frac{1}{\prod_{i=1}^{\ell} |z_i|^{1-\lceil Nb_i \rceil + Nb_i + \varepsilon}} \lesssim \frac{1}{\prod_{i=1}^{\ell} |z_i|^{1+\varepsilon}}$$

for any $\varepsilon > 0$. This yields the lemma. □

In summary, we have the following positivity result for Griffiths line bundles.

Proposition 2.5. *Let (Y, D) be a compact Kähler log pair. Let $(E = \oplus_{p+q=m} E^{p,q}, \theta, h)$ be a system of Hodge bundles over $Y - D$. Assume that its period map is immersive at one point. Then for $N \gg 0$, ${}^{\diamond}(\mathfrak{L}^{\otimes N}) \otimes \mathcal{O}_Y(-D)$ is a big line bundle on Y for $N \gg 0$. In particular, Y is projective.*

Proof. Recall that the closed positive current S is the trivial extension of the semi-positive $(1,1)$-form $\Theta_{h_{\mathfrak{L}}}(\mathfrak{L})$ over Y. By (2.1.6), one has

$$\{c_1(\mathcal{P}_1\mathfrak{L})\} = \{S\} + \sum_{i=1}^{\ell} b_i \{D_i\}.$$

Lemma 2.4 then yields

$$\{c_1(\mathcal{P}_1(\mathfrak{L}^{\otimes N}))\} = N\{S\} + \sum_{i=1}^{\ell}(Nb_i - \lceil Nb_i \rceil + 1)\{D_i\}.$$

Note that

$$\mathcal{P}_1(\mathfrak{L}^{\otimes N}) = {}^\diamond(\mathfrak{L}^{\otimes N}) \otimes \mathcal{O}_Y(D).$$

Therefore,

$$c_1({}^\diamond(\mathfrak{L}^{\otimes N}) \otimes \mathcal{O}_Y(-D)) = N\{S\} + \sum_{i=1}^{\ell}(-1 + Nb_i - \lceil Nb_i \rceil)\{D_i\}.$$

By the discussion at the beginning of this subsection, the semi-positive $(1,1)$-form $\Theta_{h_{\mathfrak{L}}}(\mathfrak{L})$ is strictly positive at the point where the period map is immersive. By Boucksom's criterion [Bou02], the cohomology class $\{S\}$ is a big $(1,1)$-class. Therefore, $N\{S\} - 2D$ is big for $N \gg 0$. Note that

$$1 + Nb_i - \lceil Nb_i \rceil \geq 0.$$

Since the sum of big class with effective class is still big, we conclude that $c_1({}^\diamond(\mathfrak{L}^{\otimes N}) \otimes \mathcal{O}_Y(-D))$ is big. This proves the lemma. □

2.2. **Special system of Hodge bundles.** Let (Y, D) be a compact Kähler log pair. Let $(F = \oplus_{p+q=m} F^{p,q}, \eta, h_F)$ be a system of Hodge bundle over $U := Y - D$, whose period map is immersive at one point. Let us denote by $r_p := \operatorname{rank} F^{p,q}$. Recall that the Griffiths line bundle for $(F = \oplus_{p+q=m} F^{p,q}, \eta, h_F)$ is

$$\mathfrak{L} := (\det F^{m,0})^{\otimes m} \otimes (\det F^{m-1,1})^{\otimes (m-1)} \otimes \cdots \otimes \det F^{1,m-1}$$

By Proposition 2.5, ${}^\diamond(\mathfrak{L}^{\otimes N}) \otimes \mathcal{O}_Y(-D)$ is a big line bundle for some $N \gg 0$. Let us denote by $r := N\big(mr_m + (m-1)r_{m-1} + \cdots + r_1\big)$.

We define a new system of Hodge bundle $(E = \oplus_{P+Q=rm} E^{P,Q}, \theta, h)$ on $U = Y - D$ by setting $(E, \theta, h) := (F, \eta, h_F)^{\otimes r}$. Precisely, $E := F^{\otimes r}$, and

$$\theta := \eta \otimes \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{(r-1)-\text{tuple}} + \mathbb{1} \otimes \eta \otimes \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{(r-2)-\text{tuple}} + \cdots + \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{(r-1)-\text{tuple}} \otimes \eta.$$

Define

$$E^{P,Q} := \oplus_{p_1+\cdots+p_r=P;\, q_1+\cdots+q_r=Q} F^{p_1,q_1} \otimes \cdots \otimes F^{p_r,q_r}. \tag{2.2.1}$$

Then

$$\theta : E^{P,Q} \to E^{P-1,Q+1} \otimes \Omega^1_U,$$

and one can easily check that $h = h_F^{\otimes r}$ is the Hodge metric for $(E = \oplus_{P+Q=rm} E^{P,Q}, \theta)$.

Note that $\det F^{p,q} = \wedge^{r_p} F^{p,q} \subset (F^{p,q})^{\otimes r_p} \subset F^{\otimes r_p}$. Hence

$$\begin{aligned}\mathfrak{L}^{\otimes N} &= (\det F^{m,0})^{\otimes Nm} \otimes (\det F^{m-1,1})^{\otimes (N(m-1))} \otimes \cdots \otimes (\det F^{1,m-1})^{\otimes N} \\ &\subset (F^{m,0})^{\otimes Nmr_m} \otimes \cdots \otimes (F^{1,m-1})^{\otimes Nr_1} \subset E^{P_0,Q_0},\end{aligned}$$

where $P_0 = N(r_m m^2 + r_{m-1}(m-1)^2 + \cdots + r_1)$ and $Q_0 = rm - P_0$. In other words, $\mathfrak{L}^{\otimes N}$ is a subbundle of E^{P_0,Q_0}. Moreover, their hermitian metrics are compatible in the following sense:

$$h_{\mathfrak{L}}^{\otimes N} = h|_{\mathfrak{L}}.$$

By the very definition of the prologation (1.4.1), one has

$${}^\diamond(\mathfrak{L}^{\otimes N}) \subset {}^\diamond E^{P_0,Q_0}.$$

In summary, we construct a *special system of log Hodge bundles* on (Y, D) as follows (we change the notations for brevity's sake).

Theorem 2.6. *Let (Y, D) be a compact Kähler log pair. Let $(F = \oplus_{p+q=m} F^{p,q}, \eta, h_F)$ be a system of Hodge bundle over $Y - D$, whose period map is immersive at one point. Then there is system of log Hodge bundles $(E = \oplus_{p+q=\ell} E^{p,q}, \theta = \oplus_{p+q=\ell} \theta_{p,q})$ on (Y, D) satisfying the following conditions.*

(i) *The Higgs field θ satisfies*

$$\theta_{p,q}: E^{p,q} \longrightarrow E^{p-1,q+1} \otimes \Omega_Y^1(\log D).$$

(ii) *(E,θ) is the canonical extension (in the sense of Definition 1.11) of some system of Hodge bundles $(\tilde{E},\tilde{\theta},h_{\rm hod})$ defined over $Y-D$.*

(iii) *There is a big line bundle L over Y such that $L \subset E^{p_0,q_0}$ for some $p_0+q_0=\ell$, and $L\otimes \mathcal{O}_Y(-D)$ is still big.* □

Remark 2.7. The interested readers can compare the Higgs bundle in Theorem 2.6 with the Viehweg-Zuo Higgs bundle in [VZ02, VZ03] (see also [PTW19]). Loosely speaking, a *Viehweg-Zuo Higgs bundle* for a log pair (Y,D) is a Higgs bundle $(E=\oplus_{p+q=m}E^{p,q},\theta)$ over $(Y,D+S)$ induced by some (geometric) $\mathbb{Z}$-PVHS defined over a Zariski open set of $Y-(D\cup S)$, where S is another divisor on Y so that $D+S$ is simple normal crossing. The extra data is that there is a sub-Higgs sheaf $(F=\oplus_{p+q=m}F^{p,q},\eta)\subset(E,\theta)$ such that the *first stage* $F^{n,0}$ is a big line bundle, and

$$\eta: F^{p,q}\longrightarrow F^{p-1,q+1}\otimes\Omega_Y^1(\log D).$$

As we explained in § 0.4.1, the positivity $F^{n,0}$ comes in a sophisticated way from the Kawamata's big line bundle $\det f_*(mK_{X/Y})$ where $f: X\longrightarrow Y$ is some algebraic fiber space between projective manifolds. For our Higgs bundle $(E=\oplus_{p+q=m}E^{p,q},\theta)$ over the log pair (Y,D) in Theorem 2.6, the global positivity is the Griffiths line bundle which is contained in some *intermediate stage* E^{p_0,q_0} of $(E=\oplus_{p+q=m}E^{p,q},\theta)$.

2.3. **Iterating Higgs fields.** Let $(E=\oplus_{p+q=\ell}E^{p,q},\theta)$ be the system of log Hodge bundles on a compact Kähler log pair (Y,D) satisfying the three conditions in Theorem 2.6. We apply ideas by Viehweg-Zuo [VZ02, VZ03] to iterate Higgs fields.

Since $\theta: E^{p,q}\longrightarrow E^{p-1,q+1}\otimes\Omega_Y^1(\log D)$, one can iterate θ k-times to obtain

$$E^{p_0,q_0}\longrightarrow E^{p_0-1,q_0+1}\otimes\Omega_Y^1(\log D)\longrightarrow\cdots\longrightarrow E^{p_0-k,q_0+k}\otimes\otimes^k\Omega_Y^1(\log D).$$

Since $\theta\wedge\theta=0$, the above morphism factors through

$$E^{p_0,q_0}\longrightarrow E^{p_0-k,q_0+k}\otimes \mathrm{Sym}^k\Omega_Y^1(\log D). \tag{2.3.1}$$

Since L is a subsheaf of E^{p_0,q_0}, it induces

$$L\longrightarrow E^{p_0-k,q_0+k}\otimes \mathrm{Sym}^k\Omega_Y^1(\log D)$$

which is equivalent to a morphism

$$\tau_k: \mathrm{Sym}^kT_Y(-\log D)\longrightarrow L^{-1}\otimes E^{p_0-k,q_0+k}. \tag{2.3.2}$$

The readers might be worried that all τ_k might be trivial so that the above construction will be meaningless. In the next subsection, we will show that this indeed cannot happen.

2.4. **An infinitesimal Torelli-type theorem.** We begin with the following technical lemma.

Proposition 2.8. *Let $(E=\oplus_{p+q=\ell}E^{p,q},\theta)$ be a system of log Hodge bundles on a compact Kähler log pair (Y,D) satisfying the three conditions in Theorem 2.6. Then there is a singular hermitian metric h_L with analytic singularities for L so that*

(i) *the curvature current*

$$\sqrt{-1}\Theta_{h_L}(L)\geq T$$

where T is the Kähler current on Y defined in Remark 1.7.

(ii) *The singular hermitian metric* $h := h_L^{-1} \otimes h_{\rm hod}$ *on* $L^{-1} \otimes E$ *is locally bounded on* Y*, and smooth outside* $D \cup \mathbf{B}_+(L-D)$*, where* $h_{\rm hod}$ *is the Hodge metric for the system of Hodge bundles* $(E = \oplus_{p+q=\ell} E^{p,q}, \theta)|_U$*. Moreover,* $h \cdot \prod_{i=1}^{\ell} |\sigma_i|_{h_i}^{-\varepsilon}$ *vanishes on* $D \cup \mathbf{B}_+(L-D)$ *for* $\varepsilon > 0$ *small enough. Here* σ_i *be the section* $H^0(Y, \mathcal{O}_Y(D_i))$ *defining* D_i*, and* h_i *is a smooth metric for the line bundle* $\mathcal{O}_Y(D_i)$*.*

Proof. By Theorem 2.6.(iii), the line bundle $L \otimes \mathcal{O}_Y(-D)$ is big and thus we can put a singular hermitian metric g_0 with analytic singularities for $L \otimes \mathcal{O}_Y(-D)$ such that g_0 is smooth on $Y \setminus \mathbf{B}_+(L \otimes \mathcal{O}_Y(-D))$ where $\mathbf{B}_+(L \otimes \mathcal{O}_Y(-D))$ is the augmented base locus of $L \otimes \mathcal{O}_Y(-D)$, and the curvature current $\sqrt{-1}\Theta_{g_0}(L-D) \geqslant \omega$ for some smooth Kähler form ω on Y. Take $g := g_0(-\prod_{i=1}^{\ell} \log|\varepsilon \cdot \sigma_i|^2_{\cdot h_i})$. Then

$$\sqrt{-1}\Theta_g(L-D) \geqslant T := \omega - \mathrm{dd}^{\mathrm{c}} \log(-\prod_{i=1}^{\ell} \log|\varepsilon \cdot \sigma_i|^2_{h_i}).$$

Note that T is a Kähler current when $0 < \varepsilon \ll 1$.

Let h_D be the canonical singular hermitian metric for D so that $\sqrt{-1}\Theta_{h_D}(\mathcal{O}_Y(D)) = [D]$. We define a singular hermitian metric on L as follows:

$$h_L := g \cdot h_D.$$

Then

$$\sqrt{-1}\Theta_{h_L}(L) = \sqrt{-1}\Theta_g(L \otimes \mathcal{O}_X(-D)) + [D] \geq T.$$

The first condition is verified.

Note that g^{-1} vanishes on $\mathbf{B}_+(L \otimes \mathcal{O}_Y(-D))$, and h_D^{-1} vanishes on D. Since $h_{\rm hod}$ is smooth over $Y - D$, $\mathbf{B}_+(L) \subset \mathbf{B}_+(L \otimes \mathcal{O}_Y(-D))$, $h := h_{\rm hod} \cdot h_L^{-1}$ thus vanishes on $\mathbf{B}_+(L) - D$. For any point $y \in D$, we pick an admissible coordinate $(W; z_1, \ldots, z_n)$ and a frame $(e_1, \ldots, e_r)$ for $E|_W$. Since (E, θ) is the canonical extension of a system of Hodge bundles $(\tilde{E}, \tilde{\theta}, h_{\rm hod})$, by (1.4.1) one has

$$|e_i|_h \lesssim \frac{1}{\prod_{i=1}^{\ell} |z_i|^{\varepsilon}}$$

for all $\varepsilon > 0$. Pick a section $e \in L(W)$ which trivialize $L|_W$. By the definition of h_L, one has

$$|e|^2_{h_L} \gtrsim \frac{1}{\prod_{i=1}^{\ell} |z_i|}$$

Hence for the frame $(e_1 \otimes e^{-1}, \ldots, e_r \otimes e^{-1})$ trivializing $E \otimes L^{-1}|_W$, one has

$$|e_i \otimes e^{-1}|_h \lesssim \prod_{i=1}^{\ell} |z_i|^{1-\varepsilon}$$

for any $\varepsilon > 0$. This shows that $h \cdot \prod_{i=1}^{\ell} |\sigma_i|_{h_i}^{-\varepsilon}$ vanishes on D when $\varepsilon > 0$ is small enough. The proposition is proved. □

Theorem 2.9 (Infinitesimal Torelli-type property)**.** *The morphism* $\tau_1 : T_Y(-\log D) \to L^{-1} \otimes E^{p_0-1,q_0+1}$ *defined in* (2.3.2) *is always generically injective.*

The proof is almost the same at that of [Den18, Theorem D]. We provide it here for completeness sake.

Proof of Theorem 2.9. By Theorem 2.6.(iii), the inclusion $L \subset E^{p_0,q_0}$ induces a global section $s \in H^0(Y, L^{-1} \otimes E^{p_0,q_0})$, which is *generically* non-vanishing over $U = Y - D$. Set

$$U_1 := \{y \in Y - (D \cup \mathbf{B}_+(L-D)) \mid s(y) \neq 0\} \tag{2.4.1}$$

which is a non-empty Zariski open set of U. Since the Hodge metric $h_{\rm hod}$ is a direct sum of metrics h_p on $E^{p,q}$, the metric h for $L^{-1} \otimes E$ is a direct sum of metrics $h_L^{-1} \cdot h_p$ on

$L^{-1}\otimes E^{p,q}$, which is smooth over $U_0:=Y-(D\cup \mathbf{B}_+(L-D))$. Let us denote D' to be the $(1,0)$-part of its Chern connection over U_1, and Θ to be its curvature form. Then over U_0 we have

$$\begin{aligned}\Theta &= -\Theta_{L,h_L}\otimes \mathbb{1}+\mathbb{1}\otimes\Theta_{h_{p_0}}(E^{p_0,q_0})\\ &= -\Theta_{L,h_L}\otimes \mathbb{1}-\mathbb{1}\otimes(\theta^*_{p_0,q_0}\wedge\theta_{p_0,q_0})-\mathbb{1}\otimes(\theta_{p_0+1,q_0-1}\wedge\theta^*_{p_0+1,q_0-1})\\ &= -\Theta_{L,h_L}\otimes \mathbb{1}-\tilde\theta^*_{p_0,q_0}\wedge\tilde\theta_{p_0,q_0}-\tilde\theta_{p_0+1,q_0-1}\wedge\tilde\theta^*_{p_0+1,q_0-1}\end{aligned}\tag{2.4.2}$$

where we set

$$\theta_{p,q}=\theta|_{E^{p,q}}:E^{p,q}\to E^{p-1,q+1}\otimes\Omega^1_Y(\log D)$$

and

$$\tilde\theta_{p,q}=\mathbb{1}\otimes\theta_{p,q}:L^{-1}\otimes E^{p,q}\to L^{-1}\otimes E^{p-1,q+1}\otimes\Omega^1_Y(\log D)$$

and define $\tilde\theta^*_{p,q}$ to be the adjoint of $\tilde\theta_{p,q}$ with respect to the metric $h_L^{-1}\cdot h$. Hence over U_1 one has

$$\begin{aligned}-dd^c\log|s|_h^2 &= \frac{\{\sqrt{-1}\Theta(s),s\}_h}{|s|_h^2}+\frac{\sqrt{-1}\{D's,s\}_h\wedge\{s,D's\}_h}{|s|_h^4}-\frac{\sqrt{-1}\{D's,D's\}_h}{|s|_h^2}\\ &\leqslant \frac{\{\sqrt{-1}\Theta(s),s\}_h}{|s|_h^2}\end{aligned}\tag{2.4.3}$$

thanks to Cauchy-Schwarz inequality

$$\sqrt{-1}|s|_h^2\cdot\{D's,D's\}_h\geqslant\sqrt{-1}\{D's,s\}_h\wedge\{s,D's\}_h.$$

Putting (2.4.2) to (2.4.3), over U_1 one has

$$\begin{aligned}\sqrt{-1}\Theta_{L,h_L}-dd^c\log|s|_h^2 &\leqslant -\frac{\{\sqrt{-1}\tilde\theta^*_{p_0,q_0}\wedge\tilde\theta_{p_0,q_0}(s),s\}_h}{|s|_h^2}\\ &\quad -\frac{\{\sqrt{-1}\tilde\theta_{p_0+1,q_0-1}\wedge\tilde\theta^*_{p_0+1,q_0-1}(s),s\}_h}{|s|_h^2}\\ &= \frac{\sqrt{-1}\{\tilde\theta_{p_0,q_0}(s),\tilde\theta_{p_0,q_0}(s)\}_h}{|s|_h^2}+\frac{\{\tilde\theta^*_{p_0+1,q_0-1}(s),\tilde\theta^*_{p_0+1,q_0-1}(s)\}_h}{|s|_h^2}\\ &\leq \frac{\sqrt{-1}\{\tilde\theta_{p_0,q_0}(s),\tilde\theta_{p_0,q_0}(s)\}_h}{|s|_h^2}\end{aligned}\tag{2.4.4}$$

where $\tilde\theta_{p_0,q_0}(s)\in H^0\big(Y,L^{-1}\otimes E^{p_0-1,q_0+1}\otimes\Omega^1_Y(\log D)\big)$. By Proposition 2.8.(ii), one has $|s|_h^2(y)=0$ for any $y\in D\cup\mathbf{B}_+(L-D)$. Therefore, there exists $y_0\in U_0$ so that $|s|_h^2(y_0)\geqslant|s|_h^2(y)$ for any $y\in U_0$. Hence $|s|_h^2(y_0)>0$, and by (2.4.1), $y_0\in U_1$. Since $|s|_h^2$ is smooth over U_0, $dd^c\log|s|_h^2$ is semi-negative at y_0 by the maximal principle. By Proposition 2.8.(i), $\sqrt{-1}\Theta_{L,h_L}$ is strictly positive at y_0. By (2.4.4) and $|s|_h^2(y_0)>0$, we conclude that $\sqrt{-1}\{\tilde\theta_{p_0,q_0}(s),\tilde\theta_{p_0,q_0}(s)\}_h$ is strictly positive at y_0. In particular, for any non-zero $\xi\in T_{Y,y_0}$, $\tilde\theta_{p_0,q_0}(s)(\xi)\neq 0$. For $k=1$, we write τ_k in (2.3.2) as

$$\tau_1:T_Y(-\log D)\to L^{-1}\otimes E^{p_0-1,q_0+1}.$$

Then over U it is defined by $\tau_1(\xi):=\tilde\theta_{p_0,q_0}(s)(\xi)$, which is thus *injective at* $y_0\in U_1$. Hence τ_1 is *generically injective*. The theorem is thus proved. □

3. Construction of negatively curved Finsler metric

The aim of this technical section is to prove Theorem 0.4 based on Theorem 2.6. We first introduce the definition of Finsler metric.

Definition 3.1 (Finsler metric)**.** Let E be a holomorphic vector bundle on a complex manifold X. A *Finsler metric* on E is a real non-negative *continuous* function $h : E \to [0, +\infty[$ such that

$$h(av) = |a|h(v)$$

for any $a \in \mathbb{C}$ and $v \in E$. The metric h is *positively definite* at a subset $U \subset X$ if $h(v) > 0$ for any nonzero $v \in E_x$ and any $x \in U$.

We mention that our definition is a bit different from that in [Kob98, Chapter 2, §3], which requires *convexity*, and the Finsler metric therein can be upper-semi continuous.

Let $(E = \oplus_{p+q=\ell} E^{p,q}, \theta)$ be a system of log Hodge bundles on a compact Kähler log pair (Y, D) satisfying the three conditions in Theorem 2.6. We adopt the same notations as those in Theorem 2.6 and § 2.4 throughout this section. Let us denote by n the largest non-negative number for k so that τ_k in (2.3.2) is not trivial. By Theorem 2.9, $n > 0$. Following [Den18, §2.3] we construct Finsler metrics $F_1, \ldots, F_n$ on $T_Y(-\log D)$ as follows. By (2.3.2), for each $k = 1, \ldots, n$, there exists

$$\tau_k : \mathrm{Sym}^k T_Y(-\log D) \to L^{-1} \otimes E^{p_0-k, q_0+k}.$$

Then it follows from Proposition 2.8.(ii) that the (Finsler) metric h on $L^{-1} \otimes E^{p_0-k,q_0+k}$ induces a Finsler metric F_k on $T_Y(-\log D)$ defined as follows: for any $e \in T_Y(-\log D)_y$,

$$F_k(e) := h\big(\tau_k(e^{\otimes k})\big)^{\frac{1}{k}}. \tag{3.0.1}$$

Let $C \subset \mathbb{C}$ be any open set of $\mathbb{C}$. For any holomorphic map $\gamma : C \to U := Y - D$, one has

$$d\gamma : T_C \to \gamma^* T_U \hookrightarrow \gamma^* T_Y(-\log D). \tag{3.0.2}$$

We denote by $\partial_t := \frac{\partial}{\partial t}$ the canonical vector fields in $C \subset \mathbb{C}$, $\bar{\partial}_t := \frac{\partial}{\partial \bar{t}}$ its conjugate. The Finsler metric F_k induces a continuous Hermitian pseudo-metric on C, defined by

$$\gamma^* F_k^2 = \sqrt{-1} G_k(t) dt \wedge d\bar{t}. \tag{3.0.3}$$

Hence $G_k(t) = |\tau_k\big(d\gamma(\partial_t)^{\otimes k}\big)|_h^{\frac{2}{k}}$, where τ_k is defined in (2.3.2).

By Theorem 2.9, there is a Zariski open set U° of U such that $U^\circ \cap \mathbf{B}_+(L) = \varnothing$, and τ_1 is injective at any point of U°. We now fix any holomorphic map $\gamma : C \to U$ with $\gamma(C) \cap U^\circ \neq \varnothing$. By Proposition 2.8.(ii), the metric h for $L^{-1} \otimes E$ is smooth and positively definite over $U - \mathbf{B}_+(L)$. Hence $G_1(t) \not\equiv 0$. Let C° be an (non-empty) open set of C whose complement $C \setminus C^\circ$ is a *discrete set* so that

- The image $\gamma(C^\circ) \subset U^\circ$.
- For every $k = 1, \ldots, n$, either $G_k(t) \equiv 0$ on C° or $G_k(t) > 0$ for every $t \in C^\circ$.
- $\gamma'(t) \neq 0$ for any $t \in C^\circ$, namely $\gamma|_{C^\circ} : C^\circ \to U^0$ is immersive everywhere.

By the definition of $G_k(t)$, if $G_k(t) \equiv 0$ for some $k > 1$, then $\tau_k(\partial_t^{\otimes k}) \equiv 0$ where τ_k is defined in (2.3.2). Note that one has $\tau_{k+1}(\partial_t^{\otimes(k+1)}) = \tilde{\theta}\big(\tau_k(\partial_t^{\otimes k})\big)(\partial_t)$, where

$$\tilde{\theta} = \mathbb{1}_{L^{-1}} \otimes \theta : L^{-1} \otimes E \to L^{-1} \otimes E \otimes \Omega_Y^1(\log D)$$

We thus conclude that $G_{k+1}(t) \equiv 0$. Hence it exists $1 \leq m \leq n$ so that the set $\{k \mid G_k(t) > 0 \text{ over } C^\circ\} = \{1, \ldots, m\}$, and $G_\ell(t) \equiv 0$ for all $\ell = m+1, \ldots, n$. From now on, *all the computations* are made over C° if not specified.

Using the same computations as those in the proof of [Den18, Proposition 2.10], we have the following curvature formula.

Theorem 3.2. *For $k = 1, \ldots, m$, over C° one has*

$$\frac{\partial^2 \log G_1}{\partial t \partial \bar{t}} \geq \Theta_{L,h_L}(\partial_t, \bar{\partial}_t) - \frac{G_2^2}{G_1} \qquad \text{if } k = 1, \tag{3.0.4}$$

$$\frac{\partial^2 \log G_k}{\partial t \partial \bar{t}} \geq \frac{1}{k}\Big(\Theta_{L,h_L}(\partial_t, \bar{\partial}_t) + \frac{G_k^k}{G_{k-1}^{k-1}} - \frac{G_{k+1}^{k+1}}{G_k^k}\Big) \qquad \text{if } k > 1. \tag{3.0.5}$$

Here we make the convention that $G_{m+1} \equiv 0$ and $\frac{0}{0} = 0$. We also write ∂_t (resp. $\bar{\partial}_t$) for $d\gamma(\partial_t)$ (resp. $d\gamma(\bar{\partial}_t)$) abusively, where $d\gamma$ is defined in (3.0.2). □

Let us mention that in [Den18, eq. (2.2.11)] we dropped the term $\Theta_{L,h_L}(\partial_t, \bar{\partial}_t)$ in (3.0.5), though it can be easily seen from the proof of [Den18, Lemma 2.7].

We will follow ideas in [Den18, §2.3] (inspired by [TY15, BPW17, Sch17]) to introduce a new Finsler metric F on $T_Y(-\log D)$ by taking convex sum in the following form

$$F := \sqrt{\sum_{k=1}^{n} k\alpha_k F_k^2}. \tag{3.0.6}$$

where $\alpha_1, \ldots, \alpha_n \in \mathbb{R}^+$ are some constants which will be fixed later.

For the above $\gamma : C \to U$ with $\gamma(C) \cap U^\circ \neq \varnothing$, we write

$$\gamma^* F^2 = \sqrt{-1} H(t) dt \wedge d\bar{t}.$$

Then

$$H(t) = \sum_{k=1}^{n} k\alpha_k G_k(t), \tag{3.0.7}$$

where G_k is defined in (3.0.3). Recall that for $k = 1, \ldots, m$, $G_k(t) > 0$ for any $t \in C^\circ$.

We first recall a computational lemma by Schumacher.

Lemma 3.3 ([Sch17, Lemma 17]). *Let $\alpha_j > 0$ and G_j be positive real numbers for $j = 1, \ldots, n$. Then*

$$\begin{aligned} &\sum_{j=2}^{n}\Big(\alpha_j \frac{G_j^{j+1}}{G_{j-1}^{j-1}} - \alpha_{j-1}\frac{G_j^j}{G_{j-1}^{j-2}}\Big) \\ &\geqslant \frac{1}{2}\Bigg(-\frac{\alpha_1^3}{\alpha_2^2}G_1^2 + \frac{\alpha_{n-1}^{n-1}}{\alpha_n^{n-2}}G_n^2 + \sum_{j=2}^{n-1}\Big(\frac{\alpha_{j-1}^{j-1}}{\alpha_j^{j-2}} - \frac{\alpha_j^{j+2}}{\alpha_{j+1}^{j+1}}\Big)G_j^2\Bigg) \end{aligned} \tag{3.0.8}$$

□

Now we are ready to compute the curvature of the Finsler metric F based on Theorem 3.2.

Theorem 3.4. *Fix a smooth Kähler metric ω on Y. There exist* universal *constants $0 < \alpha_1 < \ldots < \alpha_n$ and $\delta > 0$, such that for any holomorphic map $\gamma : C \to U = Y - D$ with C an open set of $\mathbb{C}$ and $\gamma(C) \cap U^\circ \neq \varnothing$, one has*

$$\mathrm{dd^c} \log |\gamma'(t)|_F^2 \geq \delta \gamma^* \omega. \tag{3.0.9}$$

Proof. By Theorem 2.9 and the assumption that $\gamma(C) \cap U^\circ \neq \varnothing$, $G_1(t) \not\equiv 0$. We first recall a result in [Den18, Lemma 2.9], and we write its proof here for it is crucial in what follows.

Claim 3.5. *There is a universal constant $c_0 > 0$ (*i.e.* it does not depend on γ) so that $\Theta_{L,h_L}(\partial_t, \bar{\partial}_t) \geq c_0 G_1(t)$ for all $t \in C$.*

Proof of Claim 3.5. Indeed, by Proposition 2.8.(i), it suffices to prove that

$$\frac{|\partial_t|^2_{\gamma^*(T)}}{|\tau_1(d\gamma(\partial_t))|_h^2} \geqslant c_0 \tag{3.0.10}$$

for some $c_0 > 0$, where T is a Kähler current on Y, which is a smooth complete metric over $Y - D$ of Poincaré type. It can be seen as a singular hermitian metric for $T_Y(-\log D)$. Hence for any admissible coordinate $(\mathcal{U}; z_1, \ldots, z_n)$, one has

$$|z_i \frac{\partial}{\partial z_i}|_T \sim (-\log |z_i|)^{-1}.$$

On the other hand, by Proposition 2.8.(ii) one has

$$|\tau_1(z_i \frac{\partial}{\partial z_i})|_h \lesssim C \cdot \prod_{i=1}^{\ell} |z_i|^{\varepsilon}$$

for some constant $\varepsilon > 0$. Hence one has

$$\tau_1^* h \lesssim T.$$

Since Y is compact, there exists a constant $c_0 > 0$ such that

$$T \geqslant c_0 \tau_1^* h.$$

Therefore,

$$\frac{|\partial_t|^2_{\gamma^* T}}{|\tau_1(d\gamma(\partial_t))|_h^2} = \frac{|\partial_t|^2_{\gamma^* T}}{|\partial_t|^2_{\gamma^* \tau_1^* h}} \geq c_0.$$

Hence (3.0.10) holds for any $\gamma : C \to U$ with $\gamma(C) \cap U^\circ \neq \varnothing$. The claim is proved. □

By [Sch12, Lemma 8],

$$\sqrt{-1}\partial\bar{\partial} \log(\sum_{j=1}^{n} j\alpha_j G_j) \geqslant \frac{\sum_{j=1}^{n} j\alpha_j G_j \sqrt{-1}\partial\bar{\partial} \log G_j}{\sum_{i=1}^{n} j\alpha_j G_i}. \tag{3.0.11}$$

Putting (3.0.4) and (3.0.5) to (3.0.11), and making the convention that $\frac{0}{0} = 0$, we obtain

$$\begin{aligned}
\frac{\partial^2 \log H(t)}{\partial t \partial \bar{t}} &\geq \frac{1}{H}\Big(-\alpha_1 G_2^2 + \sum_{k=2}^{n} \alpha_k \big(\frac{G_k^{k+1}}{G_{k-1}^{k-1}} - \frac{G_{k+1}^{k+1}}{G_k^{k-1}} \big) \Big) + \frac{\sum_{k=1}^{n} \alpha_k G_k}{H} \Theta_{L,h_L}(\partial_t, \bar{\partial}_t) \\
&= \frac{1}{H}\Big(\sum_{j=2}^{n} \Big(\alpha_j \frac{G_j^{j+1}}{G_{j-1}^{j-1}} - \alpha_{j-1} \frac{G_j^{j}}{G_{j-1}^{j-2}} \Big) \Big) + \frac{\sum_{k=1}^{n} \alpha_k G_k}{H} \Theta_{L,h_L}(\partial_t, \bar{\partial}_t) \\
&\overset{(3.0.8)}{\geq} \frac{1}{H}\Big(-\frac{1}{2}\frac{\alpha_1^3}{\alpha_2^2} G_1^2 + \frac{1}{2}\sum_{j=2}^{n-1} \Big(\frac{\alpha_{j-1}^{j-1}}{\alpha_j^{j-2}} - \frac{\alpha_j^{j+2}}{\alpha_{j+1}^{j+1}} \Big) G_j^2 + \frac{1}{2}\frac{\alpha_{n-1}^{n-1}}{\alpha_n^{n-2}} G_n^2 \Big) \\
&\quad + \frac{\sum_{k=1}^{n} \alpha_k G_k}{H} \Theta_{L,h_L}(\partial_t, \bar{\partial}_t) \\
&\overset{\text{Claim } 3.5}{\geq} \frac{1}{H}\Big(\frac{\alpha_1}{2}(c_0 - \frac{\alpha_1^2}{\alpha_2^2}) G_1^2 + \frac{1}{2}\sum_{j=2}^{n-1} \Big(\frac{\alpha_{j-1}^{j-1}}{\alpha_j^{j-2}} - \frac{\alpha_j^{j+2}}{\alpha_{j+1}^{j+1}} \Big) G_j^2 + \frac{1}{2}\frac{\alpha_{n-1}^{n-1}}{\alpha_n^{n-2}} G_n^2 \Big) \\
&\quad + \frac{1}{H}(\frac{1}{2}\alpha_1 G_1 + \sum_{k=2}^{n} \alpha_k G_k) \Theta_{L,h_L}(\partial_t, \bar{\partial}_t).
\end{aligned}$$

One can take $\alpha_1 = 1$, and choose the further $\alpha_j > \alpha_{j-1}$ inductively so that

$$c_0 - \frac{\alpha_1^2}{\alpha_2^2} > 0, \quad \frac{\alpha_{j-1}^{j-1}}{\alpha_j^{j-2}} - \frac{\alpha_j^{j+2}}{\alpha_{j+1}^{j+1}} > 0 \quad \forall\, j = 2, \ldots, n-1. \tag{3.0.12}$$

Hence

$$\frac{\partial^2 \log H(t)}{\partial t \partial \bar{t}} \geq \frac{1}{H}\Big(\frac{1}{2}\alpha_1 G_1 + \sum_{k=2}^{n} \alpha_k G_k\Big)\Theta_{L,h_L}(\partial_t, \bar{\partial}_t) \overset{(3.0.7)}{\geq} \frac{1}{n}\Theta_{L,h_L}(\partial_t, \bar{\partial}_t)$$

over C°. By Proposition 2.8.(i), this implies that

$$\mathrm{dd^c} \log |\gamma'|_F^2 = \mathrm{dd^c} \log H(t) \geq \frac{1}{n}\gamma^* \sqrt{-1}\Theta_{L,h_L} \geq \delta \gamma^* \omega \tag{3.0.13}$$

over C° for some positive constant δ, which does not depend on γ. Since $|\gamma'(t)|_F^2$ is continuous and locally bounded from above over C, by the extension theorem of subharmonic function, (3.0.13) holds over the whole C. Since $c_0 > 0$ is a constant which does not depend on γ, so are $\alpha_1, \ldots, \alpha_n$ by (3.0.12). The theorem is thus proved. □

In summary of results in this subsection, we obtain the following theorem.

Theorem 3.6. *Let $(E = \oplus_{p+q=\ell} E^{p,q}, \theta)$ be a system of log Hodge bundles on a compact Kähler log pair (Y, D) satisfying the three conditions in Theorem 2.6. Then there are a Finsler metric h on $T_Y(-\log D)$ which is* positively definite *on a dense Zariski open set U° of $U := Y - D$, and a smooth Kähler form ω on Y such that for any holomorphic map $\gamma : C \to U$ from any open subset C of $\mathbb{C}$ with $\gamma(C) \cap U^\circ \neq \varnothing$, one has*

$$\mathrm{dd^c} \log |\gamma'|_h^2 \geq \gamma^* \omega. \tag{3.0.14}$$

□

Proof of Theorem 0.4. Theorem 2.6 together with Theorem 3.6 imply Theorem 0.4. □

4. Big Picard theorem and Algebraic hyperbolicity

4.1. **Algebraic and Picard hyperbolicity.** In Definition 0.3 we have seen the definition of *algebraic hyperbolicity* for a compact complex manifold X, which was introduced by Demailly in [Dem97, Definition 2.2]. He proved in [Dem97, Theorem 2.1] that X is algebraically hyperbolic if it is Kobayashi hyperbolic. The notion of algebraic hyperbolicity was generalized to log pairs by Chen [Che04].

Definition 4.1 (Algebraic hyperbolicity)**.** Let (X, D) be a compact Kähler log pair. For any reduced irreducible curve $C \subset X$ such that $C \not\subset X$, we denote by $i_X(C, D)$ the number of distinct points in the set $\nu^{-1}(D)$, where $\nu : \tilde{C} \to C$ is the normalization of C. The log pair (X, D) is *algebraically hyperbolic* if there is a smooth Kähler metric ω on X such that

$$2g(\tilde{C}) - 2 + i(C, D) \geq \deg_\omega C := \int_C \omega$$

for all curves $C \subset X$ as above.

Note that $2g(\tilde{C}) - 2 + i(C, D)$ depends only on the complement $X - D$. Hence the above notion of hyperbolicity also makes sense for quasi-projective manifolds: we say that a quasi-projective manifold U is algebraically hyperbolic if it has a log compactification (X, D) which is algebraically hyperbolic.

However, unlike Demailly's theorem, it is unclear to us that Kobayashi hyperbolicity or Picard hyperbolicity of $X - D$ will imply the algebraic hyperbolicity of (X, D). In [PR07] Pacienza-Rousseau proved that if $X - D$ is hyperbolically embedded into X, the log pair (X, D) (and thus $X - D$) is algebraically hyperbolic.

Before we prove that Definition 0.1 does not depend on the compactification of U, we will need the following proposition which is a consequence of the deep extension theorem of meromorphic maps by Siu [Siu75]. The *meromorphic map* in this paper is defined in the sense of Remmert, and we refer the reader to [FG02, p. 243] for the precise definition.

Proposition 4.2. *Let Y° be a Zariski open set of a compact Kähler manifold Y. Assume that Y° is Picard hyperbolic. Then any holomorphic map $f:\Delta^p\times(\Delta^*)^q\to Y^\circ$ extends to a meromorphic map $\overline{f}:\Delta^{p+q}\dashrightarrow Y$. In particular, any holomorphic map g from a Zariski open set X° of a compact complex manifold X to Y° extends to a meromorphic map from X to Y.*

Proof. By [Siu75, Theorem 1], any meromorphic map from a Zariski open set Z° of a complex manifold Z to a compact Kähler manifold Y extends to a meromorphic map from Z to Y provided that the codimension of $Z-Z^\circ$ is at least 2. It then suffices to prove the extension theorem for any holomorphic map $f:\Delta^r\times\Delta^*\to Y^\circ$. By the assumption that Y° is Picard hyperbolic, for any $z\in\Delta^r$, the holomorphic map $f|_{\{z\}\times\Delta^*}:\{z\}\times\Delta^*\to Y^\circ$ can be extended to a holomorphic map from $\{z\}\times\Delta$ to Y. It then follows from [Siu75, p.442, (∗)] that f extends to a meromorphic map $\overline{f}:\Delta^{r+1}\dashrightarrow Y$. This proves the first part of the proposition. To prove the second part, we first apply the Hironaka theorem on resolution of singularities to assume that $X-X^\circ$ is a simple normal crossing divisor on X. Then for any point $x\in X-X^\circ$ it has an open neighborhood Ω_x which is isomorphic to Δ^{p+q} so that $X^\circ\simeq\Delta^p\times(\Delta^*)^q$ under this isomorphism. The above arguments show that $g|_{\Omega_x\cap X^\circ}$ extends to a meromorphic map from Ω_x to Y, and thus g can be extended to a meromorphic map from X to Y. The proposition is proved. □

Let us prove that Definition 0.1 does not depend on the compactification of U. It also proves the following result.

Lemma 4.3. *Let U be a Zariski open set of a compact Kähler manifold Y. If any holomorphic map $f:\Delta^*\to U$ extends to $\bar{f}:\Delta\to Y$, then Y is bimeromorphic to any other compact Kähler manifold Y' which contains U as a Zariski open set. In particular, $f:\Delta^*\to U$ also extends to a holomorphic map $\Delta\to Y'$.*

Proof. By blowing-up $Y-U$ and $Y'-U$, we can assume that both $Y-U$ and $Y'-U$ are simple normal crossing divisors. By the same arguments as those in the proof of Proposition 4.2, the identity map of U extends to meromorphic maps $a:Y\dashrightarrow Y'$ and $b:Y'\dashrightarrow Y$. Note that $a\circ b$ and $b\circ a$ are identity maps. Hence Y and Y' are bimeromorphic. Composing b with $\bar{f}$, one obtains the desired extension $\Delta\to Y'$ of $f:\Delta^*\to U$ in Y'. □

By the Chow theorem, Proposition 4.2 in particular gives an alternative proof of the fact that Picard hyperbolic variety is moreover Borel hyperbolic, proven in [JK18, Corollary 3.11].

4.2. **Proofs of Theorem A.** This subsection is devoted to the proof of Theorem A. We first recall the follwing criteria for Picard hyperbolicity established in [DLSZ19], whose proof is Nevanlinna theoretic.

Theorem 4.4 ([DLSZ19, Theorem A])**.** *Let Y be a projective manifold and let D be a simple normal crossing divisor on Y. Let $f:\Delta^*\to Y-D$ be a holomorphic map. Assume that there is a (possibly degenerate) Finsler metric h of $T_Y(-\log D)$ such that $|f'(t)|_h^2\not\equiv 0$, and*

$$\mathrm{dd^c}\log|f'(t)|_h^2\geq f^*\omega \tag{4.2.1}$$

for some smooth Kähler metric ω on Y. Then f extends to a holomorphic map $\overline{f}:\Delta\to Y$. □

We will combine Theorem 4.4 with Theorem 0.4 to prove Theorem A.

Proof of Theorem A. By Theorem 0.4, there exist finite compact Kähler log pairs $\{(X_i, D_i)\}_{i=0,\ldots,N}$ so that

(1) There are morphisms $\mu_i : X_i \to Y$ with $\mu_i^{-1}(D) = D_i$, so that each $\mu_i : X_i \to \mu_i(X_i)$ is a birational morphism, and $X_0 = Y$ with $\mu_0 = \mathbb{1}$.
(2) There are smooth Finsler metrics h_i for $T_{X_i}(-\log D_i)$ which is positively definite over a Zariski open set U_i° of $U_i := X_i - D_i$.
(3) $\mu_i|_{U_i^\circ} : U_i^\circ \to \mu_i(U_i^\circ)$ is an isomorphism.
(4) There are smooth Kähler metrics ω_i on X_i such that for any curve $\gamma : C \to U_i$ with C an open set of $\mathbb{C}$ and $\gamma(C) \cap U_i^\circ \neq 0$, one has

$$\mathrm{dd^c} \log |\gamma'|^2_{h_i} \geq \gamma^*\omega_i. \tag{4.2.2}$$

(5) For any $i \in \{0, \ldots, N\}$, either $\mu_i(U_i) - \mu_i(U_i^\circ)$ is zero dimensional, or there exists $I \subset \{0, \ldots, N\}$ so that

$$\mu_i(U_i) - \mu_i(U_i^\circ) \subset \cup_{j\in I}\mu_j(X_j)$$

Let us explain how to construct these log pairs. By the assumption, there is a $\mathbb{C}$-PVHS on $Y - D$ so that each fiber of the period map is zero dimensional. In particular, the period map is generically immersive. We then apply Theorem 0.4 to construct a Finsler metric on $T_Y(-\log D)$ which is positively definite over some Zariski open set U° of $U = Y - D$ with the desired curvature property (3.0.14). Set $X_0 = Y$, $\mu_0 = \mathbb{1}$ and $U_0^\circ = U^\circ$. Let $Z_1, \ldots, Z_m$ be all irreducible varieties of $Y - U^\circ$ which are not components of D. Then $Z_1 \cup \ldots \cup Z_m \supset U \setminus U^\circ$. For each i, we take a desingularization $\mu_i : X_i \to Z_i$ so that $D_i := \mu_i^{-1}(D)$ is a simple normal crossing divisor in X_i. We pull back the $\mathbb{C}$-PVHS to $U_i = X_i - D_i$ via μ_i. Then its period map is still generically immersive. We then apply Theorem 0.4 to construct the desired Finsler metrics in Item 4 for $T_{X_i}(-\log D_i)$. We iterate this construction, and since at each step the dimension of X_i is strictly decreasing, this algorithm stops after finite steps.

(i) We will first prove that U is Picard hyperbolic. Fix any holomorphic map $f : \Delta^* \to U$. If $f(\Delta^*) \cap U_0^\circ \neq \varnothing$, then by Theorem 4.4 and Item 4, we conclude that f extends to a holomorphic map $\overline{f} : \Delta \to X_0 = Y$.

Assume now $f(\Delta^*) \cap \mu_0(U_0^\circ) = \varnothing$. By Item 5, there exists $I_0 \subset \{0, \ldots, N\}$ so that

$$f(\Delta^*) \subset \mu_0(U_0) - \mu_0(U_0^\circ) \subset \cup_{j\in I_0}\mu_j(X_j)$$

Since $\mu_j(X_j)$ are all irreducible, there exists $k \in I_0$ so that $f(\Delta^*) \subset \mu_k(X_k)$. Note that $U_k := \mu_k^{-1}(U)$. Hence $f(\Delta^*) \subset \mu_k(U_k)$. If $f(\Delta^*) \cap \mu_k(U_k^\circ) \neq \varnothing$, then by Item 3 $f(\Delta^*)$ is not contained in the exceptional set of μ_k. Hence f can be lift to $f_k : \Delta^* \to U_k$ so that $\mu_k \circ f_k = f$ and $f_k(\Delta^*) \cap U_k^\circ \neq \varnothing$. By Theorem 4.4 and Item 4 again we conclude that f_k extends to a holomorphic map $\overline{f}_k : \Delta \to X_k$. Hence $\mu_k \circ \overline{f}_k$ extends f. If $f(\Delta^*) \cap \mu_k(U_k^\circ) = \varnothing$, we apply Item 5 to iterate the above arguments and after finitely many steps there exists X_i so that $f(\Delta^*) \subset \mu_i(U_i)$ and $f(\Delta^*) \cap \mu_i(U_i^\circ) \neq \varnothing$. By Item 3, f can be lifted to $f_i : \Delta^* \to U_i$ so that $\mu_i \circ f_i = f$ and $f_i(\Delta^*) \cap U_i^\circ \neq \varnothing$. By Theorem 4.4 and Item 4 again, f_i extends to the origin, and so is f. We prove the Picard hyperbolicity of $U = Y - D$.

(ii) Let us prove the algebraic hyperbolicity of U. Fix any reduced and irreducible curve $C \subset Y$ with $C \not\subset D$. By the above arguments, there exists $i \in \{0, \ldots, N\}$ so that $C \subset \mu_i(X_i)$ and $C \cap \mu_i(U_i^\circ) \neq \varnothing$. Let $C_i \subset X_i$ be the strict transform of C under μ_i. By Item 3, $h_i|_{C_i}$ is not identically equal to zero.

Denote by $\nu_i : \tilde{C}_i \to C_i \subset X_i$ the normalization of C_i, and set $P_i := (\mu_i \circ \nu_i)^{-1}(D) = \nu_i^{-1}(D_i)$. Then

$$d\nu_i : T_{\tilde{C}_i}(-\log P_i) \to \nu_i^* T_{X_i}(-\log D_i)$$

induces a (non-trivial) pseudo hermitian metric $\tilde{h}_i := \nu_i^* h_i$ over $T_{\tilde{C}_i}(-\log P_i)$. By (4.2.2), the *curvature current* on $\tilde{C}_i$ of $\tilde{h}_i^{-1}$ satisfies

$$\frac{\sqrt{-1}}{2\pi}\Theta_{\tilde{h}_i^{-1}}(K_{\tilde{C}_i}(\log P_i)) \geq \nu_i^*\omega_i.$$

Hence

$$2g(\tilde{C}_i) - 2 + i(C,D) = \int_{\tilde{C}_i} \frac{\sqrt{-1}}{2\pi}\Theta_{\tilde{h}_i^{-1}}(K_{\tilde{C}_i}(\log P_i)) \geq \int_{\tilde{C}_i} \nu_i^*\omega_i.$$

Fix a Kähler metric Ω_Y^1 on Y. Then there is a constant $\varepsilon_i > 0$ so that $\omega_i \geq \varepsilon_i \mu_i^* \Omega_Y^1$. We thus have

$$2g(\tilde{C}_i) - 2 + i(C,D) \geq \varepsilon_i \int_{\tilde{C}_i} (\mu_i \circ \nu_i)^*\Omega_Y^1 = \varepsilon_i \deg_{\Omega_Y^1} C,$$

for $\mu_i \circ \nu_i : \tilde{C}_i \to C$ is the normalization of C. Set $\varepsilon := \inf_{i=0,\ldots,N} \varepsilon_i$. Then we conclude that for any reduced and irreducible curve $C \subset Y$ with $C \not\subset D$, one has

$$2g(\tilde{C}) - 2 + i(C,D) \geq \varepsilon \deg_{\Omega_Y^1} C$$

where $\tilde{C} \to C$ is its normalization. This shows the algebraic hyperbolicity of U.

The proof of the theorem is accomplished. □

Remark 4.5. Let U be a quasi-projective manifold admitting an integral variation of Hodge structures with *arithmetic* monodromy group whose period map is quasi-finite. In [JL19, Theorem 4.2] Javanpeykar-Litt proved that U is *weakly bounded* in the sense of Kovács-Lieblich [KL10, Definition 2.4] (which is weaker than algebraic hyperbolicity). Though not mentioned explicitly, their proof of [JL19, Theorem 4.2] implicitly shows that such U is also algebraically hyperbolic when local monodromies of the ℂ-PVHS at infinity are unipotent. Their proof is based on the work [BBT18] as well as the Arakelov-type inequality for Hodge bundles by Peters [Pet00].

We end this section with the following remark.

Remark 4.6. Let (E,θ) be the system of log Hodge bundles on a log pair (Y,D) as that in Theorem 3.6. One can also use the idea by Viehweg-Zuo [VZ02] in constructing their *Viehweg-Zuo* sheaf (based on the negativity of kernels of Higgs fields by Zuo [Zuo00]) to prove a weaker result than Theorem 3.6: for any holomorphic map $\gamma : C \to U$ from any open subset C of $\mathbb{C}$ with $\gamma(C) \cap U^\circ \neq \varnothing$, there exists a Finsler metric h_C of $T_Y(-\log D)$ (depending on C) and a Kähler metric ω_C for Y (also depending on C) so that $|\gamma'(t)|_h^2 \not\equiv 0$ and

$$dd^c \log |\gamma'|_{h_C}^2 \geq \gamma^*\omega_C.$$

It follows from our proof of Theorem A that one can also combine Theorem 4.4 with this result, which is only weaker in appearance, to prove Theorem A. The more general result Theorem 0.4 will be used to prove Theorem 5.1.(ii) in the next section.

5. Hyperbolicity for the compactification after finite unramified cover

In this section we will prove Theorem B and Corollary C. We first prove the following theorem.

Theorem 5.1. *Let U be a quasi-compact Kähler manifold. Assume that there is a ℂ-PVHS over U whose period map is immersive at one point. Then there is a finite unramified cover $\tilde{U} \to U$ together with a compact Kähler compactification X of $\tilde{U}$ and a proper Zariski closed subvariety $Z \subsetneq X$ so that*

(i) *the variety X is of general type;*

(ii) *the variety X is Kobayashi hyperbolic modulo Z;*
(iii) *the variety X is Picard hyperbolic modulo Z;*
(iv) *the variety X is algebraically hyperbolic modulo Z.*

Let us briefly explain the idea of the proof for Theorem 5.1 and some related results. Let Y be a compact Kähler manifold compactifying U with $D := Y - U$ a simple normal crossing divisor. By Theorem 2.6 there is a special system of log Hodge bundles $(E, \theta) := (\oplus_{p+q=\ell} E^{p,q}, \oplus_{p+q=\ell} \theta_{p,q})$ on (Y, D) satisfying the properties therein. We divide the proof into five steps.

(1) The first two steps are devoted to construct a compact Kähler log pair $(X, \tilde{D})$ and a generically finite surjective log morphism $\mu : (X, \tilde{D}) \to (Y, D)$ which is unramified over U so that for each irreducible component $\tilde{D}_i$ of $\tilde{D}$,
   - either $\mathrm{ord}_{\tilde{D}_i}(\mu^* D) \gg 0$;
   - or the local monodromy of the pull back $\mathbb{C}$-PVHS over $\tilde{U}$ around $\tilde{D}_i$ is trivial.

   To find this μ, we apply the well-known result that monodromy group of a $\mathbb{C}$-PVHS is residually finite, and use the Cauchy argument principle to show the high ramification over irreducible components of $\tilde{D}$ around which the local monodromies are not trivial. Let us mention that this step is quite different from those in [Nad89, Rou16, Bru20a, Cad18] for the hyperbolicity of compactifications of quotients of bounded symmetric domains by torsion free lattice, whereas they all applied Mumford's work on toroidal compactifications of quotient of bounded symmetric domains [Mum77] so that $\mathrm{ord}_{\tilde{D}_i}(\mu^* D) \gg 0$ for all $\tilde{D}_i$. In general, we are not sure that such covering can be found in our case.
(2) The third step is to construct a new system of log Hodge bundles $(G = \oplus_{p+q=\ell} G^{p,q}, \eta)$ over $(X, \tilde{D})$ which is the canonical extension of the pull back of the $\mathbb{C}$-PVHS via μ. This system of log Hodge bundles $(G = \oplus_{p+q=\ell} G^{p,q}, \eta)$ on $(X, \tilde{D})$ satisfies the three conditions in Theorem 2.6. Moreover, some $G^{p_0, \ell - p_0}$ contains $\tilde{L} \otimes \mathcal{O}_X(\ell D_X)$ with $\tilde{L}$ a big line bundle. Here D_X is the sum of irreducible components of $\tilde{D}$ around which the local monodromies of the pull back $\mathbb{C}$-PVHS are not trivial (hence μ is highly ramified over D_X). Note that (G, η) has singularities along D_X instead of $\tilde{D}$ since the pull back $\mathbb{C}$-PVHS extends across the components where local monodromies are trivial (see (5.0.4).)
(3) The fourth step is to prove Theorem 5.1.(i). We start with $G^{p_0, \ell - p_0}$ and iterate the Higgs field η, ending at finitely many steps. By the negativity of the kernel of $\tilde{\theta}$, $\tilde{L} \otimes \mathcal{O}(\ell D_X) \subset G^{p_0, \ell - p_0}$, and (5.0.4), we can construct an ample sheaf contained in some symmetric differential $\mathrm{Sym}^\beta \Omega_X^1$ (rather than on $\mathrm{Sym}^\beta \Omega_X^1(\log \tilde{D})$!). It follows from a celebrated theorem of Campana-Păun [CP19] that, X is of general type. Let us mention that this idea of iterating Higgs fields to their kernels, originally due to Viehweg-Zuo [VZ02], has been used by Brunebarbe in [Bru20a] in which he proved similar result for quotients of bounded symmetric domains by arithmetic groups. There are also some similar results for *quotients of bounded domains* by Boucksom-Diverio [BD18] and Cadorel-Diverio-Guenancia [CDG19].
(4) The last step is to prove Theorems 5.1.(ii) and 5.1.(iii). We use the above system of log Hodge bundles (G, η) and ideas in § 3 to construct a Finsler metric F on T_X due to $\tilde{L} \otimes \mathcal{O}(\ell D_X) \subset G^{p_0, \ell - p_0}$. Such a metric F is generically positive, and has holomorphic sectional curvature bounded from above by a negative constant by Theorem 3.4. By Ahlfors-Schwarz lemma, we conclude that X is Kobayashi hyperbolic modulo a proper closed subvariety; and by Theorem 4.4 the Picard hyperbolicity modulo a proper subset of X follows. Let us mention that Rousseau [Rou16] has proved similar result for hermitian symmetric spaces, which was later refined by

Cadorel [Cad18]. Their methods use Bergman metrics for bounded symmetric domains instead of Hodge theory.

Now we start the detailed proof of Theorem 5.1.

Proof of Theorem 5.1. By Theorem 2.6, there is a system of log Hodge bundles $(E,\theta) = (\oplus_{p+q=\ell}E^{p,q}, \oplus_{p+q=\ell}\theta_{p,q})$ over (Y,D) satisfying the three conditions therein. In particular, there is a big line bundle L on Y and an inclusion $L \subset E^{p_0,\ell-p_0}$ for some $0 \le p_0 \le \ell$. Pick $m \gg 0$ so that $L - \frac{\ell+1}{m}D$ is a big $\mathbb{Q}$-line bundle.

Step 1. Fix a base point $y \in U := Y - D$. Let us denote by $\rho : \pi_1(U,y) \to GL(r,\mathbb{C})$ the monodromy representation of the corresponding ℂ-PVHS, and denote by $\Gamma := \rho(\pi_1(U,y))$ its monodromy group, which is a finitely generated linear group hence residually finite by a theorem of Malcev [Mal40]. Let us cover Y by finite admissible coordinate systems

$$\{(\mathcal{U}_\alpha; z_1^{(\alpha)},\ldots,z_d^{(\alpha)})\}_{\alpha\in S},$$

where S is a finite set, so that

$$D \cap \mathcal{U}_\alpha = (z_1^{(\alpha)}\cdots z_{k_\alpha}^{(\alpha)} = 0).$$

Write $\mathcal{U}_\alpha^* := \mathcal{U}_\alpha - D$. The fundamental group $\pi_1(\mathcal{U}_\alpha^*, y_\alpha) \simeq \pi_1((\Delta^*)^{k_\alpha}\times\Delta^{d-k_\alpha}, y_\alpha) \simeq \mathbb{Z}^{k_\alpha}$ is abelian. Pick a base point $y_\alpha \in \mathcal{U}_\alpha^*$. Let $e_1^{(\alpha)},\ldots,e_{k_\alpha}^{(\alpha)}$ be the generators of $\pi_1((\Delta^*)^{k_\alpha}\times\Delta^{d-k_\alpha}, y_\alpha)$, namely $e_i^{(\alpha)}$ is a clockwise loop around the origin in the i-th factor Δ^*. Pick a path $h_\alpha : [0,1] \to Y - D$ connecting y_α with y, and denote by $\gamma_i^{(\alpha)} \in \pi_1(Y-D,y)$ the equivalent class of the loop $h_\alpha^{-1}\cdot e_i^{(\alpha)}\cdot h_\alpha$. Denote by $T_i^{(\alpha)} := \rho(\gamma_i^{(\alpha)})$. Clearly, $T_1^{(\alpha)},\ldots,T_{\alpha_k}^{(\alpha)}$ commute pairwise.

Set $\mathfrak{S} \subset \Gamma$ to be a finite subset defined by

$$\{(T_1^{(\alpha)})^{\ell_1}\cdots(T_{k_\alpha}^{(\alpha)})^{\ell_{k_\alpha}} \mid \alpha \in S, 0 \le \ell_i < m\}. \tag{5.0.1}$$

where m be the integer chosen at the beginning. It follows from the definition of residually finite group that there is a normal subgroup $\tilde{\Gamma}$ of Γ with finite index so that

$$\mathfrak{S} \cap \tilde{\Gamma} = \{0\}. \tag{5.0.2}$$

Then $\rho^{-1}(\tilde{\Gamma})$ is a normal subgroup of $\pi_1(U,y)$ with finite index. Let $\nu : \tilde{U} \to U$ be the finite unramified cover of U so that for the induced map of the fundamental group $\nu_* : \pi_1(\tilde{U},x) \to \pi_1(U,y)$, its image is $\rho^{-1}(\tilde{\Gamma})$. Here $x \in \tilde{U}$ with $\mu(x) = y$. We consider $\pi_1(\tilde{U},x)$ as a subgroup of $\pi_1(U,y)$ of finite index. Since the monodromy representation of the pull back of the ℂ-PVHS on $\tilde{U}$ is the restriction

$$\rho|_{\pi_1(\tilde{U},x)} : \pi_1(\tilde{U},x) \to GL(r,\mathbb{C}),$$

its monodromy group is thus $\tilde{\Gamma}$.

Step 2. Note that U is quasi-projective. Hence $\tilde{U}$ is also quasi-projective. Let us take a smooth projective compactification X of $\tilde{U}$ with $\tilde{D} := X - \tilde{U}$ simple normal crossing so that $\nu : \tilde{U} \to U$ extends to a log morphism $\mu : (X,\tilde{D}) \to (Y,D)$. Write $\tilde{D} = \sum_{j=1}^n \tilde{D}_j$ where $\tilde{D}_j$'s are irreducible components of $\tilde{D}$.

Claim 5.2. *For each $j = 1,\ldots,n$, one has*

- *either* $\mathrm{ord}_{\tilde{D}_j}(\mu^*D) \ge m$,
- *or the local monodromy group of the pull-back ℂ-PVHS around $\tilde{D}_j$ is trivial.*

Proof of Claim. Since $\{\mathcal{U}^{(\alpha)}\}_{\alpha\in S}$ covers D, there is $\alpha \in S$ so that for the admissible coordinate system $(\mathcal{U}^{(\alpha)}; z_1^{(\alpha)}, \ldots, z_d^{(\alpha)})$, $\mu^{-1}(\mathcal{U}^{(\alpha)}) \cap \tilde{D}_j \neq \varnothing$. We will write $(\mathcal{U}; z_1, \ldots, z_d)$ instead of $(\mathcal{U}^{(\alpha)}; z_1^{(\alpha)}, \ldots, z_d^{(\alpha)})$, and k instead of k_α to lighten the notation. Namely, $\mathcal{U} \cap D = (z_1 \cdots z_k = 0)$. Note that $k \geq 1$.

Pick a point $x \in \tilde{D}_j - \cup_{i\neq j}\tilde{D}_i$ so that there is an admissible coordinate system $(\mathcal{W}; x_1, \ldots, x_n)$ with $\mu(\mathcal{W}) \subset \mathcal{U}$ and $\mathcal{W} \cap \tilde{D} = (x_1 = 0)$. Denote by $(\mu_1(x), \ldots, \mu_d(x))$ the expression of μ within these coordinates. Then

$$(\mu_1(x), \ldots, \mu_d(x)) = (x_1^{n_1}\nu_1(x), \ldots, x_1^{n_k}\nu_k(x), \mu_{k+1}(x), \ldots, \mu_d(x))$$

where $\nu_1(x), \ldots, \nu_k(x)$ are holomorphic functions defined on $\mathcal{W}$ so that neither of them is identically equal to zero on $(x_1 = 0)$, and $n_p \geq 0$ for $p = 1, \ldots, k$.

We thus can choose a slice $S := \{(x_1, \ldots, x_d) \mid \{|x_1| \leq \varepsilon, x_2 = \zeta_2, \ldots, x_d = \zeta_d\} \subset \mathcal{W}$ so that $\nu_i(x) \neq 0$ for any $x \in S$ and any $i = 1, \ldots, k$. Let us define a loop $e(\theta) : [0,1] \to \mathcal{W}^* := \mathcal{W} - \tilde{D}$ by $e(\theta) := (\varepsilon e^{2\pi i\theta}, \zeta_2, \ldots, \zeta_d)$ which is the generator of $\pi_1(\mathcal{W}^*, x_0)$, where $x_0 \in \mathcal{W}^*$ is a point with $\mu(x_0) = y_\alpha \in \mathcal{U}^*$. By *Cauchy's argument principle*, the winding number of $\mu_p \circ e(\theta)$ around 0 is n_p for $p = 1, \ldots, k$. Hence by the following diagram

$$\begin{tikzcd} \pi_1(\mathcal{W}^*, x_0) \arrow[r, "\nu_*"] \arrow[d, "\simeq"] & \pi_1(\mathcal{U}^*, y_\alpha) \arrow[d] \\ \mathbb{Z} \arrow[r] & \mathbb{Z}^k \end{tikzcd}$$

one has $\nu_*(1) = (n_1, \ldots, n_p)$.

Pick a path $\tilde{h} : [0,1] \to \tilde{U}$ connecting x and x_0, which lifts the above path $h_\alpha : [0,1] \to U$. Set $\tilde{\gamma}_0 \in \pi_1(\tilde{U}, x)$ to be the equivalent class of the loop $\tilde{h}^{-1} \cdot e \cdot \tilde{h}$. Then

$$\nu_*(\tilde{\gamma}_0) = [h_\alpha^{-1} \cdot (e_1^{(\alpha)})^{n_1} \cdots (e_k^{(\alpha)})^{n_k} \cdot h_\alpha] = (\gamma_1^{(\alpha)})^{n_1} \cdots (\gamma_k^{(\alpha)})^{n_k}$$

Therefore,

$$(T_1^{(\alpha)})^{n_1} \cdots (T_k^{(\alpha)})^{n_k} = \rho\big(\nu_*(\tilde{\gamma}_0)\big) \in \tilde{\Gamma}.$$

By (5.0.1) and (5.0.2), either

$$\rho\big(\nu_*(\tilde{\gamma}_0)\big) = 0,$$

or there is some $i \in \{1, \ldots, k\}$ so that $n_i \geq m$. The first case means that the local monodromy of the pull back $\mathbb{C}$-PVHS around $\tilde{D}_j$ is trivial. In the later case one has

$$\mathrm{ord}_{\tilde{D}_j}(\mu^* D) = \sum_{i=1}^{k} n_i \geq m.$$

The claim is proved. □

Step 3. Set $D_X \subset D$ to be the sum of all $\tilde{D}_j$'s so that the local monodromy group of the pull-back $\mathbb{C}$-PVHS around $\tilde{D}_j$ is *not* trivial. Then by the dichotomy in Claim 5.2, $\mu^*D - mD_X$ is an effective divisor, and the pull-back $\mathbb{C}$-PVHS on $\tilde{U}$ around $\tilde{D}_i$ with $\tilde{D}_i \not\subset D_X$ is trivial. By a theorem of Griffiths, the pull-back $\mathbb{C}$-PVHS extends to a $\mathbb{C}$-PVHS defined over $X - D_X$.

By the second condition in Theorem 2.6.(ii), (E, θ) is the canonical extension (in the sense of Definition 1.11) of some system of Hodge bundles $(\tilde{E} = \oplus_{p+q=\ell}\tilde{E}^{p,q}, \tilde{\theta}, h_{\mathrm{hod}})$ defined over $Y - D$. Hence for any admissible coordinate $(\mathcal{U}; z_1, \ldots, z_d)$ and any holomorphic frame $(e_1, \ldots, e_{r'})|_U$ for $E^{p,q}$, one has

$$|e_j|_{\mathrm{hod}} \lesssim \frac{1}{\prod_{i=1}^{k} |z_i|^{\varepsilon}}$$

for all $\varepsilon > 0$. If we take an admissibe coordinate $(\mathcal{W}; x_1, \ldots, x_d)$ with $\mathcal{W} \cap \tilde{D} = (x_1 \cdots x_c = 0)$ and $\mu(\mathcal{W}) \subset \mathcal{U}$, one can see that

$$|\mu^* e_j|_{\mu^*\mathrm{hod}} \lesssim \frac{1}{\prod_{i=1}^{c} |x_i|^{\varepsilon \cdot n_i}}$$

for all $\varepsilon > 0$. Here $n_i := \mathrm{ord}_{(x_i=0)}(\mu^*(z_1 \cdots z_k))$. It then follows from the definition of prolongation (1.4.1) that

$$\mu^* E^{p,q} \subset {}^{\diamond}(\mu^* \tilde{E}^{p,q}). \tag{5.0.3}$$

Note that $\mu^*(\tilde{E}, \tilde{\theta}, h_{\mathrm{hod}})$ is still a system of Hodge bundles over $\tilde{U}$, which corresponds to the pull back of the given ℂ-PVHS on U. Recall that the pull-back ℂ-PVHS extends to a ℂ-PVHS defined over $X - D_X$. Hence $\mu^*(\tilde{E}, \tilde{\theta}, h_{\mathrm{hod}})$ extends to a system of Hodge bundles over $X - D_X$.

We denote by $(G = \oplus_{p+q=\ell} G^{p,q}, \eta = \oplus_{p+q=\ell} \eta_{p,q})$ the canonical extension (in the sense of Definition 1.11) of $\mu^*(\tilde{E}, \tilde{\theta}, h_{\mathrm{hod}})$ (which is defined over $X - D_X$) over the log pair (X, D_X), which is thus a system of log Hodge bundles on (X, D_X). In particular, one has

$$\eta_{p,q} : G^{p,q} \to G^{p-1,q+1} \otimes \Omega_X^1(\log D_X). \tag{5.0.4}$$

By Lemma 1.9.(i) one has

$$G^{p,q} = {}^{\diamond}(\mu^* \tilde{E}^{p,q}). \tag{5.0.5}$$

Since L is a subsheaf of $E^{p_0, \ell - p_0}$, by (5.0.3) and (5.0.5) one has

$$\mu^* L \subset \mu^* E^{p_0,q_0} \subset G^{p_0,q_0}.$$

Recall that $\mu^* D - m D_X$ is an effective divisor, and $L - \frac{\ell+1}{m} D$ is a big ℚ-line bundle. Write $\tilde{L} := \mu^* L - \ell D_X$. Then $\tilde{L}$ and $\tilde{L} - D_X$ are both big line bundles. The above inclusion yields

$$\tilde{L} \otimes \mathcal{O}_X(\ell D_X) \subset G^{p_0,q_0}. \tag{5.0.6}$$

Step 4. Now we iterate η by k-times as in § 2.3 to obtain a morphism

$$G^{p_0, \ell - p_0} \to G^{p_0 - k, \ell - p_0 + k} \otimes \mathrm{Sym}^k \Omega_X^1(\log D_X). \tag{5.0.7}$$

The inclusion (5.0.6) then induces a morphism

$$\kappa_k : \tilde{L} \otimes \mathcal{O}_X(\ell D_X) \to G^{p_0 - k, \ell - p_0 + k} \otimes \mathrm{Sym}^k \Omega_X^1(\log D_X). \tag{5.0.8}$$

Write k_0 for the largest k so that κ_{k_0} is non-trivial. Then $0 \le k_0 \le p_0 \le \ell$. Let us denote by N_p the kernel of $\theta_{p, \ell - p}$. Hence κ_{k_0} admits a factorization

$$\kappa_{k_0} : \tilde{L} \otimes \mathcal{O}_X(\ell D_X) \to N_{p_0 - k_0} \otimes \mathrm{Sym}^{k_0} \Omega_X^1(\log D_X).$$

We first note $k_0 > 0$; or else, there is a morphism from the big line bundle $\tilde{L} \otimes \mathcal{O}_X(\ell D_X)$ to N_{p_0}, whose dual $N_{p_0}^*$ is weakly positive in the sense of Viehweg by [Bru17] (see also [Den20, Theorem 4.6])[2]. Hence κ_{k_0} induces

$$\tilde{L} \to N_{p_0 - k_0} \otimes \mathrm{Sym}^{k_0} \Omega_X^1(\log D_X) \otimes \mathcal{O}_X(-\ell D_X) \subset N_{p_0 - k_0} \otimes \mathrm{Sym}^{k_0} \Omega_X^1$$

due to $k_0 \le p_0 \le \ell$. In other words, there exists a non-trivial morphism

$$\tilde{L} \otimes N_{p_0 - k_0}^* \to \mathrm{Sym}^{k_0} \Omega_X^1.$$

Recall that $N_{p_0 - k_0}^*$ is weakly positive. The torsion free coherent sheaf $\tilde{L} \otimes N_{p_0 - k_0}^*$ is big in the sense of Viehweg. Hence there is $\alpha > 0$ so that

$$\mathrm{Sym}^{\alpha}(\tilde{L} \otimes N_{p_0 - k_0}^*) \otimes \mathcal{O}_X(-A)$$

[2] When the monodromy of ℂ-PVHS is unipotent, this was proved by Zuo in [Zuo00].

is generically globally generated for some ample divisor A. One thus has a non-trivial morphism

$$\mathcal{O}_X(A) \to \mathrm{Sym}^{\alpha k_0}\Omega^1_X.$$

By a theorem of Campana-Păun [CP19, Corollary 8.7], X is of general type.

Step 5. Let us prove that X is both pseudo Picard and pseudo Kobayashi hyperbolic. Note that κ_k in (5.0.8) induces a morphism

$$\tau_k : \mathrm{Sym}^k T_X(-\log D_X) \to G^{p_0-k,\ell-p_0+k} \otimes \tilde{L}^{-1} \otimes \mathcal{O}_X(-\ell D_X)$$

By Theorem 2.9 we know that τ_1 is injective on a Zariski open set $\tilde{U}' \subset \tilde{U}$. τ_k induces a morphism

$$\tilde{\tau}_k : \mathrm{Sym}^k T_X \to \mathrm{Sym}^k T_X(-\log D_X) \otimes \mathcal{O}_X(\ell D_X) \to G^{p_0-k,\ell-p_0+k} \otimes \tilde{L}^{-1}$$

which coincides with τ_k over $\tilde{U}$. Hence $\tilde{\tau}_1$ is also injective over $\tilde{U}'$. By Proposition 2.8, we can take a singular hermitian metric $h_{\tilde{L}}$ for $\tilde{L}$ so that $h := h_{\tilde{L}}^{-1} \otimes \tilde{h}_{\rm hod}$ on $G \otimes \tilde{L}^{-1}$ is locally bounded on Y, and smooth outside $D_X \cup \mathbf{B}_+(\tilde{L} - D_X)$, where $\tilde{h}_{\rm hod}$ is the Hodge metric for the system of Hodge bundles $(G, \eta)|_{X-D_X}$. Moreover, h vanishes on $D_X \cup \mathbf{B}_+(\tilde{L} - D_X)$. This metric h on $G \otimes \tilde{L}^{-1}$ induces a Finsler metric F_k on T_X defined as follows: for any $e \in T_{X,x}$,

$$F_k(e) := h\big(\tilde{\tau}_k(e^{\otimes k})\big)^{\frac{1}{k}}$$

We apply the same method in § 3 to construct a new Finsler metric F on T_X by taking convex sum in the following form

$$F := \sqrt{\sum_{i=1}^{k_0} \alpha_i F_i^2}$$

where $\alpha_1, \ldots, \alpha_{k_0} \in \mathbb{R}^+$ are certain constants. This Finsler metric F on T_X is positively definite over $\tilde{U}^\circ := \tilde{U}' - \mathbf{B}_+(\tilde{L} - D_X)$ as $\tilde{\tau}_1$ is injective over $\tilde{U}'$ and h is smooth on $\tilde{U} - \mathbf{B}_+(\tilde{L} - D_X)$. Denote by $Z := X \setminus \tilde{U}^\circ$, which is a proper Zariski closed subvariety of X. By Theorem 3.4 one can choose $\alpha_1, \ldots, \alpha_{k_0} \in \mathbb{R}^+$ properly so that for any $\gamma : C \to X$ with C an open set of $\mathbb{C}$ and $\gamma(C) \cap \tilde{U}^\circ \neq \varnothing$, one has

$$\mathrm{dd^c} \log |\gamma'(t)|_F^2 \geq \gamma^*\omega \tag{5.0.9}$$

for some fixed smooth Kähler form ω on X. Indeed, it follows from the proof of Theorem 3.4 that there is an open subset C° of C whose complement is a discrete set such that (5.0.9) holds over C°. By Definition 3.1, $|\gamma'(t)|_F^2$ is continuous and locally bounded from above over C, and by the extension theorem of subharmonic function, (5.0.9) holds over the whole unit disk C. Applying Theorem 4.4 to (5.0.9), we conclude that X is Picard hyperbolic modulo Z. Hence Theorem 5.1.(iii) follows.

Let C be an irreducible compact curve in X not contained in Z. Write $h_{\tilde{C}}$ the induced singular hermitian metric for $T_{\tilde{C}}$ by F, where $\tilde{C}$ is the normalization of C. Then by (5.0.9) one has

$$2g(\tilde{C}) - 2 = -\sqrt{-1}\Theta_{h_{\tilde{C}}}(T_{\tilde{C}}) \geq \deg_\omega(C).$$

This proves Theorem 5.1.(iv).

By Definition 3.1 again, there is $\varepsilon > 0$ so that $\omega \geq \varepsilon F^2$. Hence (5.0.9) implies that

$$\frac{\partial^2 \log |\gamma'(t)|_F^2}{\partial t \partial \bar{t}} \geq \varepsilon |\gamma'(t)|_F^2$$

for any $\gamma : \Delta \to X$ with $\gamma(\Delta) \cap \tilde{U}^\circ \neq \varnothing$. In other words, the holomorphic sectional curvature of F is bounded from above by the negative constant $-\varepsilon$ (see [Kob98, Theorem

2.3.5]). By the Ahlfors-Schwarz lemma, we conclude that X is Kobayashi hyperbolic modulo Z (see [Den18, Lemma 2.4]). This proves Theorem 5.1.(ii).

The theorem is proved. □

During the proof of the above proof, we indeed obtained the following result.

Theorem 5.3. *Let (Y, D) be a compact Kähler log pair and let $(E, \theta) = (\oplus_{p+q=\ell} E^{p,q}, \oplus_{p+q=\ell}\theta_{p,q})$ be a system of log Hodge bundles on (Y, D) satisfying the following conditions.*

(1) (E, θ) is the canonical extension of some system of Hodge bundles over $Y - D$ of weight ℓ.
(2) There is a big line bundle L over Y such that $L \otimes \mathcal{O}_X(\ell D) \subset E^{p_0,q_0}$ for some $p_0 + q_0 = \ell$.
(3) The line bundle $L \otimes \mathcal{O}_X(-D)$ is still big.

Then there is a proper Zariski closed subset $Z \subsetneq Y$ so that

(i) *Y is Kobayashi hyperbolic modulo Z;*
(ii) *Y is Picard hyperbolic modulo Z;*
(iii) *Y is algebraically hyperbolic modulo Z.*
(iv) *Y is of general type;*

□

Now we are able to prove Theorem B.

Proof of Theorem B. By Step 1-3 in the proof of Theorem 5.1, we can construct a projective log pair (X, D) and a log morphism $\mu : (X, \tilde{D}) \to (Y, D)$ which is a finite étale cover over U. Over (X, D), there is a system of log Hodge bundles $(G = \oplus_{p+q=\ell} G^{p,q}, \eta = \oplus_{p+q=\ell}\eta_{p,q})$ satisfying the following properties:

(1) (G, η) is the canonical extension of some system of Hodge bundle on $X - D_X$, where D_X is a reduced simple normal crossing divisor supported on D.
(2) There is a big line bundle $\tilde{L}$ on X so that $\tilde{L} \otimes \mathcal{O}(-D_X)$ is also big,
(3) There is an inclusion $\tilde{L} \otimes \mathcal{O}_X(\ell D_X) \subset G^{p_0,\ell-p_0}$ for some $p_0 > 0$.

Let $\tilde{Z}$ be any irreducible Zariski closed subvariety of X of positive dimension which is not contained in $\tilde{D}$. Take a resolution of singularities $g : Z \to \tilde{Z}$ so that $D_Z := \nu^{-1}(D_X)$ is simple normal crossing. Then $g : (Z, D_Z) \to (X, D_X)$ is a log morphism which is generically finite.

Note that $L_Z := g^*\tilde{L}$ is big. Since $g^*D_X - D_Z$ is an effective divisor, $L_Z \otimes \mathcal{O}_Z(-D_Z)$ is also big. By (3), one has

$$L_Z \otimes \mathcal{O}_Z(\ell D_Z) \subset g^*\Big(\tilde{L} \otimes \mathcal{O}_X(\ell D_X)\Big) \subset g^*G^{p_0,\ell-p_0}.$$

For the $\mathbb{C}$-PVHS corresponding to $(G, \eta)|_{X-D_X}$, we pull it back to $Z - D_Z$ via g and denote by $(\tilde{E}, \tilde{\theta})$ the induced system of Hodge bundles on $Z - D_Z$. Set $(E = \oplus_{p+q=\ell} E^{p,q}, \theta)$ to be the canonical extension of such system of Hodge bundles. In the same vein as the proof of (5.0.3), one has

$$g^*G^{p_0,\ell-p_0} \subset E^{p_0,\ell-p_0}.$$

In summary, we construct a system of log Hodge bundles $(E = \oplus_{p+q=\ell} E^{p,q}, \theta)$ on (Z, D_Z) satisfying the three conditions in Theorem 5.3. By Theorem 5.3, Z is of general type. We proved Theorem B.(i).

Let us prove Theorem B.(ii). For any $\tilde{\gamma} : \Delta^* \to X$ whose image is not contained in $\tilde{D}$, let $\tilde{Z}$ be its Zariski closure. Take a desingularization $\nu : Z \to \tilde{Z}$ as above, and let $\gamma : \Delta^* \to Z$ be the lift of γ. By the above argument and Theorem 5.3, γ extends to a holomorphic map $\overline{\gamma} : \Delta \to Z$. Therefore, $\nu \circ \overline{\gamma}$ extends $\tilde{\gamma}$. We proved Theorem B.(ii). It is easy to see that Theorem B.(ii) implies Theorem B.(iii).

The proof of Theorem B.(iv) is exactly the same as that in Theorem A. We will not repeat the arguments and leave it to the interested readers. □

We now show how to deduce Corollary C from Theorem B.

Proof of Corollary C. By the work of Baily-Borel and Mok, we know that U is quasi-projective. By the work of Deligne, U admits a $\mathbb{C}$-PVHS whose period map is immersive everywhere (see *e.g.* [Mil13, Theorem 7.10]). The corollary immediately follows from Theorem B.(ii). □

Indeed, using the Bergman metric in [Rou16] one can also obtain a proof for Corollary C by applying Theorem 4.4.

Remark 5.4. Corollary C unifies the previous result by Nadel who proved that X is Brody hyperbolic modulo $X - \tilde{U}$. Applying Theorem B.(i), it also reproves theorems by Brunebarbe [Bru20a] and Cadorel [Cad18]: any positive dimensional irreducible subvariety of X not contained in $X - \tilde{U}$ is of general type. However, since our proof does not rely on special properties of bounded symmetric domains (we neither use Mumford's work on toroidal compactifications, nor the existence of variation of Hodge structures of Calabi-Yau type over quotients of bounded symmetric domains by arthmetic groups), we certainly loose the effectivity result regarding the level structures of the unramified coverings, which are also main results in [Nad89, Bru20a, Cad18].

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