APPLICATIONS OF THE OHSAWA-TAKEGOSHI EXTENSION THEOREM TO DIRECT IMAGE PROBLEMS

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Abstract. In the first part of the paper, we study a Fujita-type conjecture by Popa-Schnell, and obtain an effective bound on the generic global generation of direct images of twisted pluricanonical bundles. We also point out the relation between the Seshadri constant and the optimal bound. In the second part, we give an affirmative answer to a question by Demailly-Peternell-Schneider in a more general setting. As byproducts of our proofs, we extend a result by Fujino-Gongyo on images of weak Fano manifolds to the Kawamata log terminal settings, and refine a theorem by Broustet-Pacienza on the rational connectedness of the image.

1. Introduction

The first goal of this paper is to study the following conjecture by Popa-Schnell on the global generation of push-forwards of pluricanonical bundles twisted by ample line bundles.

Conjecture 1.1 ([PS14, Conjecture 1.3])**.** *Let $f : X \to Y$ be a surjective morphism of projective manifolds, with $\dim Y = n$, and let A be an ample line bundle on Y. Then, for every $k \geq 1$, the sheaf*

$$f_*(K_X^{\otimes k}) \otimes A^\ell$$

is globally generated for any $\ell \geq k(n+1)$.

In [PS14], Popa-Schnell proved the conjecture in the case when A is an ample and globally generated line bundle, and in general when $\dim X = 1$. In a recent preprint [Dut17], Dutta was able to remove the global generation assumption on A making a statement about generic global generation with weaker bound on the twist, as in the work of Angehrn and Siu [AS95], on the effective freeness of adjoint bundles. Her theorem is as follows:

Theorem 1.2 (Dutta)**.** *Let (X, Δ) be a Kawamata log terminal (*klt *for short) $\mathbb{Q}$-pair, and let Y be a projective manifold with $\dim Y = n$ equipped with an ample line bundle A. Assume that $f : X \to Y$ is a surjective morphism. Then for any positive integer m such that $m(K_X + \Delta)$ is a Cartier divisor, the sheaf*

$$f_*\big(m(K_X + \Delta)\big) \otimes A^\ell$$

is generated by global sections at a general point $y \in Y$, either

(a) for all $\ell \geq m\binom{n+1}{2} + m$.

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or

(b) for all $\ell \geq m(n+1)$ when $n \leq 4$.

Let us mention that $\binom{n+1}{2}$ is the Angehrn-Siu type bound in their work on the Fujita conjecture [AS95].

Inspired by the work of Demailly [Dem92b] in studying the Fujita conjecture, and that on the pluricanonical extension theorem by Berndtsson-Păun [BP10] and Cao [Cao16], we obtain a better bound for Conjecture 1.1.

Theorem A. *Under the same assumptions as those in Theorem 1.2, the direct image $f_*\big(m(K_X+\Delta)\big)\otimes A^\ell$ is generated by global sections at a general point $y\in Y$ if any of the following cases holds:*

(a) *for all $\ell \geq m(n+1)+n^2-n$;*
(b) *for all $\ell \geq n^2+2$ when K_Y is pseudo-effective;*
(c) *for all $\ell \geq m(n+1)$ when the Seshadri constant $\varepsilon(A,z)\geq 1$ for very general points $z\in Y$.*

Hence the expected lower bounds for ℓ in Conjecture 1.1 can be achieved if the condition in Theorem A.(c) is satisfied, which is indeed a conjecture of Ein-Lazarsfeld (see Conjecture 2.6).

In [DPS01], Demailly-Peternell-Schneider raised the following question.

Problem 1.3 ([DPS01, Problem 4.13])**.** *Let $f: X\to Y$ be a surjective morphism between normal projective $\mathbb{Q}$-Gorenstein varieties. If $-K_X$ is pseudo-effective and its non-nef locus does not project onto Y, is $-K_Y$ pseudo-effective?*

The second part of the paper is to give an affirmative answer to the above question (see Theorem B.(a) below). Moreover, we give a precise description for the non-nef locus of $-K_Y$ (see Theorem B.(b)).

Theorem B. *Let (X,D) be a $\mathbb{Q}$-pair which is log canonical (*lc *for short). Let Y be a normal projective $\mathbb{Q}$-Gorenstein variety, and let $f: X\to Y$ be a surjective morphism.*

(a) *Assume that $-(K_X+D)$ is pseudo-effective, and the non-nef locus $\mathrm{NNef}\big(-(K_X+D)\big)$ does not project onto Y. Then $-K_Y$ is also pseudo-effective.*
(b) *If we further assume that both X and Y are smooth, with Δ a (not necessarily effective) $\mathbb{Q}$-divisor on Y, such that $-(K_X+D)-f^*\Delta$ is pseudo-effective, with its non-nef locus does not map onto Y. Then $-K_Y-\Delta$ is pseudo-effective with its non-nef locus contained in $f\big(\mathrm{NNef}(-K_X-D-f^*\Delta)\big)\bigcup Z\bigcup Z_D$, where Z is the minimal proper subvariety on Y such that $f: X\setminus f^{-1}(Z)\to Y\setminus Z$ is smooth, and Z_D is an at most countable union of proper subvarieties containing Z such that for every $y\notin Z_D$, the pair $\big(f^{-1}(y), D_{\restriction f^{-1}(y)}\big)$ is lc.*

As a consequence of Theorem B.(b), we reprove the following theorem by Fujino-Gongyo [FG14, Theorem 1.1].

Theorem 1.4 (Fujino-Gongyo)**.** *Let $f: X\to Y$ be a smooth fibration between projective manifolds. Let D be an effective $\mathbb{Q}$-divisor on X such that (X,D) is lc,* $\mathrm{Supp}(D)$ *is simple normal crossing and is relatively normal crossing over Y. Let Δ be a (not necessarily effective) $\mathbb{Q}$-divisor on Y. Assume that $-(K_X+D)-f^*\Delta$ is nef. Then so is $-K_Y-\Delta$.*

As a byproduct of our proof, we obtain the following variant of Theorem B.

Theorem C. *Let (X, D) be a $\mathbb{Q}$-pair, let Y be a normal projective $\mathbb{Q}$-Gorenstein variety, and let $f : X \to Y$ be a surjective projective morphism.*

(a) *Assume that both X and Y are smooth, with (X, D) klt. Let Δ be a (not necessarily effective) $\mathbb{Q}$-divisor over Y, such that $-K_X - D - f^*\Delta$ is big and its non-nef locus $\mathrm{NNef}(-K_X - D - f^*\Delta)$ does not dominate Y, then $-K_Y - \Delta$ is big.*

(b) *Assume that $-(K_X + D)$ is big and the restriction of f to $\mathrm{NNef}(-K_X - D) \bigcup \mathrm{Nklt}(X, D)$ does not dominate Y, then $-K_Y$ is big. Here $\mathrm{Nklt}(X, D)$ denotes the* non-klt locus *of (X, D) (see Definition 2.2).*

Theorem C has several consequences. Combining Theorems C.(a) and B.(b), we first prove a generalization of another theorem by Fujino-Gongyo [FG12, Theorem 1.1].

Corollary D. *Let $f : X \to Y$ be a smooth fibration between projective manifolds. Assume that Δ is a (not necessarily effective) $\mathbb{Q}$-divisor over Y, (X, D) is klt, and $(X_y, D_{\restriction X_y})$ is lc for every $y \in Y$. If $-K_X - D - f^*\Delta$ is big and nef, so is $-K_Y - \Delta$.*

Lastly, we apply Theorem C.(b) to refine a theorem by Broustet-Pacienza on the rational connectedness of the image.

Corollary E. *Let (X, D) be a $\mathbb{Q}$-pair with $-(K_X + D)$ big, and let Y be a normal projective $\mathbb{Q}$-Gorenstein variety. Assume that $f : X \to Y$ is a surjective morphism such that the restriction of f to $\mathrm{NNef}(-K_X - D) \bigcup \mathrm{Nklt}(X, D)$ does not dominate Y. Then $-K_Y$ is big and Y is* rationally connected modulo $\mathrm{NNef}(-K_Y) \cup \mathrm{Nklt}(-K_Y)$*, that is, there exists an irreducible component V of $\mathrm{NNef}(-K_Y) \cup \mathrm{Nklt}(-K_Y)$ such that for any general point y of Y, there exists a rational curve R_y passing through y and intersecting V.*

2. Technical Preliminaries

2.1. Base locus.

Definition 2.1. A $\mathbb{Q}$-*pair* consists of a normal projective variety X, together with a Weil $\mathbb{Q}$-divisor Δ on X, such that the $\mathbb{Q}$-divisor $K_X + \Delta$ is $\mathbb{Q}$-Cartier on X.

In general, if not specially mentioned, we always require that the divisor Δ in Definition 2.1 is effective.

Definition 2.2 (Non-klt locus)**.** For a $\mathbb{Q}$-pair (X, Δ), the non-klt locus of is defined by

$$\mathrm{Nklt}(X, \Delta) := \{x \in X \mid \mathscr{J}(X, \Delta)_x \neq \mathscr{O}_{X,x}\},$$

where $\mathscr{J}(X, \Delta)$ denotes to be the multiplier ideal sheaf of (X, Δ) (see [Laz04b, §9.3.G] for the definition).

Following [BBP13, Definition 1.5], we recall the definitions of stable base locus, non-nef locus and restricted base locus for pseudo-effective $\mathbb{Q}$-divisors on normal projective varieties.

Definition 2.3. Let X be a normal projective variety. Let D be a pseudo-effective $\mathbb{Q}$-divisor on X. The *non-nef locus* of D is defined as

$$\mathrm{NNef}(D) := \{c_X(v) \mid v(\|D\|) > 0\},$$

where $c_X(v)$ denotes the center on X of a given divisorial valuation v. The stable base locus of D is defined by

$$\mathbf{B}(D) := \bigcap_m \mathrm{Bs}(mD),$$

where m is taken over all positive integers such that mD is integral. The *restricted base locus* of D is defined as

$$\mathbf{B}_-(D) := \bigcup_{m>0} \mathbf{B}(D + \frac{1}{m}A),$$

where A is an ample divisor, and $\mathbf{B}(\bullet)$ denotes *stable base locus* of the $\mathbb{Q}$-divisor.

It was proved in [BBP13, Lemma 1.6] that,

$$\mathrm{NNef}(D) \subseteq \mathbf{B}_-(D),$$

and equality was shown to hold when X is smooth.

2.2. **Seshadri constants.** Motivated in part by his study of linear series in connection with the Fujita conjecture, Demailly [Dem92b] introduced the *Seshadri constant* $\varepsilon(L, x)$ to measure the local positivity of the nef line bundle L at a point x. Since it is now a standard tool in algebraic geometry, we will not recall its definition and we refer the readers to [Dem92b, §6] or [Laz04a, §5.1] for further details.

In [Dem92b] Demailly introduced another constant $\gamma(L, x)$ for any nef line bundle L in terms of singular hermitian metrics. Let us first begin with the following definition.

Definition 2.4. (i) A function $\psi : X \to]-\infty, +\infty]$ on a complex manifold X of dimension m is said to be quasi-plurisubharmonic (quasi-psh for short) if ψ is locally the sum of a psh function and of a smooth function (or equivalently, if $\sqrt{-1}\partial\overline{\partial}\psi$ is locally bounded from below). In addition, we say that ψ has neat analytic singularities if every point $x \in X$ possesses an open neighborhood U on which ψ can be written

$$\psi = c \log \sum_j |g_j|^2 + w(z)$$

where $g_j \in \mathscr{O}(U)$, $c \geq 0$ and $w(z) \in \mathscr{C}^\infty(U)$.

(ii) A singular metric h with analytic singularities on the line bundle L is said to have *isolated logarithmic pole of coefficient* $\nu > 0$ at a point $x \in X$, if on a neighborhood U of x with a coordinate system $(z_1, \ldots, z_n)$ centered at x, the local weight φ of h can be written

$$\varphi = \nu \log \sum_{i=1}^n |z_i|^2 + w(z)$$

where $\nu > 0$ and $w(z) \in \mathscr{C}^\infty(U)$. In this setting, we set $\nu(h, x) := \nu$.

Following [Dem92b] we set

$$\gamma(L,x) := \sup_h \nu(h,x),$$

where the supremum is taken over all singular hermitian metrics h of L with positive curvature current, whose local weight φ has neat singularities and logarithmic poles at x.

The numbers $\varepsilon(L,x)$ and $\gamma(L,x)$ will be seen to carry a lot of useful information about the local positivity of L. In case L is big and nef, these two constants coincide outside a certain proper subvariety of X.

Theorem 2.5 ([Dem92b, Theorem 6.4])**.** *Let L be a big and nef line bundle over X. Then we have*

$$\varepsilon(L,x) = \gamma(L,x)$$

for any $x \notin \mathbf{B}_+(L)$, where $\mathbf{B}_+(L)$ is the augmented base locus of L (see [Laz04b, Definition 10.2.2]). In particular, if L is ample, then $\varepsilon(L,x) = \gamma(L,x)$ holds everywhere. □

After the work [Dem92b], Ein-Lazarsfeld systematically studied the Seshadri constant, and they first proved that for any ample line bundle L on a projective surface Y, the Seshadri constant $\varepsilon(L,y) \geq 1$ for a very general point on Y [EL93]. They further conjectured that this should be true in general.

Conjecture 2.6 (Ein-Lazarsfeld)**.** *Let Y be any projective manifold, and L any ample line bundle on Y. Then the Seshadri constant*

$$\varepsilon(L,y) \geq 1$$

at a very general point $y \in Y$.

In [EKL95], Ein-Küchle-Lazarsfeld gave the existence of universal generic bounds for the Seshadri constants in a fixed dimension. However, the bound is suboptimal by a factor of $\dim Y$.

Theorem 2.7 (Ein-Küchle-Lazarsfeld)**.** *Let L be an ample line bundle on an irreducible projective variety Y of dimension n. Then for any given $\delta > 0$, the locus*

$$\{y \in Y \mid \varepsilon(L,y) > \frac{1}{n+\delta}\}$$

contains a Zariski-dense open set in Y. □

2.3. **L^2-Extension Theorem.** Before we state Ohsawa-Takegoshi type Extension Theorem of Demailly, we begin with a definition in [Dem16]. Following Demailly and Nadel, the multiplier ideal sheaf $\mathscr{J}(\psi)$ is the sheaf of germs of holomorphic functions f such that $|f|^2 e^{-2\psi}$ is locally summable. We say that the singularities of ψ are log canonical along the zero variety $Y := V(\mathscr{J}(\psi))$ if $\mathscr{J}\big((1-\varepsilon)\psi\big)_{\restriction Y} = \mathscr{O}_{X\restriction Y}$ for every $\varepsilon > 0$.

If ψ possesses both neat and log canonical singularities, it is easy to show that the zero scheme $V(\mathscr{J}(Y))$ is a reduced variety. In this case, by Ohsawa one can also associate in a natural way a measure $dV_{Y^\circ,\omega}[\psi]$ on the set $Y^\circ := Y^{\mathrm{reg}}$ of regular points of Y as follows. If $g \in \mathscr{C}_c(Y^\circ)$ is a compactly supported continuous function on Y°, and $\tilde{g}$ compactly supported extension of g to X, we set

$$\int_{Y^\circ} g\, dV_{Y^\circ,\omega}[\psi] := \limsup_{t\to-\infty} \int_{x\in X, t<\psi(x)<t+1} \tilde{g}(x)\, dV_{X,\omega}. \tag{2.1}$$

Here ω is a Kähler metric on X, and $dV_{X,\omega} = \frac{\omega^m}{m!}$. In [Dem16] Demailly proved that the limit does not depend on the continuous extension $\tilde{g}$, and one gets in this way a measure with smooth positive density with respect to the Lebesgue measure, at least on an (analytic) Zariski open set in Y°.

We are now in a position to recall the Ohsawa-Takegoshi type extension Theorem by Demailly. We only quote a special case of his very general theorem.

Theorem 2.8 ([Dem16, Theorem 2.8])**.** *Let X be a smooth projective manifold, and ω a Kähler metric on X. Let L be a holomorphic line bundle equipped with a (singular) hermitian metric h on X, and let $\psi : X \to]-\infty, +\infty]$ be a quasi-psh function on X with neat analytic singularities. Assume that Y be a closed analytic subvariety of X so that $V(\mathscr{J}(\psi)) = Y \sqcup Z$, where Z is another closed analytic subvariety of X. If ψ has log canonical singularities along Y, and*

$$\mathrm{i}\Theta_{L,h} + \alpha\sqrt{-1}\partial\bar{\partial}\psi \geq 0$$

for all $\alpha \in [1, 1+\delta]$ and some $\delta > 0$, then for every section $s \in H^0\big(Y^\circ, (K_X \otimes L)_{\restriction Y^\circ}\big)$ on $Y^\circ := Y^{\mathrm{reg}}$ such that

$$\int_{Y^\circ} |s|^2_{\omega,h} dV_{Y^\circ,\omega}[\psi] < +\infty,$$

there is $S \in H^0\big(X, K_X \otimes L \otimes \mathscr{J}(\psi)\big)$ so that $S_{\restriction Y^\circ} = s_{\restriction Y^\circ}$, and it satisfies certain L^2-estimates. □

A direct consequence of Theorem 2.8 is the following extension theorem for surjective morphisms.

Corollary 2.9. *Let $f : X \to Y$ be a surjective morphism between smooth manifolds, and let L be any pseudo-effective line bundle L over X with a positively curved singular hermitian metric h. For any ample line bundle A on Y, and any regular value y of f, if the Seshadri constant*

$$\varepsilon(A, y) > \dim Y =: n, \tag{2.2}$$

and the restriction of h to $X_y := f^{-1}(y)$ is not identically zero, then any section s of

$$H^0\big(X_y, (K_X + L + f^*A)_{\restriction X_y} \otimes \mathscr{J}(h_{\restriction X_y})\big).$$

can always be extended to a global one

$$S \in H^0\big(X, K_X + L + f^*A\big)$$

with certain L^2 estimates.

Proof. Since A is ample over Y, there exists a smooth hermitian metric h_0 on A with the curvature $\sqrt{-1}\Theta_{A,h_0} \geq \omega$, where ω is some Kähler form on Y.

By the assumption that the Seshadri constant $\varepsilon(A, y) > n$, it follows from Theorem 2.5 that we can find a global quasi-psh function φ with neat singularities on Y such that

(i) $\sqrt{-1}\Theta_{A,h_0} + \sqrt{-1}\partial\bar{\partial}\varphi \geq 0$;

(ii) on a neighborhood W of y, we have

$$\varphi = (1+\delta)n \log \sum |z - y|^2 + w(z),$$

where $\delta > 0$ and $w(z) \in \mathscr{C}^\infty(W)$ with $w(y) = 0$. Now set $\psi := \frac{1}{1+\delta}\varphi \circ f$, which is a quasi-psh function with neat singularities on X. Since y is a regular value of f, the scheme-theoretical inverse image $X_y := f^{-1}(y)$ is a finite disjoint union of closed smooth submanifolds of codimension n in X, and the multiplier ideal sheaf $\mathscr{J}(\psi)_{\restriction f^{-1}(W)} = \mathcal{I}_{X_y}$, where $\mathcal{I}_{X_y}$ is the ideal sheaf of the variety X_y. Hence ψ has log canonical singularities along X_y, and we have

$$\mathrm{i}\Theta_{L,h} + \mathrm{i}\Theta_{f^*A, f^*h_0} + \alpha\sqrt{-1}\partial\bar{\partial}\psi \geq 0$$

for all $\alpha \in [1, 1+\delta]$. Then for any section s of

$$H^0\big(X_y, (K_X + L + f^*A)_{\restriction X_y} \otimes \mathscr{J}(h_{\restriction X_y})\big),$$

one applies Theorem 2.8 to extend s to a global section

$$S \in H^0\big(X, (K_X + L + f^*A) \otimes \mathscr{J}(h)\big).$$

The proof is accomplished. □

2.4. **L^2-extension theorem for twisted pluricanonical bundles.** We recall the following twisted pluricanonical extension theorem, which was used by J. Cao in [Cao16, Theorem 2.10] to prove the local triviality of Albanese maps of projective manifolds with nef anticanonical bundles. It is based on previous work by Berndtsson and Păun.

Theorem 2.10 (Cao)**.** *Let $f : X \to Y$ be a surjective morphism between projective manifolds X and Y. Let A_Y be a line bundle on Y such that the difference $A_Y - K_Y$ is an ample line bundle, and let L be a pseudo-effective line bundle on X equipped with a positively curved singular metric h_L. Assume that there exists some regular value z of f, we have*

(i) *all the sections of the bundle $mK_{X_z} + L$ extend near z,*

(ii) $H^0\big(X_z, (mK_{X_z} + L_{\restriction X_z}) \otimes \mathscr{J}(h^{\frac{1}{m}}_{L\restriction X_z})\big) \neq \varnothing$.

Then for any $y \in Y$ such that

(a) *y is the regular value of f,*

(b) *the Seshadri constant $\varepsilon(A_Y - K_Y, y) > n$,*

(c) *all the sections of the bundle $(mK_X + L)_{\restriction X_y}$ extend locally near y,*

the restriction map

$$H^0(X, mK_{X/Y} + L + f^*A_Y) \to H^0\big(X_y, (mK_{X_y} + L_{\restriction X_y}) \otimes \mathscr{J}(h^{\frac{1}{m}}_{L\restriction X_y})\big) \tag{2.3}$$

is surjective.

Since [Cao16, Theorem 2.10] does not state the precise locus for y where (2.3) holds (though it is implicit), and the positivity condition for $A_Y - K_Y$ therein is slightly stronger than Condition (b), we provide a quick proof here.

Proof of theorem 2.10. Thanks to [BP10, A.2.1], Conditions (i) and (ii) imply that there exists a m-relative Bergman type metric $h_{m,B}$ on $mK_{X/Y} + L$ with respect to h_L such that $\sqrt{-1}\Theta_{h_{m,B}}(mK_{X/Y} + L) \geq 0$. Thus $h := h^{\frac{m-1}{m}}_{m,B} \cdot h^{\frac{1}{m}}_L$ defines a possible singular metric on

$$\widetilde{L} := \frac{m-1}{m}(mK_{X/Y} + L) + \frac{1}{m}L = (m-1)K_{X/Y} + L,$$

with $i\Theta_h(\widetilde{L}) \geq 0$.

Take any $s \in H^0\big(X_y, (mK_{X_y} + L_{\upharpoonright X_y}) \otimes \mathscr{J}(h_{L\upharpoonright X_y}^{\frac{1}{m}})\big)$. By Conditions (a), (b) and (c), as well as the construction of the m-relative Bergman kernel metric, $|s|^2_{h_{m,B}}$ is $\mathscr{C}^0$-bounded. Hence

$$\begin{aligned} \int_{X_y} |s|^2_{\omega,h} dV_{X_y,\omega} &= \int_{X_y} |s|^{\frac{2(m-1)}{m}}_{h_{m,B}} |s|^{\frac{2}{m}}_{\omega,h_L^{\frac{1}{m}}} dV_{X_y,\omega} \\ &\leq C \int_{X_y} |s|^{\frac{2}{m}}_{\omega,h_L^{\frac{1}{m}}} dV_{X_y,\omega} < +\infty. \end{aligned}$$

We then can apply Corollary 2.9 to $K_X + \widetilde{L} + f^*(A_Y - K_Y)$, to extend s to a section in $H^0(X, K_{X/Y} + \widetilde{L} + f^*A_Y)$. In conclusion, the restriction

$$H^0(X, mK_{X/Y} + L + f^*A_Y) \to H^0\big(X_y, (mK_{X_y} + L_{\upharpoonright X_y}) \otimes \mathscr{J}(h_{L\upharpoonright X_y}^{\frac{1}{m}})\big)$$

is surjective and the theorem is proved. □

3. On the conjecture of Popa-Schnell

Let us first recall the following fundamental result in birational geometry:

Theorem 3.1. *Let A be an ample line bundle over a projective n-fold Y, then the adjoint line bundle $K_Y + (n+1)A$ is semi-ample.* □

Based on the Mori theory, one observes that $n+1$ is the maximal length of extremal rays of smooth projective n-folds, which shows that $K_Y + (n+1)A$ is nef. By the base-point-free theorem, one can even show that $K_Y + (n+1)A$ is semi-ample. It is noticeable that in his work on the Fujita conjecture [Dem96], Demailly gave an analytic proof for the nefness of $K_Y + (n+1)A$.

Proof of Theorem A. Take a a log resolution $\mu : X' \to X$ of (X, Δ) such that

$$K_{X'} = \mu^*(K_X + \Delta) + \sum_i a_i E_i - \sum_j b_j F_j,$$

where $a_i, b_j \in \mathbb{Q}_+$, and $\sum_{i,j} E_i + F_j$ is a divisor with simple normal crossing support. By the assumption that (X, Δ) is klt and Δ is effective, each E_i is an exceptional divisor and $0 < b_j < 1$ for each b_j. Then $f' : X' \to Y$ is a surjective morphism between projective manifolds, where $f' := f \circ \mu$.

It follows from Theorem 3.1 that $K_Y + (n+1)A$ can be equipped with a smooth hermitian metric h_1 with semi-positive curvature. If we further assume that K_Y is pseudo-effective, $(m-1)K_Y + A$ is big for any $m \geq 1$ and thus can be equipped with a singular hermitian metric h with neat singularities such that $\sqrt{-1}\Theta_h \geq \varepsilon\omega$ for some hermitian form ω over Y. Observe that if $m(K_X + \Delta)$ is a Cartier divisor, then $ma_i, mb_j \in \mathbb{Z}^+$ for each a_i, b_j. Let us equip the divisor $\sum_j mb_jF_j$ with the canonical singular hermitian metric h_2 so that $\sqrt{-1}\Theta_{h_2} = \sum_j mb_j[F_j]$. For any point $y \in Y$, we denote by X'_y the fiber of $f' : X' \to Y$. In general, we write $P := (m-1)f^*\big(K_Y + (n+1)A\big) + \sum_j mb_jF_j$, which is equipped with the positively curved singular hermitian metric $h_P := f^*h_1^{m-1} \cdot h_2$; and when K_Y is pseudo-effective, we denote by $P := f'^*\big((m-1)K_Y + A\big) + \sum_j mb_jF_j$, which is endowed with the positively curved singular hermitian metric $h_P := f^*h \cdot h_2$. Recall that $\sum_j F_j$ is simple normal crossing and $0 < b_j < 1$. Then for general $y \in Y$, $\mathscr{J}(h_{P\restriction X'_y}^{\frac{1}{m}}) = \mathscr{O}_{X'_y}$. By Theorem 2.7, when $\ell \geq n^2+1$, the Seshadri constant

$$\varepsilon\big(\ell L, y\big) > n$$

for a general $y \in Y$. We apply Theorem 2.10 to show the surjectivity

$$H^0\big(X', mK_{X'/Y} + P + f'^*(K_Y + \ell A)\big) \to H^0\big(X'_y, (mK_{X'} + \sum_j mb_j F_j)_{\restriction X_y}\big)$$

for a general point $y \in Y$. Hence for general $y \in Y$,

$$H^0\big(X', \mu^*(mK_X + m\Delta + \ell f^*A) + \sum_i ma_i E_i\big) \to H^0\Big(X'_y, m\big(\mu^*(K_X+\Delta) + \sum_i a_i E_i\big)_{\restriction X'_y}\Big)$$

is surjective for any $\ell \geq m(n+1) + n^2 - n$ in the general cases, and for $\ell \geq n^2+2$ when K_Y is pseudo-effective. Since each E_i is an exceptional divisor of the birational morphism $\mu : X' \to X$, the natural inclusion

$$H^0\big(X', \mu^*(mK_X + m\Delta + \ell f^*A)\big) \xrightarrow{\simeq} H^0\big(X', \mu^*(mK_X + m\Delta + \ell f^*A) + \sum_i ma_i E_i\big)$$

is an isomorphism. Hence one also has the surjectivity

$$H^0\big(X', \mu^*(mK_X + m\Delta + \ell f^*A)\big) \to H^0\big(X'_y, \mu^*(mK_X + m\Delta)_{\restriction X'_y}\big)$$

for the above ℓ. In other words, the direct image

$$f'_*\big(\mu^*(mK_X + m\Delta + \ell f^*A)\big) = f_*(mK_X + m\Delta) \otimes A^\ell$$

is generated by global sections at general points of Y when $\ell \geq m(n+1) + n^2 - n$, and for $\ell \geq n^2+2$ when K_Y is pseudo-effective.

It follows from the above proof that, when Conjecture 2.6 holds for Y, one can take $\ell \geq m(n+1)$. This completes the proof of the theorem. □

Remark 3.2. After the present paper was made available on arXiv, very recently Iwai [Iwa17] and Dutta-Murayama [DM19] pursued our general strategies in the proof of Theorem A in combination with the Angehrn-Siu methods so that they were able to slightly improve the lower bound in Theorem A.(a) to $\ell \geq m(n+1) + \frac{n^2-n}{2}$.

4. On the question of Demailly-Peternell-Schneider

This section is devoted to prove Theorem B, and in particular we give an affirmative answer to Problem 1.3.

Proof of Theorem B. Let us first prove Theorem B.(b). We denote by Z the minimal proper subvariety of Y so that f is smooth over $Y \setminus Z$. Take a sufficiently divisible $q \in \mathbb{N}$ such that both $q\Delta$ and qD are Cartier divisors, and a sufficient ample line bundle A_X on X such that

(1) $A_X + qD$ is ample and is equipped with a smooth hermitian metric h with $\sqrt{-1}\Theta_h(A_X + qD) \geq 3\omega$ for some Kähler metric ω.
(2) The direct image $f_*(A_X)$ is a torsion free coherent sheaf which is locally free and globally generated over the Zariski open set $Y \setminus Z$.

Then $f_*(A_X)$ is locally free outside a subvariety $W \subseteq Z$ of codimension at least 2. Denote by r the generic rank of $f_*(A_X)$, and define

$$\det f_*(A_X) := \Lambda^r f_*(A_X)^{\star\star}$$

to be the bidual of $\Lambda^r f_*(A_X)$, which is an invertible sheaf over Y. Then there is a coherent ideal sheaf $\mathcal{I}$ supported on W such that

$$\Lambda^r f_*(A_X) = \det f_*(A_X) \otimes \mathcal{I}.$$

Pick a very ample line bundle A_Y on Y such that $A_Y - K_Y$ generates $(n+1)$-jets everywhere and $rA_Y + \det f_*(A_X)$ is also ample. In particular, the Seshadri constant $\varepsilon(A_Y - K_Y, y) > n$ for any y.

It follows from [BDPP13] that for any pseudo-effective line bundle E,

$$\mathbf{B}_-(E) = \bigcup_{m\in\mathbb{N}} \bigcap_T E_+(T),$$

where T runs over the set $c_1(E)[-\frac{1}{m}\omega]$ of all closed real $(1,1)$-currents $T \in c_1(E)$ such that $T \geq -\frac{1}{m}\omega$, and $E_+(T)$ denotes the locus where the Lelong numbers of T are strictly positive. By [Bou02], there is always a current $T_{\min,m}$ which achieves minimum singularities and minimum Lelong numbers among all members of $c_1(E)[-\frac{1}{m}\omega]$, and thus

$$\mathbf{B}_-(E) = \bigcup_{m\in\mathbb{N}} E_+(T_{\min,m}).$$

By Demailly's regularization theorem in [Dem92a], for every $m \in \mathbb{N}$, we can find a closed (1,1)-current $T_m \in c_1(E)$ with neat singularities such that $T_m \geq -\frac{2}{m}\omega$, and thus

$$E_+(T_{\min,2m}) \subset E_+(T_m) \subset E_+(T_{\min,m}).$$

In other words, there exists a singular hermitian metric $\tilde{h}_m$ on E with neat singularities, such that the curvature current

$$\sqrt{-1}\Theta_{\tilde{h}_m}(E) = T_m \geq -\frac{2}{m}\omega.$$

Set $E := -q(K_X + D) - f^*(q\Delta)$. Since $\mathbf{B}_-(E) = \mathbf{B}_-\big(-(K_X + D) - f^*\Delta\big)$ does not project onto Y, thus for any $m \in \mathbb{N}$, $Z_m := f\big(E_+(T_m)\big)$ is a proper subvariety of Y, and the singular hermitian metric $\tilde{h}_m^{\otimes m} h$ of $-mq(K_X + D) - mqf^*\Delta + A_X + qD$ is smooth on $X \setminus f^{-1}(Z_m)$.

Let h_D be the canonical singular hermitian metric of the effective divisor qD so that the curvature current

$$\sqrt{-1}\Theta_{h_D} = [qD] \geq 0,$$

where $[qD]$ denotes to be the (1,1)-current of integration associated to qD.

Denote by Z_D the minimal set containing Z, such that for every $y \notin Z_D$, the pair $(X_y, D_{\restriction X_y})$ is also lc. Then Z_D is an at most countable union of proper subvarieties of Y. Indeed, since (X, D) is lc, for each $m \in \mathbb{N}$, the set

$$Y_m := \{y \notin Z \mid (X_y, (1 - \frac{1}{m})D_{\restriction X_y}) \text{ is klt}\}$$

is an Zariski open set of Y. Therefore, one has

$$Z_D = \bigcup_{m=1}^{\infty} Y \setminus Y_m.$$

For the singular hermitian metric $h_m := \tilde{h}_m^{\otimes m} h h_D^{\otimes m-1}$ on $-mqK_X - mqf^*\Delta + A_X$, the multiplier ideal sheaf

$$\mathscr{J}(h_{m\restriction X_y}^{\frac{1}{qm}}) = \mathscr{J}((1-\frac{1}{m})D_{\restriction X_y}) = \mathscr{O}_{X_y} \tag{4.1}$$

for any $y \in Y_m \setminus Z_m$, and the curvature current $\sqrt{-1}\Theta_{h_m} \geq \omega$.

Denote $L := -mqK_X - mqf^*\Delta + A_X$. For any $y \in Y_m \setminus Z_m$, all the sections of the bundle $(mqK_X + L)_{\restriction X_y} = A_{X\restriction X_y}$ extend locally near y, and thus Conditions (a), (b) and (c) in Theorem 2.10 are all satisfied. It then follows from (4.1) and Theorem 2.10 that the restriction

$$H^0(X, mqK_{X/Y} + L + f^*A_Y) \to H^0(X_y, A_{X\restriction X_y})$$

is surjective for any $y \in Y_m \setminus Z_m$. In other words, the direct image sheaf

$$f_*\big(mqK_{X/Y} - mqK_X - mqf^*\Delta + A_X + f^*A_Y\big) = (-K_Y - \Delta)^{\otimes mq} \otimes A_Y \otimes f_*(A_X) \tag{4.2}$$

is generated by global sections over $Y_m \setminus Z_m$. By the assumption that $f_*(A_X)$ is locally free over $Y \setminus Z$, we conclude that the top exterior power

$$\Lambda^r\big((-K_Y - \Delta)^{\otimes mq} \otimes A_Y \otimes f_*(A_X)\big) = (-K_Y - \Delta)^{\otimes rmq} \otimes A_Y^{\otimes r} \otimes \det f_*(A_X) \otimes \mathcal{I}$$

is also generated by global sections over $Y_m \setminus Z_m$. In particular, for every $m \in \mathbb{N}$, the base locus

$$\mathrm{Bs}\big(mrq(-K_Y - \Delta) + rA_Y + \det f_*(A_X)\big) \subset Z_m \bigcup Y \setminus Y_m. \tag{4.3}$$

By our choice of A_Y, $rA_Y + \det f_*(A_X)$ is an ample line bundle on Y. Let m tend to infinity, we conclude the pseudo-effectivity of $-K_Y - \Delta$. Moreover, it follows from (4.3) that the restricted base locus

$$\mathbf{B}_-(-K_Y - \Delta) \subset \bigcup_{m=1}^{\infty} Z_m \bigcup Y \setminus Y_m = f\big(\mathbf{B}_-(-K_X - D - f^*\Delta)\big) \bigcup Z_D.$$

Hence Theorem B.(b) is proved.

Let us prove Theorem B.(a). Let $p : Y' \to Y$ be a log-resolution of (Y, Δ), and $\mu : X' \to X$ be a log resolution of (X, D), such that the induced rational map $f' : X' \to Y'$ is in fact a morphism. We have the following commutative diagram:

$$\begin{array}{ccc} X' & \xrightarrow{\mu} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{p} & Y. \end{array}$$

Since (X, D) is lc, $K_{X'} + D' = \mu^*(K_X + D) + F$, where D' and F are both effective $\mathbb{Q}$-divisors without common components and $(D' + F)_{\mathrm{red}}$ is simple normal crossing. Moreover, the coefficients of any prime divisor appearing in D' is at most 1, and F

is exceptional. For any $q \in \mathbb{N}$ such that $q(K_X + D)$ is a Cartier divisor, both qF and qD' are also Cartier. By [BBP13, Lemma 2.6], one has

$$\mu\Big(\mathrm{NNef}\big(-\pi^*(K_X + D)\big)\Big) \subset \mathrm{NNef}\big(-(K_X + D)\big).$$

By the assumption of the theorem, we have

$$f'\big(\mathrm{NNef}(-K_{X'} - D' + F)\big) \subsetneq Y'. \tag{4.4}$$

Repeating the same proof as that of Theorem B.(b) with $K_X + D + f^*\Delta$ replaced by $K_{X'} + D' - F$, one can show that, after one fixes certain ample divisors $A_{X'}$ and $A_{Y'}$ over X' and Y', for any $m \in \mathbb{N}$, the restriction

$$H^0(X', mqK_{X'/Y'} - mqK_{X'} + mqF + A_{X'} + f'^*A_{Y'}) \to H^0(X'_y, (mqF + A_{X'})_{\upharpoonright X'_y})$$

is surjective for a general point y in Y. Since qF is an effective exceptional divisor, the natural inclusion

$$H^0(X', mqK_{X'/Y'} - mqK_{X'} + A_{X'} + f'^*A_{Y'}) \xrightarrow{\simeq} H^0(X', mqK_{X'/Y'} - mqK_{X'} + mqF + A_{X'} + f'^*A_{Y'})$$

is an isomorphism, and thus the restriction

$$H^0(X', mqK_{X'/Y'} - mqK_{X'} + A_{X'} + f'^*A_{Y'}) \to H^0(X'_y, A_{X' \upharpoonright X'_y})$$

is also surjective for a general point y in Y'. By the same proof as above, we conclude that $-K_{Y'}$ is pseudo-effective, and it follows from Lemma 4.1 below that $-K_Y$ is pseudo-effective as well. We finish the proof of Theorem B.(a). $\square$

Lemma 4.1. *Let $\mu : Y' \to Y$ be a birational morphism from a projective manifold Y' to the normal $\mathbb{Q}$-Gorenstein variety Y. When $-K_{Y'}$ (resp. $K_{Y'}$) is pseudo-effective or big, so is $-K_Y$ (resp. K_Y).*

Proof. For any sufficiently divisible $m \in \mathbb{N}$ such that mK_Y is Cartier, there exists effective exceptional divisors E and F on Y' such that

$$\mu^*(-mK_Y) = -mK_{Y'} + E - F.$$

Take an ample divisor A over Y such that, for some effective exceptional divisors G, $\mu^*A - G$ is also ample over Y'. When $-K_{Y'}$ is pseudo-effective, $-mK_{Y'} + \mu^*(A) - G$ is big, and thus for a sufficiently large $\ell \in \mathbb{N}$, there exists a non-zero section

$$s \in H^0\big(Y', \ell\mu^*(A - mK_Y) - \ell G + \ell F - \ell E\big).$$

Write s_E and s_G for canonical sections of $H^0\big(Y', \mathscr{O}_{Y'}(E)\big)$ and $H^0\big(Y', \mathscr{O}_{Y'}(G)\big)$ defining G and E. The non-zero section $s \cdot s_E^\ell \cdot s_G^\ell \in H^0\big(Y', \ell\mu^*(A - mK_Y) + \ell F\big)$ gives rise to a non-zero section $s' \in H^0(Y, \ell A - \ell mK_Y)$. Since m can be chosen arbitrarily large, we conclude that $-K_Y$ is pseudo-effective. The proofs of the remaining assertions are similar, and we left them as an exercise to the readers. $\square$

Let us explain how to derive Theorem 1.4 from Theorem B.(b). Assume that f is a smooth fibration, $\mathrm{Supp}(D)$ is both simple normal crossing and relatively normal crossing over Y. When (X, D) is lc, $(X_y, D_{\upharpoonright X_y})$ is also lc for every $y \in Y$. In particular, $Z_D = \varnothing$. If $-K_X - D - f^*\Delta$ is nef, $\mathbf{B}_-(-K_X - D - f^*\Delta) = \varnothing$. By Theorem B.(b), $\mathbf{B}_-(-K_Y - \Delta) = \varnothing$, which is equivalent to the nefness of $-K_Y - \Delta$.

Remark 4.2. In [CZ13], M. Chen and Q. Zhang proved a similar result as Theorem B.(a), under the stronger assumption that $-(K_X + D)$ is nef. In a very recent preprint [Ou17], W. Ou extended the theorem by Chen-Zhang to the rational dominant maps, which was a crucial step in his proof of the *generic nefness conjecture* for tangent sheaves by T. Peternell [Pet12, Conjecture 1.5].

5. On the images of varieties with big anti-canonical bundle

5.1. On the images of weak klt Fano pairs. In this subsection, we prove the first part of Theorem C.

Proof of Theorem C.(a). Take a very ample line bundle A_Y over Y such that A_Y generates $n+1$ jets everywhere. Since $-K_X - D - f^*\Delta$ is big, we can find a a sufficiently divisible $a \in \mathbb{N}$ such that $-a(K_X + D + f^*\Delta) - 2f^*A_Y$ is an effective line bundle. Fix any effective divisor $E \in |-a(K_X + D + f^*\Delta) - 2f^*A_Y|$. Since (X, D) is klt, then there exists a sufficiently large and divisible integer $m > a$ such that the multiplier ideal sheaves

$$\mathscr{J}\big(\frac{1}{m-1}E_{\restriction X_y}\big) = \mathscr{J}\big(\frac{m}{m-1}D_{\restriction X_y}\big) = \mathscr{O}_{X_y} \tag{5.1}$$

for general fibers X_y, and both mD and $m\Delta$ are integral. We can also find a singular hermitian metric h_1 with neat singularities on $-(m^2-a)(K_X + D + f^*\Delta)$ such that $\sqrt{-1}\Theta_{h_1} \geq \tilde{\omega}$ for some Kähler metric $\tilde{\omega}$ on X. Take some small rational number $\varepsilon > 0$ such that $\mathscr{J}(h^{\varepsilon}_{1\restriction X_y}) = \mathscr{O}_{X_y}$ for general fibers X_y.

On the other hand, since the non-nef locus $\mathrm{NNef}(-K_X - D - f^*\Delta)$ does not project onto Y, it follows from the proof of Theorem B that, one can find a singular hermitian metric h_ε over $-(m^2-a)(K_X + D + f^*\Delta)$ with neat singularities, such that $\sqrt{-1}\Theta_{h_\varepsilon} \geq -\varepsilon\tilde{\omega}$ and the singularities of h_ε does not project onto Y. Set $h := h_1^{\varepsilon} h_\varepsilon^{1-\varepsilon}$ which is also a hermitian metric on $-(m^2-a)(K_X + D + f^*\Delta)$, then we have $\sqrt{-1}\Theta_h \geq \varepsilon^2\tilde{\omega}$ and the multiplier ideal sheaf

$$\mathscr{J}(h_{\restriction X_y}) = \mathscr{O}_{X_y} \tag{5.2}$$

for general fibers X_y.

We equip the line bundle $-m^2(K_X + D + f^*\Delta) - 2f^*A_Y + m^2D$ with the singular hermitian metric $h_0 := h_E h h_D^{\otimes m^2}$, where h_E (resp. h_D) is the canonical singular hermitian metric of $-a(K_X + D + f^*\Delta) - 2f^*A_Y$ (resp. D) such that

$$\sqrt{-1}\Theta_{h_E} = [E] \quad (\text{resp.} \sqrt{-1}\Theta_{h_D} = [D]).$$

Then the multiplier ideal sheaf $\mathscr{J}(h^{\frac{1}{m^2}}_{0\restriction X_y}) = \mathscr{O}_{X_y}$ for the general smooth fiber X_y satisfying (5.1) and (5.2). Indeed, for any $s \in \mathscr{O}_{X_y,z}$, let φ_E, φ_D and φ be the weights of the metric h_E, h_D and h on a small neighborhood $U \subseteq X_y$ of a point $z \in X_y$. Then by the Hölder inequality we have

$$\int_U |s|^2 e^{-\frac{\varphi_E+\varphi}{m^2}+\varphi_D} \leq \big(\int_U |s|^2 e^{-\varphi}\big)^{\frac{1}{m^2}} \cdot \big(\int_U |s|^2 e^{-\frac{\varphi_E}{m-1}}\big)^{\frac{m-1}{m^2}} \cdot \big(\int_U |s|^2 e^{-\frac{m}{m-1}\varphi_D}\big)^{\frac{m-1}{m}} < +\infty.$$

By Theorem 2.10 with $L := -m^2(K_X + f^*\Delta) - 2f^*A_Y$ endowed with the positively-curved singular hermitian metric h_0, the restriction

$$H^0\big(X, m^2K_{X/Y} + (-m^2K_X - m^2f^*\Delta - 2f^*A_Y) + f^*A_Y\big) \twoheadrightarrow H^0\big(X_y, \mathscr{O}_{X_y}\big) \simeq \mathbb{C}^{\ell}, \tag{5.3}$$

is surjective for general $y \in Y$, where ℓ denotes the number of connected components of X_y. In particular, we have the non-vanishing

$$H^0\big(X, f^*(-m^2K_Y - m^2\Delta - A_Y)\big) \neq 0.$$

Let us show that $-m^2K_Y - m^2\Delta - A_Y$ is pseudo-effective. Take a Stein factorization of f

$$X \xrightarrow{f'} Y' \xrightarrow{p} Y, \quad f = p \circ f'$$

so that $p : Y' \to Y$ is a finite surjective morphism and $f'_*(\mathscr{O}_X) = \mathscr{O}_{Y'}$. Then f' induces an isomorphism

$$f'_* : H^0\big(X, f^*(-m^2K_Y - m^2\Delta - A_Y)\big) \xrightarrow{\simeq} H^0\big(Y', p^*(-m^2K_Y - m^2\Delta - A_Y)\big),$$

which implies that the line bundle $p^*(-m^2K_Y - m^2\Delta - A_Y)$ is effective. Since $p : Y' \to Y$ is a finite surjective morphism, by [Bou02, Proposition 4.2], $-m^2K_Y - m^2\Delta - A_Y$ is a pseudo-effective line bundle, which in particular implies that the $\mathbb{Q}$-divisor $-K_Y - \Delta$ is big. □

A projective manifold X is said *weakly Fano* if $-K_X$ is big and nef. In the series of articles [FG12, FG14], Fujino-Gongyo studied the image of weak Fano manifolds.

Theorem 5.1 (Fujino-Gongyo). *Let $f : X \to Y$ be a smooth fibration between two smooth manifolds X and Y. If X is weak Fano, then so is Y.*

Based on Theorem C.(a), we can extend Theorem 5.1 to *weak klt Fano pairs*, *i.e.* klt $\mathbb{Q}$-pairs (X, D) with $-(K_X + D)$ big and nef.

Proof of Corollary D. Since f is a smooth fibration, (X, D) is klt, and $(X_y, D_{\upharpoonright X_y})$ is also klt for every $y \in Y$, from the very definition of Z_D in Theorem B.(b) we see that $Z_D = \varnothing$. By the nefness of $-(K_X + D) - f^*\Delta$, the set

$$\mathbf{B}_-\big(-(K_X + D) - f^*\Delta\big) = \varnothing.$$

By Theorem B.(b) we conclude that $-K_Y - \Delta$ is nef. The bigness of $-K_Y - \Delta$ follows from Theorem C.(a). The proof is accomplished. □

Note that Theorem 5.1 follows by setting $D = 0$ and $\Delta = 0$ in Corollary D. If we only assume that $-K_X$ is big, then the following example in [FG12] shows that, even if f is smooth, $-K_Y$ is not big.

Example 5.2 (Fujino-Gongyo). Let $E \subset \mathbb{P}^2$ be a smooth cubic curve. Consider $f : X = \mathbb{P}_E(\mathscr{O}_E \oplus \mathscr{O}_E(1)) \to E = Y$. Then, we see that $-K_X$ is big. However, $-K_Y$ is not big since E is a smooth elliptic curve.

5.2. **On the rational connectedness of the image.** By Mori's bend-and-break, Fano varieties are uniruled; in fact by [Cam92, KMM92] a stronger result holds: the projective Fano variety is rationally connected. Afterwards, Q. Zhang [Zha06] and Hacon-McKernan [HM07] proved that the same conclusion holds for weak klt Fano pairs. This was generalized by Broustet-Pacienza [BP11, Theorem 1.2], who proved that a klt pair (X, D) with $-(K_X+D)$ big is rationally connected modulo the non-nef locus of $-(K_X+D)$: there exists an irreducible component V of $\mathbf{B}_-(-K_X-D)$ such that for any general point x of X there exists a rational curve R_x passing through X and intersecting V. As an application, they proved the following theorem.

Theorem 5.3 (Broustet-Pacienza)**.** *Let (X, D) be a $\mathbb{Q}$-pair such that $-(K_X + D)$ is big. Let $f : X \dashrightarrow Y$ be a dominant rational map with connected fibers such that the restriction of f to $\mathrm{NNef}\big(-(K_X + D)\big) \bigcup \mathrm{Nklt}(X, D)$ does not dominate Y, then Y is uniruled.*

In this subsection, we will refine their results in a more general setting. We first prove Theorem C.(b).

Proof of Theorem C.(b). Let $p : Y' \to Y$ be a log-resolution of singularities of Y, and let $\pi : X' \to X$ be a log resolution of (X, D), such that the induced rational map $f' : X' \to Y'$ is in fact a morphism. We have the following commutative diagram:

$$\begin{array}{ccc} X' & \xrightarrow{\pi} & X \\ \Big\downarrow{\scriptstyle f'} & & \Big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow{p} & Y. \end{array}$$

Then we can write $K_{X'} = \pi^*(K_X + D) + \sum_i a_i F_i - \sum_j b_j E_j$, where $a_i, b_j \in \mathbb{Q}_+$, and $\sum_{i,j} F_i + E_j$ is a simple normal crossing divisor. Moreover, each F_i is π-exceptional. Write $D' := \sum_j b_j E_j$ and $F := \sum_i a_i F_i$. By Definition 2.2, one has

$$\mathrm{Nklt}(X, D) = \bigcup_{\{j | b_j \geq 1\}} \pi(E_j),$$

which does not project onto Y by the assumption. Hence

$$f'\big(\mathrm{Nklt}(X', D')\big) = f'\Big(\bigcup_{\{j | b_j \geq 1\}} E_j\Big)$$

is a proper subvariety of Y'.

On the other hand, by [BBP13, Lemma 2.6], one has

$$\pi\big(\mathrm{NNef}(-K_{X'} - D' + F)\big) = \pi\Big(\mathrm{NNef}\big(-\pi^*(K_X + D)\big)\Big) \subseteq \mathrm{NNef}\big(-(K_X + D)\big).$$

It then follows from the assumption that

$$f'\big(\mathrm{NNef}(-K_{X'} - D' + F) \bigcup \mathrm{Nklt}(X', D')\big) \subsetneq Y' \tag{5.4}$$

We do the same proof as that of Theorem C.(a) with $K_X + D + f^*\Delta$ replaced by $K'_X + D' - F$, and we can prove that for sufficiently large and divisible $m \in \mathbb{N}$, the restriction

$$H^0\big(X', m^2 K_{X'/Y'} + (-m^2 K_{X'} - 2f'^* A_{Y'} + m^2 F) + f'^* A_{Y'}\big) \twoheadrightarrow H^0\big(X'_y, m^2 F_{\restriction X'_y}\big) \neq 0$$

is surjective as (5.3), where $A_{Y'}$ is a sufficiently ample line bundle on Y'. In particular, we have the non-vanishing

$$H^0\big(X', f'^*(-m^2 K_{Y'} - A_{Y'}) + m^2 F\big) \neq 0.$$

Since X is normal and F is a π-exceptional effective divisor, the natural isomorphism

$$H^0\big(X', f'^*(-m^2 K_{Y'} - A_{Y'})\big) \xrightarrow{\simeq} H^0\big(X', f'^*(-m^2 K_{Y'} - A_{Y'}) + m^2 F\big) \neq 0$$

is an isomorphism. Take a Stein factorization of f'

$$X' \xrightarrow{\tilde{f}} \tilde{Y} \xrightarrow{\mu} Y', \quad f' = \mu \circ \tilde{f}$$

where $\mu : \tilde{Y} \to Y'$ is a finite surjective morphism and $\tilde{f}_* \mathscr{O}_{X'} = \mathscr{O}_{\tilde{Y}}$. Then we have an isomorphism

$$\tilde{f}_* : H^0\big(X, f'^*(-m^2 K_{Y'} - A_{Y'})\big) \xrightarrow{\simeq} H^0\big(\tilde{Y}, \mu^*(-m^2 K_{Y'} - A_{Y'})\big),$$

which implies that the line bundle $\mu^*(-m^2 K_{Y'} - A_{Y'})$ is effective. Since $p : Y' \to Y$ is a finite surjective morphism, by [Bou02, Proposition 4.2] again, $-m^2 K_{Y'} - A_{Y'}$ is pseudo-effective, and thus $-K_{Y'}$ is big. By Lemma 4.1, we conclude that $-K_Y$ is also big. □

Proof of Corollary E. By Theorem C.(b) we conclude that $-K_Y$ is big. By [BP11, Theorem 1.2], Y is rationally connected modulo $\mathrm{NNef}(-K_Y) \bigcup \mathrm{Nklt}(-K_Y)$. The corollary is thus proved. □

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