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Hyperbolicity and fundamental groups of quasi-projective varieties

Ya Deng

(joint work with Benoit Cadorel, Katsutoshi Yamanoi)

The concept of pseudo Picard hyperbolicity and pseudo Brody hyperbolicity has been introduced for complex algebraic varieties. A complex quasi-projective normal variety X is said to be pseudo Picard hyperbolic if there exists a proper Zariski closed subset $Z \subsetneqq X$ such that any holomorphic map $f : \mathbb{D}^* \to X$ from the punctured disk $\mathbb{D}^*$ with an essential singularity at the origin is contained in Z. Similarly, X is called pseudo Brody hyperbolic if there exists a proper Zariski closed subset $Z \subsetneqq X$ such that any non-constant holomorphic map $f : \mathbb{C} \to X$ is contained in Z. It is worth noting that pseudo Picard hyperbolicity implies pseudo Brody hyperbolicity, which is a weaker form of hyperbolicity.

Another concept that has been studied extensively is the notion of log general type. A variety X is said to be *strongly of log general type* if there exists a proper Zariski closed subset $Z \subsetneqq X$ such that any closed positive-dimensional subvariety V of X that is not of log general type is contained in Z.

In a recent paper by Cadorel, Yamanoi, and the reporter [2], the strong version of the Green-Griffiths-Lang conjecture has been studied for varieties that admit a big and reductive representation of their (topological) fundamental group $\pi_1(X)$. This conjecture states that the four hyperbolicity properties, namely, pseudo Picard hyperbolicity, pseudo Brody hyperbolicity, log general type, and strongly of log general type are equivalent for a given variety X. We were able to prove this conjecture for the aforementioned class of varieties.

Theorem 1 ([2, Theorem 0.4])**.** Let X be a complex smooth quasi-projective variety and $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ be a big and reductive representation. Then for any automorphism $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, the strong Green-Griffiths-Lang conjecture holds for the conjugate variety $X^\sigma := X \times_\sigma \mathbb{C}$, i.e. the following properties are equivalent:

(1) X^σ is of log general type.
(2) X^σ is strongly of log general type.
(3) X^σ is pseudo Picard hyperbolic.
(4) X^σ is pseudo Brody hyperbolic.

Recall that a representation $\varrho : \pi_1(X) \to G(\mathbb{C})$ is said to be *big*, or *generically large* in [10], if for any closed subvariety $Z \subset X$ containing a *very general* point of X, $\varrho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)])$ is infinite, where Z^{norm} denotes the normalization of Z. It is worth noting that a stronger notion of largeness exists, where ϱ is called *large* if $\varrho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)])$ is infinite for any closed subvariety Z of X.

We introduce four special subsets of X that measure the non-hyperbolicity locus from different perspectives.

Definition 2 (Special subsets)**.** Let X be a smooth quasi-projective variety.

(1) $\mathrm{Sp}_{\mathrm{sab}}(X) := \overline{\bigcup_f f(A_0)}^{\mathrm{Zar}}$, where f ranges over all non-constant rational maps $f : A \dashrightarrow X$ from all semi-abelian varieties A to X such that f is regular on a Zariski open subset $A_0 \subset A$ whose complement $A \backslash A_0$ has codimension at least two;
(2) $\mathrm{Sp}_{\mathrm{h}}(X) := \overline{\bigcup_f f(\mathbb{C})}^{\mathrm{Zar}}$, where f ranges over all non-constant holomorphic maps from $\mathbb{C}$ to X;
(3) $\mathrm{Sp}(X) := \overline{\bigcup_V V}^{\mathrm{Zar}}$, where V ranges over all positive-dimensional closed subvarieties of X which are not of log general type;
(4) $\mathrm{Sp}_{\mathrm{p}}(X) := \overline{\bigcup_f f(\mathbb{D}^*)}^{\mathrm{Zar}}$, where f ranges over all holomorphic maps from the punctured disk $\mathbb{D}^*$ to X with essential singularity at the origin.

Another strong version of the Green-Griffiths-Lang conjecture asserts that the four special subsets defined in Definition 2 should coincide. We establish this conjecture under the assumption that $\pi_1(X)$ admits a large and reductive representation, as stated in the following theorem.

Theorem 3 ([2, Theorem 0.6])**.** Let X be a smooth quasi-projective variety and $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ be a large and reductive representation. Then for any automorphism $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$,

(a) the four special subsets defined in Definition 2 are the same, i.e.,

$$\mathrm{Sp}(X^\sigma) = \mathrm{Sp}_{\mathrm{sab}}(X^\sigma) = \mathrm{Sp}_{\mathrm{h}}(X^\sigma) = \mathrm{Sp}_{\mathrm{p}}(X^\sigma).$$

(b) These special subsets are conjugate under automorphism σ, i.e.,

$$\mathrm{Sp}_\bullet(X^\sigma) = \mathrm{Sp}_\bullet(X)^\sigma,$$

where $\mathrm{Sp}_\bullet$ denotes any of Sp, $\mathrm{Sp}_{\mathrm{sab}}$, Sp_{h} or Sp_{p}.

(c) $\mathrm{Sp}(X^\sigma)$ is a proper Zariski closed subset of X^σ if and only if X is of log general type.

In [2], we also prove the following result:

Theorem 4 ([2, Theorem 0.1]). Let X be a complex quasi-projective normal variety and let $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ be a big representation such that the Zariski closure of $\varrho(\pi_1(X))$ is a semisimple algebraic group. Then, for any automorphism $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, the variety X^σ is strongly of log general type and pseudo Picard hyperbolic.

We remark that the condition in Theorem 4 is sharp. Theorem 4 are new even in the case where X is projective. When the variety X in Theorem 4 is projective, Campana-Claudon-Eyssidieux [5, Theorem 1] proved that X is of general type and Yamanoi [12, Proposition 2.1] proved that X does not admit Zariski dense entire curves $f : \mathbb{C} \to X$.

It is noteworthy that the condition of bigness for the representations ϱ in Theorem 4 is not particularly restrictive, as demonstrated by the following result:

Corollary 5 ([2, Corollary 0.2]). *Let X be a complex quasi-projective normal variety and let G be a semisimple algebraic group over $\mathbb{C}$. If $\varrho : \pi_1(X) \to G(\mathbb{C})$ is a Zariski dense representation, then there exist a finite étale cover $\nu : \widehat{X} \to X$, a birational and proper morphism $\mu : \widehat{X}' \to \widehat{X}$, a dominant morphism $f : \widehat{X}' \to Y$ with connected general fibers, and a big and Zariski dense representation $\tau : \pi_1(Y) \to G(\mathbb{C})$ such that*

(a) $f^*\tau = (\nu \circ \mu)^*\varrho$.

(b) *the variety Y is pseudo Picard hyperbolic and strongly of log general type.*

In particular, X is neither weakly special nor Brody special.

Note that by Campana [4], a quasi-projective variety X is *weakly special* if for any finite étale cover $\widehat{X} \to X$ and any birational modification $\widehat{X}' \to \widehat{X}$, there exists no dominant morphism $\widehat{X}' \to Y$ with Y a positive-dimensional quasi-projective normal variety of log general type. By [8] a quasi-projective variety is *Brody special* if it contains a Zariski dense entire curve.

Corollary 5 generalizes the previous work by Mok [11], Corlette-Simpson [6], and Campana-Claudon-Eyssidieux [5], in which they proved similar factorisation results.

On the other hand, Campana's abelianity conjecture [4, 11.2] predicts that a smooth quasi-projective variety X that is *special* or *Brody special* has a virtually abelian fundamental group. When a special variety X is projective, it is known that all linear quotients of $\pi_1(X)$ are virtually abelian (cf. [3, Theorem 7.8]). The same conclusion is valid for any Brody special smooth projective variety X (cf. [12, Theorem 1.1]). While it is natural to expect similar results for smooth quasi-projective varieties, we construct in [2] an example of a quasi-projective surface that is special and Brody special, whose fundamental group is linear and nilpotent but not virtually abelian. This provides a counterexample to Campana's conjecture in the general case. In the same work, we prove the following theorem:

Theorem 6 ([2, Theorem 0.8])**.** Let X be a special or Brody special smooth quasi-projective variety. Let $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ be a linear representation. Then $\varrho(\pi_1(X))$ is *virtually nilpotent.*

To prove the above theorems, in [2] we develop new features in non-abelian Hodge theories in both archimedean and non-archimedean settings, geometric group theory, and Nevanlinna theory. Along the way, two difficult theorems are established, which are of significant interest in their own right. One such technique is a reduction theorem for Zariski dense representations $\varrho : \pi_1(X) \to G(K)$, where G is a reductive algebraic group defined over a non-Archimedean local field K.

Theorem 7 ([2, Theorem 0.11])**.** Let X be a complex quasi-projective manifold, and let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a reductive representation where K is a non-archimedean local field. Then there exists a quasi-projective normal variety S_ϱ and a dominant morphism $s_\varrho : X \to S_\varrho$ with connected general fibers, such that for any connected Zariski closed subset T of X, the following properties are equivalent:

(a) the image $\rho(\mathrm{Im}[\pi_1(T) \to \pi_1(X)])$ is a bounded subgroup of $G(K)$.
(b) For every irreducible component T_o of T, the image $\rho(\mathrm{Im}[\pi_1(T_o^{\mathrm{norm}}) \to \pi_1(X)])$ is a bounded subgroup of $G(K)$.
(c) The image $s_\varrho(T)$ is a point.

When X is projective, this theorem was proved in [9, 7]. One of the building blocks of the proof of Theorem 7 is based on previous results by Brotbek, Daskalopoulos, Mese, and the reporter [1] on the existence of harmonic mappings to Bruhat-Tits buildings (an extension of Gromov-Schoen's theorem to quasi-projective cases) and the construction of logarithmic symmetric differential forms via these harmonic mappings.

Another significant building block is the following theorem.

Theorem 8 ([2, Theorem 0.13])**.** Let X be a quasi-projective variety. Assume that there is a morphism $a : X \to A$ such that $\dim X = \dim a(X)$ where A is a semi-abelian variety (e.g., when X has maximal quasi-Albanese dimension). Then the following properties are equivalent:

(a) X is of log general type.
(b) X is strongly of log general type.
(c) X is pseudo Picard hyperbolic.
(d) X is pseudo Brody hyperbolic.

The proof of Theorem 8 is heavily based on Nevanlinna theory.

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The Riemann-Schottky problem via singularities of theta divisors

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(joint work with Christian Schnell)

This talk is about the classical Riemann-Schottky problem: determine which complex principally polarized abelian varieties (p.p.a.v.) arise as Jacobians of complex curves. This problem has a long history, going back to the work of Riemann, and there are many results. For a recent summary, see Grushevsky's survey [8]. More precisely, in this talk we would like to approach this problem using singularities of theta divisors, which can be traced back to the work of Andreotti-Mayer [1]. There is a precise question posed by Casalaina-Martin in 2008 [4, Question 4.7].

Question 1. Let (A, Θ) be a principally polarized abelian variety. If (A, Θ) is indecomposable as p.p.a.v., it is true that

$$\dim \operatorname{Sing}_m(\Theta) \leq \dim A - 2m + 1, \quad \forall m \geq 2? \tag{1}$$

Here $\operatorname{Sing}_m(\Theta) := \{x \in \Theta \mid \operatorname{mult}_x(\Theta) \geq m\}$. Moreover, if the equality is achieved in (1) by any $m \geq 2$, is it true that A is either the Jacobian of a hyperelliptic curve or the intermediate Jacobian of a cubic threefold?

If this question is true, then it implies a conjecture of Debarre, proposed by Grushevsky [8, Conjecture 5.5] and a conjecture of Grushevsky [8, Conjecture 5.12]. Here are some evidences.

(1) If $A = \operatorname{Jac}(C)$ for a smooth projective curve C, then it is true by the Riemann Singularity Theorem and Marten's work on Brill-Noether varieties.