[23] Chenyang Xu, *Finiteness of algebraic fundamental groups*, Compos. Math. (3) **150** (2014), 409–414.
[24] Chenyang Xu and Ziquan Zhuang, *Uniqueness of the minimizer of the normalized volume function*, preprint (2020), arXiv:2005.08303.
[25] De-Qi Zhang, *The fundamental group of the smooth part of a log Fano variety*, Osaka J. Math. **32** (1995), 637–644.

Big Picard theorem for varieties admitting variation of Hodge structures

Ya Deng

In 1972, Armand Borel [3] and Kobayashi-Ochiai [8] proved generalized big Picard theorem for any hermitian locally symmetric variety X: any holomorphic map from the punctured disk to X extends to a holomorphic map of the disk into any projective compactification of X. In particular, they proved that any analytic map from a quasi-projective variety to X is algebraic. Their theorem motivated Ariyan Javenpeykar to propose the notion of "Borel hyperbolicity" for quasi-projective varieties in [7].

Definition 1. *A complex quasi-projective variety Y is* Borel hyperbolic *if any holomorphic map from another quasi-projective variety X to Y is an algebraic morphism.*

Inspired by the above-mentioned generalized big Picard theorem of Borel and Kobayashi-Ochiai, we proposed the following notion of hyperbolicity in [5].

Definition 2. *A complex quasi-projective variety Y is* Picard hyperbolic *if any holomorphic map from the punctured unit disk to Y extends to a holomorphic map from the unit disk to $\overline{Y}$, where $\overline{Y}$ is some (thus any) projective compactification of Y.*

It has been proved in [7] that a complex quasi-projective variety Y is Borel hyperbolic if it is Picard hyperbolic.

Period domains, introduced by Griffiths in 1969, are classifying spaces for Hodge structures. They are transcendental generalizations of hermitian locally symmetric varieties. In the workshop "Oberwolfach Komplex analysis 2017", Ariyan Javenpeykar proposed the following conjecture.

Conjecture 3. *Let Y be a quasi-projective manifold, which admits a polarized variation of Hodge structures, whose period map is quasi-finite. Then Y is Borel hyperbolic.*

He further conjectured that the moduli spaces of polarized manifolds with semi-ample canonical sheaf is Borel hyperbolic.

Conjecture 3 was first investigated by Bakker, Brunebarbe and Tsimerman in their work [1] on the Griffiths conjecture. They proved Conjecture 3 when the monodromy group of the variation of Hodge structures is arithmetic. Their work is based on delicate results in o-minimality. In particular, they have to use the very recent deep theorem by Bakker, Klingler and Tsimerman [2] on the definability of period maps.

In [5], we proved the following generalized big Picard theorem for varieties admitting complex variation of Hodge structures. In particular, we proved Conjecture 3 completely.

Theorem 4. *Let Y be a quasi-projective manifold, which admits a polarized variation of Hodge structures, whose period map is quasi-finite. Then Y is Picard hyperbolic.*

We use purely complex analytic methods to prove Theorem 4. Based on Nevanlinna theory, we first establish some criterion for Picard hyperbolic in [6].

Lemma 5. *Let X be a quasi-projective manifold. Assume that $\gamma : \Delta^* \to X$. If there is a Finsler metric h for $T_{\overline{X}}(-\log D)$ so that $dd^c \log |\gamma'(t)|^2_h \geq \gamma^*\omega$ where ω is a Kähler form on $\overline{X}$. Then γ extends to $\Delta \to \overline{X}$.*

Such criterion was first applied in [6] to prove the Picard hyperbolicity of moduli spaces of polarized manifolds with semi-ample canonical sheaf. Let us mention that, prior to that, in [4] we proved the Brody hyperbolicity of these moduli spaces, which is indeed a conjecture by Viehweg and Zuo [9].

In [5], we construct some special Higgs bundles over the variety Y in Theorem 4. We then apply these Higgs bundles to construct the desired negatively curved Finsler metric as Lemma 5 so that we can apply the above criterion to prove the Picard hyperbolicity of Y.

Let us mention that our techniques unifies Picard hyperbolicity of hermitian locally symmetric varieties X by Borel and Kobayashi-Ochiai.

References

[1] B. Bakker, Y. Brunebarbe, and J. Tsimerman. *o-minimal GAGA and a conjecture of Griffiths*, *arXiv e-prints* (2018) arXiv:1811.12230.

[2] B. Bakker, B. Klingler, and J. Tsimerman. *Tame topology of arithmetic quotients and algebraicity of Hodge loci*, *arXiv e-prints* (2018) arXiv:1810.04801.

[3] A. Borel. *Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem*, *J. Differential Geometry* **6** (1972), 543–560.

[4] Y. Deng. *On the hyperbolicity of base spaces for maximal variational families of smooth projective varieties.* To appear in J. Eur. Math. Soc. (JEMS).

[5] Y. Deng. *Big Picard theorem and algebraic hyperbolicity for varieties admitting a variation of Hodge structures*, *arXiv e-prints* (2020) arXiv:2001.04426.

[6] Y. Deng, S. Lu, R. Sun, and K. Zuo. *Picard theorems for moduli spaces of polarized varieties*, *arXiv e-prints* (2019) arXiv:1911.02973.

[7] A. Javanpeykar and R. A. Kucharczyk. *Algebraicity of analytic maps to a hyperbolic variety*, *arXiv e-prints* (2018) arXiv:1806.09338.

[8] S. Kobayashi and T. Ochiai. *Satake compactification and the great Picard theorem*, J. Math. Soc. Japan **23** (1971), 340–350.

[9] E. Viehweg and K. Zuo. *On the Brody hyperbolicity of moduli spaces for canonically polarized manifolds*, Duke Math. J. **118 (1)** (2003), 103–150.