PICARD THEOREMS FOR MODULI SPACES OF POLARIZED VARIETIES

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Abstract. We obtain a general big Picard theorem for the case of complex Finsler pseudometric of negative curvature on log-smooth pairs (X, D). In particular, we show, after a full recall and discussion of the construction of Viehweg and Zuo in their studies of Brody hyperbolicity in the moduli context, that the big Picard theorem holds for the moduli stack $\mathcal{M}_h$ of polarized complex projective manifolds of semi-ample canonical bundle and Hilbert polynomial h, i.e., for an algebraic variety U, a compactification Y and a quasi-finite morphism $U \to \mathcal{M}_h$ induced by an algebraic family over U of such manifolds, that any holomorphic map from the punctured disk $\mathbb{D}^*$ to U extends to a holomorphic map $\mathbb{D} \to Y$. Borel hyperbolicity of $\mathcal{M}_h$ is then a useful corollary: that holomorphic maps from algebraic varieties to U are in fact algebraic. We also show the related algebraic hyperbolicity property of $\mathcal{M}_h$ at the end.

Contents

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1. Introduction

In this paper, we obtain the Picard extension property of holomorphic curves on moduli spaces of polarized complex projective manifolds. More precisely, consider an algebraic family of polarized complex projective manifolds with semi-ample canonical divisors given by $(f : V \to U, \mathcal{L}) \in \mathcal{M}_h(U)$, where h is the Hilbert polynomial. Suppose that the classifying map from the parameter space U to the coarse moduli space M_h induced by the family, say moduli map, is quasi-finite. Let $\bar{U}$ be an algebraic compactification of U, $\bar{C}$ a complex curve and $\gamma : C \to U$ a holomorphic map from a Zariski open subset C of $\bar{C}$. We show that γ has a holomorphic extension $\bar{\gamma} : \bar{C} \to \bar{U}$. We also extend this to the case when C is replaced by a complex space and holomorphic by the word meromorphic, obtaining in particular the algebraicity of γ when C is an algebraic variety. Our proof proceeds in two major steps, comprising the two parts of substance of the paper: the first extracts the necessary curvature conditions via a detailed recall of the Viehweg-Zuo paper [VZ03] while the second uses Nevanlinna theory to prove our key technical theorem, of independent interest as no algebraic conditions are imposed. We offer two proofs of our technical theorem, one inspired by the method of Griffiths-King [GK73] (which followed the metric approach of Chern and others) in dealing with moduli spaces, and a more 'modern' one of reducing to the classical logarithmic derivative lemma, whose proof is inspired by the fundamental vanishing theorem of Siu-Yeung [SY97] and Demailly [Dem97b]. At the end, we also show a related but easier hyperbolicity property of the moduli space: that U as above is birational to $X \setminus D$ for a projective a log-smooth pair (X, D) that is algebraically hyperbolic in the sense of Demailly.

It is well understood that big-Picard-type theorems are strongly related to hyperbolicity. In fact, a theorem of Kwack and Kobayashi (K^2) gives in general that every holomorphic map $\gamma : \mathbb{D}^* \to U$ extends to a holomorphic map $\bar{\gamma} : \mathbb{D} \to \bar{U}$ if U is hyperbolically embedded in some compactification $\bar{U}$ (cf. [Kob98, Theorem (6.3.7)] or [Lan87, II§2]), a notion invented by Kobayashi expressly for this purpose. Motivated by the Shafarevich problem from the 60's (cf. [Vie01, Kov03] for an introduction) and its higher dimensional generalizations, the hyperbolicity properties of the moduli space of polarized varieties have been extensively studied with many major advances.

To start off, the moduli space of canonically polarized manifolds (i.e., those with ample canonical line bundle) has been shown to be Brody hyperbolic (i.e., having no non-constant holomorphic maps from $\mathbb{C}$) by Eckart Viehweg and the fourth named author in [VZ03]. This was done by the construction (via iterating the Kodaira-Spencer map) of certain Hodge-theoretic objects, the so-called Viehweg-Zuo sheaves, and its consequent negatively curved (complex) Finsler pseudometrics.

In [TY15] and [TY18], To and Yeung proved that the moduli spaces of canonically polarized manifolds and of Ricci flat manifolds are Kobayashi hyperbolic by constructing a negatively curved Finsler metric on U directly. This is stronger than Brody hyperbolicity for a noncompact U, but still weaker than having a hyperbolic embedding (cf. [Lan87, Chapter 2] or [Kob98, Chapter 3, §3]). More recently, the first named author in [Den18a] proved in addition that the moduli space of polarized varieties whose canonical divisors are semi ample and big is Kobayashi hyperbolic and that the moduli space without the bigness condition is Brody hyperbolic, both of which were conjectured by Viehweg and the fourth named author, [VZ03, Question 0.2]. He did this using negatively curved Finsler (pseudo)metrics on U obtained by the addition at the end of the Viehweg-Zuo construction two natural ingredients in the hyperbolic context: a pointwise argument plus a convex linear combination of components of the Viehweg-Zuo Finsler pseudo-metrics.

We remark that without having a hyperbolic embedding, i.e., a good compactification, one cannot get the big Picard theorem by directly applying the classical K^2 theorem mentioned. For moduli spaces of canonically polarized varieties, the KSBA compactifications seem to be a natural candidate (see the survey [Kol13]). As a generalization to Borel's theorem, we can ask the following:

Question 1.1. *Let $f : X \to Y$ be a KSBA stable family over a projective variety Y. Denote by U the smooth locus of f in Y. Is U hyperbolically embedded into Y?*

Now, for some moduli spaces such as the moduli space of abelian varieties or of K3 surfaces, there do exist good compactifications that yield hyperbolic embeddings. Recall that those moduli spaces are locally symmetric varieties. Such a variety admits the famous Baily-Borel compactification (cf.[BB64]), a hyperbolic embedding which is also a minimal compactification in the sense of Mori's theory. The big Picard theorem on those moduli spaces follows from a theorem of Borel [Bor72]:

Theorem 1.2 (Borel)**.** *Let X be a torsion-free arithmetic quotient of a bounded symmetric domain. Denote by X^* the Baily-Borel compactification of X. Then X is hyperbolically embedded into X^*.*

Inspired by this theorem, Javanpeykar and Kucharczyk in [JK18] formulated the following notion:

Definition 1.3. *A finite type scheme X over $\mathbb{C}$ is* Borel hyperbolic *if, for every finite type reduced scheme S over $\mathbb{C}$, any holomorphic map from S to X is algebraic.*

It is easy to see that Borel hyperbolicity implies Brody hyperbolicity. And hyperbolic embeddability implies Borel hyperbolicity by the K^2 theorem. Although [JK18] refers to the work [Bor72] of Armand Borel, we note that Emil Borel in e.g. [Bor97] almost a century earlier has done seminal works on hyperbolicity, obtaining also such extension theorems in some general logarithmic (i.e. quasiprojective) settings. We have thus welcomed the term Borel hyperbolicity in these contexts.

By the works of Viehweg-Zuo, To-Yeung *et al.* we already know that the moduli spaces in question have certain hyperbolicity properties. So the following question naturally arises:

Are the moduli spaces thus far considered (that are not locally symmetric) Borel hyperbolic?

Very recently, Bakker, Brunebarbe and Tsimerman have obtained sweeping partial result in this direction, see [BBT18, Corollary 7.1]. For a family with local Torelli injectivity (i.e. the period map is quasi-finite), the Borel hyperbolicity is a direct corollary of a conjecture of Griffiths on the quasi-projectivity of images of period maps. More precisely, let B be a smooth quasi-projective variety which admits a polarized variation of Hodge structures (PVHS) and $\Phi : B \to \Gamma\backslash D$ the induced period map. Here D is the period domain (namely the classifying space of Hodge structures with fixed Hodge numbers) and Γ is the monodromy group of the PVHS on B. In [Gri70b] Griffiths conjectured that the image $\Phi(B) \subset \Gamma\backslash D$ is a quasi-projective variety. Note that the quotient space $\Gamma\backslash D$ is in general a highly transcendental object. The paper [BBT18] confirms this conjecture assuming that Γ is arithmetic as a corollary of its deep results on the o-minimal GAGA theorem.

We remark that this Griffiths conjecture with arbitrary monodromy group Γ was established for the cases of $\dim \Phi(B) = 1$ (cf. [Som73], [CDK95]) and $\dim B = 2$ (cf. [GGLR19, Theorem 1.2.6]).

Nevertheless, families of polarized varieties where the local Torelli injectivity fails abounds.

1.1. **The big Picard theorem and Borel hyperbolicity.** Let $f : X \to Y$ be an analytic family of projective manifolds over a projective base Y with degeneration locus $S \subset Y$. In his seminal papers [Gri68a, Gri68b, Gri70a], Griffiths introduced the notion of polarized variation of Hodge structure on $U = Y \setminus S$ by taking the Betti cohomology of the smooth fibers endowed with Hodge structures. Schmid, Deligne and Cattani-Kaplan-Schmid ([Sch73, Del87, CKS86]) have studied the asymptotic behavior of the Hodge structures and the Hodge metric near the degeneration locus. Their results are of fundamental importance in the study of the geometry of families. Kawamata and Viehweg's positivity theorems on the direct image $f_*\omega^{\nu}_{X/Y}$ of powers of the relative dualizing sheaf are examples that play crucial roles in the investigation of the Iitaka conjecture. Another is Viehweg's work on constructing the moduli space of varieties with semi-ample dualizing sheaves.

As mentioned, the Torelli-type theorem fails in general for such a family. As a substitute, Viehweg and the fourth named author constructed in [VZ03] a non-trivial comparison map between the usual Kodaira-Spencer map and the Kodaira-Spencer map on the Hodge bundles associated to some new family built from certain cyclic coverings of X. Consequently, using the semi-negativity of the kernels of the Kodaira-Spencer maps on the Hodge bundles (proven in [Zuo00]) and the positivity results on the direct image sheaves, the maximal non-zero iteration of Kodaira-Spencer map yields the "*bigness*" of the so-called *Viehweg-Zuo subsheaves* in some symmetric power of $\Omega^1_Y(\log S)$. This subsheaf gives rise analytically to a negatively curved complex Finsler pseudometric on $U = Y \setminus S$.

In a fundamental paper [GK73], Griffiths and King studied the higher dimensional generalization of value distribution theory, also known as Nevanlinna theory. With it, they obtained a Nevanlinna-theoretic proof of Borel's theorem via negative curvature (cf.[GK73, Corollary (9.22)]). The existence of negativily curved complex Finsler pseudometrics in our setting makes this proof of Griffiths-King, suitably generalized, applicable. We extract the key technical theorem of our proof:

Theorem A (Criterion for big Picard theorem)**.** *Let X be a projective manifold and ω a Kähler metric on X and let D be a simple normal crossing divisor on X. Let $\gamma : \mathbb{D}^* \to X \setminus D$ be a holomorphic map from the punctured unit disk $\mathbb{D}^*$. Assume that there is a Finsler pseudometric h of $T_X(-\log D)$ (in the sense of Definition 2.15) such that $|\gamma'(z)|^2_h \not\equiv 0$, $\log|\gamma'(z)|^2_h$ is locally integrable and that the following inequality holds in the sense of currents*

$$\mathrm{dd^c}\log|\gamma'(z)|^2_h \geq \gamma^*\omega. \tag{1.1}$$

Then γ extends to a holomorphic map $\bar{\gamma} : \mathbb{D} \to X$.

We also present a more modern proof of this theorem via the classical logarithmic derivative lemma. Let us mention that the criterion in Theorem A was recently applied by the first named author in [Den20] to prove the big Picard theorem for varieties admitting a quasi-finite period map. We believe that it will have further applications.

As the construction of [VZ03] generalizes directly from the canonically polarized to the semiample case, thereby providing for an h giving the inequality in (1.1), our main theorem follows:

Theorem B (Big Picard theorem)**.** *Let $(f : V \to U, \mathcal{L}) \in \mathcal{M}_h(U)$ be a family of polarized manifolds with semi-ample canonical divisors with h the Hilbert polynomial. Suppose that the classifying map $U \to M_h$ from U to the coarse moduli space M_h induced by the family is quasi-finite. Then any holomorphic map $\gamma : \mathbb{D}^* \to U$ from the punctured disk $\mathbb{D}^*$ extends to the origin.*

We can even generalize Theorem B to the case when the domain of γ is higher dimensional.

Corollary C. *Let $(f : V \to U, \mathcal{L}) \in \mathcal{M}_h(U)$ be the polarized family as that in Theorem B. Let Y be a smooth projective compactification of U. Then any holomorphic map $\gamma : \mathbb{D}^p \times (\mathbb{D}^*)^q \to U$ extends to a meromorphic map $\overline{\gamma} : \mathbb{D}^{p+q} \dashrightarrow Y$. In particular, U is Borel hyperbolic.*

Indeed, by the first part of Corollary C, any holomorphic map $g : W \to U$ from a quasi-projective variety W to the base manifold U in Theorem B, extends to a meromorphic map of their projective compactifications, which is thus rational by the Chow theorem. This proves the Borel hyperbolicity of the base space U given in Theorem B.

In fact Theorem B can be modified to a Green-Griffiths conjecture type statement: for a family of polarized manifolds $f : V \to U$ with maximal variation of moduli, there is a proper subvariety of the base U, so that any punctured disk whose image is not contained in this proper subvariety satisfies the big Picard theorem. See Remark 3.5 for details.
It seems that there is no reason to expect the big Picard theorem holds true for the base spaces of families which are not of maximal variation. But we do have the following conjecture:

Conjecture 1.4 (Relative Isotriviality Conjecture)**.** *Let $(f : V \to U, \mathcal{L}) \in \mathcal{M}_h(U)$ be a family of polarized manifolds with fixed Hilbert polynomial h. And $\varphi : U \to \mathcal{M}_h$ is the classifying map induced by the family f. Suppose there is an entire curve $\gamma : \mathbb{C} \to U$ in the base space. Then the image $\gamma(\mathbb{C})$ is contained in some fiber of the classifying map φ.*

Note that this implies a hyperbolic version of *Campana's isotriviality conjecture* [**?**, Conjecture 13.21].

Conjecture 1.5. *$(f : V \to U, \mathcal{L}) \in \mathcal{M}_h(U)$ be a polarized family of smooth projective manifolds with semi-ample canonical bundle over a quasi-projective manifold which admits a dense entire curve. Then f is isotrivial, namely the moduli map $U \to M_h$ associated to f is constant.*

Note that Conjecture 1.4 has been proved by the first named author for families of canonically polarized manifolds in [Den19b].

1.2. **Algebraic hyperbolicity.** *Algebraic hyperbolicity* for a compact complex manifold X was introduced by Demailly in [Dem97a, Definition 2.2], and he proved in [Dem97a, Theorem 2.1] that X is algebraic hyperbolic if it is Kobayashi hyperbolicity. The notion of algebraic hyperbolicity was generalized to the case of log-pairs (X, D) by Chen [Che04]. We go further:

Definition 1.6 (Algebraic hyperbolicity)**.** *Let X be a projective manifold and Δ an algebraic subset. For a reduced irreducible curve $C \subset X$ with $C \not\subset \Delta$ and $\nu : \tilde{C} \to C$ its normalization, let $i_X(C, \Delta)$ be the number of points in $\nu^{-1}(\Delta)$. The pair (X, Δ) is* algebraically hyperbolic *if there is a Kähler metric ω on X such that, for all curves $C \subset X$ as above,*

$$2g(\tilde{C}) - 2 + i(C, \Delta) \geq \deg_\omega C := \int_C \omega. \tag{1.2}$$

It is easy to see, just as observed by Demailly in the case $\Delta = \emptyset$, that if $X \setminus \Delta$ is hyperbolically embedded into X, the log pair (X, Δ) is algebraically hyperbolic, c.f. [PR07].

Note that $2g(\tilde{C}) - 2 + i(C, \Delta)$ depends only on the complement $X \setminus \Delta$. Hence the above notion of hyperbolicity also makes sense for quasi-projective varieties: we say that a quasi-projective variety U is algebraically hyperbolic if it is birational to $Y \setminus D$ for a smooth log pair (Y, D) so that a Kähler metric ω exists on Y satisfying inequality 1.2 for all curves $C \subset Y$ such that $C \setminus D$ is finite over U.

The last main result in this paper is the algebraic hyperbolicity of the moduli spaces considered.

Theorem D (Algebraic hyperbolicity)**.** *Let* $(f : V \to U, \mathcal{L}) \in \mathcal{M}_h(U)$ *be the polarized family as that in Theorem B. Then the base* U *is algebraically hyperbolic.*

This result generalizes the *Arakelov-type inequality* by Möller-Viehweg-Zuo in [MVZ06] and the *weak boundedness* of moduli stacks of canonically polarized manifolds by Kovács-Lieblich in [KL10] (see Remark 5.1).

1.3. **Outline.** In Section 2, we give a detailed recollection of the construction of Viehweg and the fourth named author in [VZ02, VZ03], as well as the curvature property of the Finsler pseudometric they produced, as applied in our more general setting. Sections 3 and 4 give two very distinct Nevanlinna theoretic proofs of our main theorems, the first being self-contained and based on curvature methods while the second is based on the classical lemma on logarithmic derivatives, discuss other such proofs and prove the natural generalization of the theorems to higher dimensions that results in Borel hyperbolicity. Section 5 proves our theorem on algebraic hyperbolicity.

1.4. **Notation.** In general we follow the notations of [VZ02, VZ03] on the construction of Higgs bundles. We write $u \gtrsim v$ if there exists a constant $c > 0$ such that $u(s) \geq c \cdot v(s)$ for all $s \in S$.

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2. Recollections on the Viehweg-Zuo construction

In [VZ03, VZ02] Viehweg and the fourth named author constructed two graded logarithmic Higgs bundles for a given smooth family of polarized manifolds $f : V \to U$, which are used to prove the hyperbolicity of the base manifold U. We recall the construction briefly with some simplifications.

2.1. **Cyclic covering and the comparison map.** Let $V \to U$ be a smooth algebraic family of polarized manifolds with semi-ample canonical divisors with U nonsingular. Denote by $f : X \to Y$ a partial good compactification of the original family, meaning that:

1) X and Y are quasi-projective manifolds, and $U \subset Y$.
2) $S := Y \setminus U$ and $\Delta := f^*S$ are normal crossing divisors.
3) f is a log smooth projective morphism between the log pairs (X, Δ) and (Y, S), and $f^{-1}(U) \to U$ coincides with the original family $V \to U$.

4) Y has a non-singular projective compactification $\bar{Y}$ such that $\bar{Y} \setminus U$ is a normal crossing divisor and $\mathrm{codim}(\bar{Y} \setminus Y) \geq 2$.

We say that $f|_U$ is of maximal variation if $\mathrm{Var}(f) = \dim Y$, cf. [Kaw85, Section 1] for the definition. Note that if start of with such a family but whose base is not smooth, which is the case of that considered in Section 1 in general when the classifying map is only quasi-finite, we can always pull back the family by a desingularization of the base to one that is again of maximal variation whose base is. Since all the arguments from Viehweg-Zuo and thus for our theorems allow for such birational modifications, we assume from now on that the base U from Section 1 is non-singular.

To use the strategy from Viehweg-Zuo [VZ03, VZ02], one needs to construct some cyclic covering over the total space X. Thus we first collect the following result about the positivity of the direct image sheaf of the relative canonical bundle that is to be used to construct the cyclic covering.

Theorem 2.1 ([VZ03, Corollary 4.3] or [VZ02, Proposition 3.9])**.** *Denote by $\mathcal{L} := \Omega^n_{X/Y}(\log \Delta)$ the sheaf of top relative log differential forms (n is the relative dimension). Suppose that $f|_U$ is of maximal variation. Then there exists an ample line bundle A on $\bar{Y}$ and an integer $\nu \gg 1$ such that $\mathcal{L}^\nu \otimes f^*A^{-\nu}$ is globally generated over $V_0 := f^{-1}(U_0)$, where U_0 is some open dense subset of Y.*

Remark 2.2. To get this positivity result, Viehweg-Zuo used the self-fiber product trick. Though the dimension of the fibers of the new family could increase in general, it does not affect the proof of the hyperbolicity of the *original base space*.

It follows from this theorem that the invertible sheaf $\mathcal{L}^\nu \otimes f^*A^{-\nu}$ has plenty of nontrivial sections. For such a section s, we can get a cyclic covering of X by taking the ν-th roots of s. We choose Z to be a desingularization of this covering and denote the induced morphisms by $\psi : Z \to X$ and $g : Z \to Y$. The new family g has in general a larger discriminant locus than S since the restriction of the zero divisor $H = (s)$ on a general fiber of f is non-singular. Let T denote the discriminant of H over Y. Then the restriction of g over $Y \setminus (S \cup T)$ is smooth, which we denote by $g_0 : Z_0 \to U_0$.

2.1.1. *The Higgs bundle coming from variation of Hodge structures.* Consider the VHS on U_0 induced by the local system $R^n g_{0*}\mathbb{C}_{Z_0}$. By blowing up the closure $\bar{S}+\bar{T}$ of $S+T$ in $\bar{Y}$ and replacing $\bar{S}+\bar{T}$ by its preimage, we assume that $\bar{S}+\bar{T}$ is of simple normal crossings. Deligne's quasi-canonical extension then applies and we get a locally free sheaf $\mathcal{V}$ on $\bar{Y}$ with the Gauss-Manin connection:

$$\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_{\bar{Y}}(\log(\bar{S} + \bar{T})).$$

By the nilpotent orbit theorem in [Sch73] and [CKS86], the Hodge filtrations $\{\mathcal{F}^p\}$ extends as a subbundle filtration of $\mathcal{V}$ so that the associated Hodge bundle $E := \mathrm{Gr}_{\mathcal{F}^\bullet}\mathcal{V}$ is locally free on $\bar{Y}$. The induced Higgs map

$$\theta := \mathrm{Gr}_{\mathcal{F}^\bullet}\nabla : E \to E \otimes \Omega^1_{\bar{Y}}(\log(\bar{S} + \bar{T}))$$

has logarithmic poles along $\bar{S} + \bar{T}$. One can write the Hodge bundle explicitly as higher direct image sheaves of log forms if the divisor $\bar{S} + \bar{T}$ is smooth (cf. [Zuc84]). More precisely, we have

$$E^{p,q}|_{Y_0} \cong R^q g_* \Omega^p_{Z/Y}(\log \Pi)|_{Y_0}$$

for $Y_0 := Y \setminus \mathrm{Sing}(\bar{S} + \bar{T})$ and($q := n - p$), where $\Pi := g^{-1}(S \cup T)$ (Π is assumed to be normal crossing after birational modification of Z). It is apparent that $\mathrm{codim}(\bar{Y} \setminus Y_0) \geq 2$.

Remark 2.3. In the construction above we have already changed the birational model of U since we have to blow up T inside U. As mentioned, this is allowed in our application.

2.1.2. *The Higgs bundle coming from deformation theory.* The Hodge bundle (E,θ) has extra, artificially introduced logarithmic poles along T. To study the hyperbolicity of the original base space U, we shall construct a Higgs bundle directly from the original family, whose Higgs map has logarithmic poles only along the boundary S.

As in [VZ03] and [VZ02], we shall use the tautological short exact sequences

$$0 \to f^*\Omega^1_Y(\log S) \otimes \Omega^{p-1}_{X/Y}(\log \Delta) \to \mathfrak{gr}(\Omega^p_X(\log \Delta)) \to \Omega^p_{X/Y}(\log \Delta) \to 0 \tag{2.1}$$

where

$$\mathfrak{gr}(\Omega^p_X(\log \Delta)) := \Omega^p_X(\log \Delta)/f^*\Omega^2_Y(\log S) \otimes \Omega^{p-2}_{X/Y}(\log \Delta).$$

Note that the short exact sequence can be established only when $f : (X,\Delta) \to (Y,S)$ is log smooth. Denote by $\mathcal{L} = \Omega^n_{X/Y}(\log \Delta)$ as before. We define

$$F_0^{p,q} := R^q f_*(\Omega^p_{X/Y}(\log \Delta) \otimes \mathcal{L}^{-1})/\text{torsion}$$

together with the edge morphisms

$$\tau_0^{p,q} : F_0^{p,q} \to F_0^{p-1,q+1} \otimes \Omega^1_Y(\log S)$$

induced by the exact sequence (2.1) tensored with $\mathcal{L}^{-1}$.

Remark 2.4. It is easy to see that $\tau_0^{n,0}|_U$ is nothing but the Kodaira-Spencer map of the family. So the Higgs maps $\tau_0^{p,q}$ can be regarded as the *generalized Kodaira-Spencer maps.*

We denote by $F^{p,q}$ the reflexive hull of $F_0^{p,q}$ on $\bar{Y}$. The Higgs maps $\tau_0^{p,q}$ extends automatically since $\operatorname{codim}(\bar{Y} \setminus Y) \geq 2$. So we get the Higgs sheaf (F,τ) defined on $\bar{Y}$.

2.1.3. *The comparison maps.* In [VZ03, VZ02] Viehweg and Zuo constructed the following comparison maps $\rho^{p,q}$, which connect (F,τ) and (E,θ).

Lemma 2.5. *Using the same notations introduced above, let*

$$\iota : \Omega^1_{\bar{Y}}(\log \bar{S}) \to \Omega^1_{\bar{Y}}(\log(\bar{S}+\bar{T}))$$

be the natural inclusion. Then there exists morphisms $\rho^{p,q} : F^{p,q} \to A^{-1} \otimes E^{p,q}$ such that the following diagram commutes.

$$\begin{array}{ccc} F^{p,q} & \xrightarrow{\tau^{p,q}} & F^{p-1,q+1} \otimes \Omega^1_{\bar{Y}}(\log \bar{S}) \\ \big\downarrow{\scriptstyle \rho^{p,q}} & & \big\downarrow{\scriptstyle \rho^{p-1,q+1}\otimes\iota} \\ A^{-1} \otimes E^{p,q} & \xrightarrow{\mathrm{id}\otimes\theta^{p,q}} & A^{-1} \otimes E^{p-1,q+1} \otimes \Omega^1_{\bar{Y}}(\log(\bar{S}+\bar{T})) \end{array} \tag{2.2}$$

Remark 2.6. Note that our comparison map $\rho^{p,q}$ is defined only on Y_0 *a priori*, that is, a morphism between $F^{p,q}|_{Y_0} \cong R^q f_*(\Omega^p_{X/Y}(\log \Delta) \otimes \mathcal{L}^{-1})|_{Y_0}/\text{torsion}$ and $E^{p,q}|_{Y_0} \cong R^q g_* \Omega^p_{Z/Y}(\log \Pi)|_{Y_0}$. Since $F^{p,q}$ is reflexive, $E^{p,q}$ is locally free and $\operatorname{codim}(\bar{Y} \setminus Y_0) \geq 2$, the comparison map $\rho^{p,q}$ extends to $\bar{Y}$.

2.1.4. *The injectivity of the comparison map.* In order to use the comparison map to construct a negatively curved Finsler metric, one need to show the pointwise injectivity of the comparison map

$$(\rho^{n-1,1} \otimes \iota) \circ \tau^{n,0} : \mathcal{O}_{\bar{Y}} = F^{n,0} \to A^{-1} \otimes E^{n-1,1} \otimes \Omega^1_{\bar{Y}}(\log \bar{S})$$

as well as that of the induced map

$$\tau^1 : T_{\bar{Y}}(-\log \bar{S}) \to A^{-1} \otimes E^{n-1,1}. \tag{2.3}$$

Denote by $\rho^{p,q}_y$ the restriction of $\rho^{p,q}$ at a point $y \in Y$. In [VZ03], the following fact on the injectivity of $\rho^{p,q}_y$ are obtained:

1) $\rho^{n,0}_y$ is always injective for every $y \in U \setminus T$.
2) If the family is canonically polarized, then $\rho^{p,q}_y$ is injective for each (p,q) with $p+q=n$ and for every $y \in U \setminus T$.

We note that while the injectivity of $\rho^{p,q}$ follows from the Kodaira-Akizuki-Nakano vanishing theorem for all p,q, for the injectivity of $\rho^{n-1,1}$, it suffices to use only the Bogomolov-Sommese vanishing theorem [PTW18]. As the latter holds true for varieties of general type, one obtains

Theorem 2.7 (Viehweg-Zuo)**.** *Let $(f : V \to U, \mathcal{L}) \in \mathcal{M}_h(U)$ be a family of polarized varieties with semi-ample and big canonical divisors as that in Theorem B. Then the map $(\rho^{n-1,1} \otimes \iota) \circ \tau^{n,0}$ is injective at all the points in $U \setminus T$ where the Kodaira-Spencer map is injective, evalued in each tangent vector.*

In the case of semi-ample canonical divisors, Viehweg-Zuo showed that

Theorem 2.8 (Viehweg-Zuo)**.** *Let $(f : V \to U, \mathcal{L}) \in \mathcal{M}_h(U)$ be the polarized family as that in Theorem B. Then the map $(\rho^{n-1,1} \otimes \iota) \circ \tau^{n,0}$ along any algebraic curve $\gamma : C \to U$ does not vanish.*

We now outline the proof of this theorem, which used a global argument relying on the Griffiths curvature computation for Hodge metrics. Suppose that $(\rho^{n-1,1} \otimes \iota) \circ \tau^{n,0}$ vanishes along $\gamma(C)$. Then the image $\mathcal{O}_Y = F^{n,0} \subset E^{n,0} \otimes A^{-1}$ lies in $\mathrm{Ker}(\theta^{n-1,1}) \otimes A^{-1}$. Note that $\mathrm{Ker}(\theta^{n-1,1})$ is semi-negatively curved for the degenerated Hodge metric (cf. [Zuo00]), which essentially follows from the Griffiths curvature computation. By taking integration of the curvature form of Hodge metric restricted to $\mathcal{O}_Y$ one shows that the trivial line bundle is strictly negative because of the curvature decreasing property for holomorphic subbundles. This is of course a contradiction.

Very recently, the first named author observed that the argument of Viehweg-Zuo can be made pointwise, combining with a usual maximal principle argument. His argument runs as following: instead of taking integration of the curvature form of Hodge metric, he evaluated the curvature form on a special point y_0 in U, where the norm function of the constant section of $\mathcal{O}_Y$ with respect to the Hodge metric of $E^{n,0} \otimes A^{-1}$ restricted to $\mathcal{O}_Y$ via $\rho^{n,0} : \mathcal{O}_Y \hookrightarrow E^{n,0} \otimes A^{-1}$ (in fact a natural modification by Popa-Taji-Wu of the Viehweg-Zuo's metric on U, in order to remove its singularity at T) takes the maximal value. This implies that the curvature form on $\mathcal{O}_Y$ is semi-positive at this point, evaluated at all the tangent vectors. On the other hand, if the map $(\rho^{n-1,1} \otimes \iota) \circ \tau^{n,0}$ at this specific point evaluated in some tangent vector vanishes, then the Griffiths curvature formula and the strict negativity of A^{-1} implies that the the curvature form on $\mathcal{O}_Y$ at y_0 is strictly negative along this tangent vector, which gives us a contradiction.

Theorem 2.9 ([Den18b])**.** *Let $f : V \to U$ is a smooth projective family of manifolds with semi-ample canonical sheaf, which is of maximal variation. Then the map τ^1 defined in (2.3) is a* *vector bundle injection on a Zariski open subset U° of U. In particular, the analytic version of Theorem 2.8 holds true: the map $(\rho^{n-1,1}\otimes\iota)\circ\tau^{n,0}$ along any holomorphic curve $\gamma : C\to U$ with $\gamma(C)\cap U^\circ\neq\varnothing$.*

Conjecture 2.10. *Let $(f : V\to U,\mathcal{L})\in\mathcal{M}_h(U)$ be the polarized family as that in Theorem B. Then the map $(\rho^{n-1,1}\otimes\iota)\circ\tau^{n,0}$ is a vector bundle injection for all the points in $U\setminus(S+T)$.*

Conjecture 2.10 has been verified for such a family with members of Kodaira dimension one in a joint paper of the third and fourth named authors with Xin Lu [LSZ].

2.2. **Maximal non-zero iteration of Kodaira-Spencer maps.** We iterate the Higgs maps to get

$$\tau^{n-q+1,q-1}\circ\cdots\circ\tau^{n,0} : F^{n,0}\to F^{n-q,q}\otimes\bigotimes^{q}\Omega^1_{\bar Y}(\log\bar S).$$

This composition factors through

$$\tau^q : F^{n,0}\to F^{n-q,q}\otimes Sym^q\,\Omega^1_{\bar Y}(\log\bar S)$$

because the Higgs field τ satisfies $\tau\wedge\tau=0$. As $\mathcal{O}_{\bar Y}$ is a subsheaf of $F^{n,0}$, the composition of maps

$$\begin{array}{c} Sym^q\,T_{\bar Y}(-\log\bar S)\xrightarrow{\subset}F^{n,0}\otimes Sym^q\,T_{\bar Y}(-\log\bar S)\xrightarrow{\tau^q\otimes\mathrm{id}}F^{n-q,q}\otimes Sym^q\,\Omega^1_{\bar Y}(\log\bar S)\otimes Sym^q\,T_{\bar Y}(-\log\bar S)\\ \big\downarrow{\scriptstyle \mathrm{id}\otimes<,>}\\ F^{n-q,q}\end{array}$$

makes sense, which we will still denote as τ^q by abuse of notations.

Composing this τ^q with the comparison map $\rho^{n-q,q}$, we get the *iterated Kodaira-Spencer map*

$$Sym^q\,T_{\bar Y}(-\log\bar S)\xrightarrow{\tau^q}F^{n-q,q}\xrightarrow{\rho^{n-q,q}}A^{-1}\otimes E^{n-q,q}. \tag{2.4}$$

2.2.1. *Maximal non-zero iteration.* We define the *maximal non-zero iteration of Kodaira-Spencer map* to be the the m-th iterated Kodaira-Spencer map with $\rho^{n-m,m}\circ\tau^m(Sym^m\,T_{\bar Y}(-\log\bar S))\neq0$ with m being the largest number satisfying this property. More precisely,

$$\begin{array}{ccc} Sym^q\,T_{\bar Y}(-\log\bar S) & \xrightarrow{\rho^{n-m,m}\circ\tau^m\neq0} & A^{-1}\otimes E^{n-m,m}\\ & \searrow{\scriptstyle =0} & \big\downarrow{\scriptstyle \mathrm{id}\otimes\theta^{n-m,m}}\\ & & A^{-1}\otimes E^{n-m-1,m+1}\otimes\Omega^1_{\bar Y}(\log(\bar S+\bar T)). \end{array}$$

Note that $\{0\}\neq\mathrm{Im}(\rho^{n-m,m}\circ\tau^m)\subset A^{-1}\otimes\mathrm{Ker}(\theta^{n-m,m})$. We call this number m the *maximal length of iteration* or the *maximal iteration length*.

Lemma 2.11. *Keeping the assumptions as above, we have that $\rho^{n-1,1}\circ\tau^1$ is injective at the generic point, evaluated at each tangent vector.*

Proof. This follows from Theorem 2.8 when the canonical divisor of a general fiber of the family is semi-ample and big and from Theorem 2.9 when the canonical divisor of a general fiber of the family is semi-ample. □

Corollary 2.12. *The maximal non-zero iteration of Kodaira-Spencer map on Y exists. And its length m is bounded by $1 \leq m \leq n$.*

Proof. $\rho^{n-1,1} \circ \tau^1$ is non-zero by Lemma 2.11. The upper bound of m follows from $\theta^{0,n} = 0$. □

2.2.2. *Maximal non-zero iteration of Kodaira-Spencer map along an analytic curve.* Let $V \to U$ be the smooth family as that in Theorem 2.9. In our application we shall consider an analytic map γ from a complex analytic curve C to the base manifold U so that $\gamma(C) \cap U^\circ \neq \varnothing$, where U° is the dense Zariski open set of U in Theorem 2.9. All the Higgs bundles can be pulled back to C, as well as the iteration process. We define the composition

$$\tau_\gamma^{p,q} : \gamma^* F^{p,q} \xrightarrow{\gamma^* \tau^{p,q}} \gamma^* F^{p-1,q+1} \otimes \gamma^* \Omega^1_{\bar{Y}}(\log \bar{S}) \xrightarrow{\mathrm{id} \otimes \mathrm{d}\gamma} \gamma^* F^{p-1,q+1} \otimes \Omega^1_C$$

as the *Higgs map along* γ. We define θ_γ similarly. Then $(\gamma^* F, \tau_\gamma)$ and $(\gamma^* E, \theta_\gamma)$ are holomorphic Higgs bundles on the curve C. The iterated Kodaira-Spencer maps on C are also defined similarly:

$$T_C^{\otimes q} \xrightarrow{\tau_\gamma^q} \gamma^* F^{n-q,q} \xrightarrow{\gamma^* \rho^{n-q,q}} \gamma^* A^{-1} \otimes \gamma^* E^{n-q,q}.$$

Corollary 2.13. *The maximal non-zero iteration of Kodaira-Spencer map along γ exists, i.e., $m \geq 1$.*

Proof. Since $\gamma(C) \subset U$ is Zariski dense, we know from Lemma 2.11 that $\rho^{n-1,1} \circ \tau^1$ is injective at all the points of $\gamma(C)$ contained in a Zariski open subset of U, evaluated at all tangent directions at those points. This implies that at least $\gamma^*(\rho^{n-1,1} \circ \tau^1_\gamma)(T_C)$ is non-zero. □

By its definition, the maximal non-zero iteration of Kodaira-Spencer map along γ of the length m has the properties: $\gamma^*(\rho^{n-m,m} \circ \tau^m_\gamma)(T_C^{\otimes m}) \neq 0$ and $\mathrm{Im}(\gamma^*(\rho^{n-m,m} \circ \tau^m_\gamma)) \subset \gamma^* A^{-1} \otimes \mathrm{Ker}(\theta_\gamma)$. Those properties are crucial for constructing a negatively curved Finsler metric along γ in section 2.3.

Remark 2.14. One should be warned that the maximal length of iteration along γ could be *shorter* than the maximal length of iteration of the original family. This is because the iterated Kodaira-Spencer maps $Sym^q\, T_{\bar{Y}}(-\log \bar{S}) \xrightarrow{\rho^{n-q,q} \circ \tau^q} A^{-1} \otimes E^{n-q,q}$ with $q > 1$ are not injective in general.

2.3. The Finsler (pseudo)metric.

Let us begin with the following definition.

Definition 2.15 (Finsler metric)**.** *Let E be a holomorphic vector bundle on a complex manifold X. A* Finsler pseudometric *on E is a* continuous *function $h : E \to [0, +\infty[$ such that*

$$h(av) = |a| h(v)$$

for any $a \in \mathbb{C}$ and $v \in E$. We call h a Finsler metric if it is nondegenerate, i.e, if $h(v) = 0$ only when $v = 0$.

As [VZ03], we construct a Finsler pseudometric via the maximal iterated Kodaira-Spencer map

$$Sym^m\, T_{\bar{Y}}(-\log \bar{S}) \xrightarrow{\rho^{n-m,m} \circ \tau^m} A^{-1} \otimes E^{n-m,m} \tag{2.5}$$

given as follows. Consider $g_{A^{-1}} \otimes g_{hod}$ on $A^{-1} \otimes E^{n-m,m}$, where g_A is the Fubini-Study metric of the ample line bundle A and g_{hod} is the Hodge metric on the Hodge bundle E. Pulling it back via $\rho^{n-m,m} \circ \tau^m$ and taking m-th root, we get our desired Finsler pseudometric on $T_{\bar{Y}}(-\log \bar{S})$.

2.3.1. *Modification along the boundary.* In fact, Viehweg and the fourth named author used some modified version of the Finsler pseudometric described above in order to have the right kind of curvature property. This method of modifying metric appeared first in the second named author's thesis about extending meromorphic maps [Lu91].

First we construct an auxiliary function associated to the boundary divisor $\bar{S}$. Denote by $\bar{S}_1, \ldots, \bar{S}_p$ the components of $\bar{S}$. Let L_i be the line bundle with section s_i such that $\bar{S}_i = \mathrm{div}(s_i)$. Equip each L_i with a smooth hermitian metric g_i. Let $l_i := -\log |s_i|^2_{g_i}$ and $l_S := l_1 l_2 \cdots l_p$. Recall that the Hodge metric g_{hod} has extra degeneration along $\bar{T}$ since the Hodge bundle (E, θ) has logarithmic poles along $\bar{T}$. To control the asymptotic behaviour of g_{hod} near $\bar{T}$, we construct another auxiliary function associated to the divisor $\bar{T}$, in a similar manner as for l_S: Denote by $\bar{T}_1, \ldots, \bar{T}_q$ the non-singular components of $\bar{T}$. Let L_i' be the line bundle with section t_i such that $\bar{T}_i = \mathrm{div}(t_i)$. Equip each L_i' with a smooth hermitian metric g_i'. Let $l_i' := -\log |t_i|^2_{g_i'}$ and $l_T := l_1' l_2' \cdots l_q'$.

Now for each positive integer α, we define a new singular hermitian metric $g_\alpha := g_A \cdot l_S^\alpha \cdot l_T^\alpha$ on A.

Before entering our setting of the Viehweg-Zuo construction, we remark that by suitably modifying a (pseudo)metric on the tautological line bundle of the projective log tangent bundle satisfying the hypothesis of Theorem A, one can already obtain the strongly negative curvature property that the holomorphic sectional curvature is bounded from above by a negative constant.

Proposition 2.16. *Let (X, D), $\gamma : \mathbb{D}^* \to X \setminus D$ be the same as in Theorem A. Let h be the Finsler pseudometric on $T_X(-\log D)$ (or equivalently, a semi-norm on $\mathcal{O}_{\mathbb{P}_{alg}(T_X(-\log D)^\vee)}(-1)$) satisfying the curvature inequality (1.1). Then there is a Finsler pseudometric h_α defined as $h \cdot l_D^{-\alpha}$, where l_D is the above auxiliary function of the boundary divisor D and α is some positive integer, such that its holomorphic sectional curvature is bounded from above by a negative constant, i.e.:*

$$-dd^c \log |\gamma'(z)|^2_{h_\alpha} \lesssim -\mu_\alpha$$

where μ_α is the semi-positive $(1,1)$ form associated with the possibly degenerate hermitian metric $(\mathbb{P}\gamma')^{-1}(|\ |_{h_\alpha} \circ \gamma_*)^2 = ((\mathbb{P}\gamma')^{-1} h_\alpha)|\gamma_*|^2$ on $\mathbb{D}^*$ and the inequality above holds in the sense of currents.

Proof. By direct computation we have

$$\begin{aligned} dd^c \log |f'(z)|^2_{h_\alpha} &= dd^c \log |f'(z)|^2_h - \alpha f^* \Sigma \frac{\mathrm{dd}^c l_i}{l_i} + \frac{\sqrt{-1}}{2\pi} \alpha f^* \Sigma \frac{\partial l_i \wedge \overline{\partial l_i}}{l_i^2} \\ &\geq dd^c \log |f'(z)|^2_h - \alpha f^* \Sigma \frac{\mathrm{dd}^c l_i}{l_i} \geq f^* \left(\omega_{FS} - \alpha \Sigma \frac{\mathrm{dd}^c l_i}{l_i} \right). \end{aligned} \tag{2.6}$$

Here we have used the inequality $dd^c \log |f'(z)|^2_h \geq f^* \omega_{FS}$. And by the same argument in [Lu91, §4, Proposition 1] (see also the proof of Lemma 7.1 in [VZ03]), we can find a positive definite Hermitian form ω_α on $T_X(-\log D)$ such that

$$\omega_{FS} - \alpha \Sigma \frac{\mathrm{dd}^c l_i}{l_i} \geq l_D^{-2} \cdot \omega_\alpha.$$

Note that $l_D^{-2} \cdot \omega_\alpha$ can also be regarded as a semi-norm on the dual of the tautological line bundle. So if we choose $\alpha > 2$, then $l_D^{-2} \cdot \omega_\alpha \gtrsim h_\alpha$ by the compactness of X. Therefore, we get the desired bound on the negative holomorphic sectional curvature $-dd^c \log |f'(z)|^2_{h_\alpha} \lesssim -(\mathbb{P}f')^{-1} h_\alpha$. □

Now we come back to our setting of the Viehweg-Zuo construction. We first note that

Lemma 2.17. *$\Theta(A, g_\alpha)$ dominates the Kähler form $\omega_{FS} := \Theta(A, g_A)$ on $\bar{Y}$ as currents.*

Proof. This follows from the computation

$$\begin{aligned}\Theta(A,g_\alpha) &= \Theta(A, g_A\cdot l_S^\alpha\cdot l_T^\alpha)\\ &= \Theta(A,g_A) - \alpha\Sigma\frac{\mathrm{dd}^c l_i}{l_i} - \alpha\Sigma\frac{\mathrm{dd}^c l_i'}{l_i'} + \frac{\sqrt{-1}}{2\pi}\alpha\Sigma\frac{\partial l_i\wedge\overline{\partial l_i}}{l_i^2} + \frac{\sqrt{-1}}{2\pi}\alpha\Sigma\frac{\partial l_i'\wedge\overline{\partial l_i'}}{l_i'^2}\\ &\geq \Theta(A,g_A) - \alpha\Sigma\frac{\mathrm{dd}^c l_i}{l_i} - \alpha\Sigma\frac{\mathrm{dd}^c l_i'}{l_i'} \geq c\cdot\Theta(A,g_A),\end{aligned}$$

which holds for some positive constant c. Note that one can rescale g_i (respectively g_i') to make l_i (respectively l_i') sufficiently large and leave $\mathrm{dd}^c l_i$ (respectively $\mathrm{dd}^c l_i'$) unchanged. □

Remark 2.18. (1) The same computations as above show, by the hypothesis on h, that $dd^c\log|f'(z)|^2_{h_\alpha}$ dominates $dd^c\log|f'(z)|^2_h$ and $f^*\omega_{FS}$ as currents. (2) As we mentioned above, the second named author used this type of modification of metrics to prove his extension theorem (cf. [Lu91, §4]). Later, Viehweg and the fourth named author applied it in [VZ03, §7] to the Viehweg-Zuo metric of the family over U and obtained the curvature estimate in Lemma 2.17. Since the family concerned in [VZ03] is canonically polarized, one can move the branch divisor of the cyclic covering such that the discriminant locus T intersects with the analytic curve $\gamma(C)$ only at the smooth part of T, and the intersection is transversal. Then, the monodromy of the pull-back local system around γ^*T is finite, and the pull-back Hodge metric γ^*g_{hod} is bounded (see section 5 of [VZ03] for details). Popa-Taji-Wu also observed that a similar modification along $\bar{T}$ applies without violating the curvature estimate and chose α sufficiently large such that the singular hermitian metric $g_\alpha^{-1}\otimes g_{hod}$ is bounded (in fact continuous) near $\bar{T}$ without the canonically polarized hypothesis (cf. [PTW18, §3.1]). This means that the Finsler pseudometric on the log tangent bundle induced by $g_\alpha^{-1}\otimes g_{hod}$ is bounded, which is important for our curvature estimate. Note that here we use the property that the Hodge metrics have at most logarithmic growth along $\bar{S}+\bar{T}$, which is guaranteed by the study of the higher dimensional asymptotic behavior of the Hodge metric in [CKS86].

2.3.2. *The curvature inequalities.* Now we consider an analytic map γ from a Riemann surface C (for instance, $C=\mathbb{D}^*$) to the base manifold U. Let m be the maximal length of iteration along γ.

It is very natural to use the hermitian metric $g_\alpha^{-1}\otimes g_{hod}$ and the iterated Kodaira-Spencer map to construct a Finsler pseudometric F_α on $T_{\bar{Y}}(-\log\bar{S})$:

$$|v|^2_{F_\alpha} := |\rho^{n-m,m}\circ\tau^m(v^{\otimes m})|^{2/m}_{g_\alpha^{-1}\otimes g_{hod}},\ \text{for } v\in T_{\bar{Y}}(-\log\bar{S}). \tag{2.7}$$

We first state a curvature inequality associated to F_α *which validates the hypothesis in our criterion for big Picard theorem.* This inequality is given, though not explicitly, in [VZ03, §7] and holds in general once we have Theorem 2.9 in hand. We repeat the proof here only for consistency.

Theorem 2.19 (The curvature inequality)**.** *Let $V\to U$ be the same family as that in Theorem 2.9. Fix the analytic map $\gamma: C\to U$ so that $\gamma(C)\cap U^\circ\neq\varnothing$ where U° is the dense Zariski open set of U in Theorem 2.9 so that $\tau^1|_{U^\circ}: T_{U^\circ}\to A^{-1}\otimes E^{n-1,1}|_{U^\circ}$ is injective. Then for any positive integer α, the Finsler (pseudo)metric F_α constructed above satisfies the following curvature inequality*

$$\mathrm{dd}^c\log|\gamma'(z)|^2_{F_\alpha}\gtrsim\gamma^*\omega_{FS}.$$

Proof. By the Poincaré-Lelong formula, we know that $\mathrm{dd}^c \log|\gamma'(z)|^2_{F_\alpha} = -\Theta(T_C, (\mathrm{d}\gamma)^*F_\alpha) + R \geq -\Theta(T_C, (\mathrm{d}\gamma)^*F_\alpha)$, where R is the ramification divisor of γ. Denote by N the saturation of the image of $\mathrm{d}\gamma : T_C \to \gamma^*T_Y(-\log S)$. Then we have the following curvature current estimate

$$\begin{aligned} \Theta(T_C, (\mathrm{d}\gamma)^*F_\alpha) &\leq \Theta(N, \gamma^*F_\alpha) = \frac{1}{m}\Theta(N^{\otimes m}, \gamma^*F_\alpha^{\otimes m}) \\ &\leq \frac{1}{m}\gamma^*\Theta(Sym^m T_Y(-\log S), (\rho^{n-m,m} \circ \tau^m)^*(g_\alpha^{-1} \otimes g_{hod}))|_{N^{\otimes m}} \\ &\leq \frac{1}{m}\gamma^*\Theta(A^{-1} \otimes E, g_\alpha^{-1} \otimes g_{hod})|_{\gamma^*(\rho^{n-m,m}\circ\tau^m)(N^{\otimes m})} \\ &= -\frac{1}{m}\gamma^*\Theta(A, g_\alpha) + \frac{1}{m}\gamma^*\Theta(E, g_{hod})|_{\gamma^*A \otimes \gamma^*(\rho^{n-m,m}\circ\tau^m)(N^{\otimes m})}. \end{aligned} \tag{2.8}$$

Recall that m is the maximal length of the iteration along γ so that $\gamma^*A \otimes \gamma^*(\rho^{n-m,m} \circ \tau^m)(N^{\otimes m})$ lies in the kernel of θ_γ. Therefore, as the last term in the estimate (2.8) is semi-negative by the Griffiths curvature computation (see [Sch73, Lemma (7.18)] or [Zuo00]), we have as currents that

$$\Theta(T_C, (\mathrm{d}\gamma)^*F_\alpha) \leq -\frac{1}{m}\gamma^*\Theta(A, g_\alpha).$$

And hence, we have

$$\mathrm{dd}^c \log|\gamma'(z)|^2_{F_\alpha} \geq \frac{1}{m}\gamma^*\Theta(A, g_\alpha) \gtrsim \gamma^*\omega_{FS},$$

the last inequality being given by Lemma 2.17 □

Using the curvature inequality in Theorem 2.19 and the estimate in Proposition 2.16, Viehweg and the fourth named author showed that F_α is strongly negatively curved along the analytic curve.

Theorem 2.20 (Viehweg-Zuo)**.** *Keeping the assumptions on the family $V \to U$ the same as in Theorem 2.9. Fix the analytic map $\gamma : C \to U$. Then there exists a positive integer α (depending on the maximal length along γ) and $c_\alpha > 0$ (depending on α) such that the curvature of F_α satisfies*

$$K_{F_\alpha}(v) \leq -c_\alpha$$

for any nonzero tangent vector $v := \gamma'(z)$ of γ.

Remark 2.21. (1) Although $C = \mathbb{C}$ in [VZ03], all the arguments above work for a general Riemann surface C, except the final step in [VZ03, Lemma 7.9] where the Ahlfors-Schwarz lemma is used.
(2) Since $(\mathrm{d}\gamma)^*F_\alpha$ is locally bounded on T_C by construction, the inequality above entails an inequality in the sense of currents just as that in Theorem 2.16.

Before entering the next section, we list two crucial points to the the arguments we present there:

- *logarithmic growth of the Hodge metric near boundary:* In fact this is the crucial point of Viehweg-Zuo's curvature estimates; and those estimates are crucial to our argument.
- *local boundedness of the Finsler pseudometric near $\bar{T}$:* It is used in the definition of our Nevanlinna characteristic function.

3. Big Picard theorem via negative curvature

In this section we shall prove Theorem A and Theorem B by using the negative curvature method inspired by the argument in §9 of Griffiths-King [GK73]. Now, we have two negatively curved Finsler pseudometrics: h_α of Proposition 2.16 and F_α of Theorem 2.20. Since h_α shares the same curvature properties as F_α thanks to Proposition 2.16, we only present the proof of Theorem B using F_α. The proof of Theorem A using h_α is verbatim.

We identify $\mathbb{D}^*$ with the inverted punctured unit disk $\mathbb{D}^\circ := \{z \in \mathbb{C};\ |z| \geq 1\}$ in order to match the usual notations in Nevanlinna theory for entire curves. We set $\mathbb{D}_{r_0,r} := \{r_0 \leq |z| < r\} \subset \mathbb{D}^*$ and $\mathbb{D}_r := \{z \in \mathbb{C}; |z| < r\}$. Denote by $\gamma : \mathbb{D}^\circ \to U$ the analytic map in question. Then we want to show that γ extends over the point at infinity. We fix an $r_0 > 1$ from now on.

By the constructions in Subsection 2.3, there is a Finsler pseudometric F_α on $T_Y(-\log S)$ (resp. h_α on $T_X(-\log D)$), or equivalently a semi-norm on the tautological line bundle $\mathcal{O}_{\mathbb{P}_{\rm alg}(T_Y(-\log S)^\vee)}(-1)$, with the following inequalities of curvature currents

$$\begin{aligned}\gamma^*\omega_{FS} &\lesssim dd^c\log\,(\mathbb{P}\gamma')^{-1}|\ |^2_{F_\alpha}\\ \omega_\gamma &\lesssim dd^c\log\,(\mathbb{P}\gamma')^{-1}|\ |^2_{F_\alpha},\end{aligned}$$

by Theorems 2.19 and 2.20 (or by the hypothesis in Theorem A and Proposition 2.16) respectively. Here ω_γ is the semi-positve (1,1)-form associated to the semi-norm $(\mathbb{P}\gamma')^{-1}(|\ |_{F_\alpha}\circ\gamma_*)$ on $T_{\mathbb{D}^\circ}$.

Remark 3.1. For the construction of F_α, remember that one needs to change the birational model of U in the construction of those two Higgs bundles (F,τ) and (E,θ). In our application, we can always assume that the image of γ is Zariski dense by replacing $\bar{Y}$ by the Zariski closure of $\gamma(\mathbb{D}^\circ)$. Then the analytic map lifts to $\tilde{\gamma} : \mathbb{D}^\circ \to \tilde{U}$, where $\tilde{U}$ is the new birational model for the new zariski closure base space. Clearly, it suffices to prove the extension property for $\tilde{\gamma}$.

By the above argument, the following Nevanlinna characteristic functions

$$T_{\gamma^*\omega_{FS}}(r) := \int_{r_0}^r \frac{d\rho}{\rho}\int_{\mathbb{D}_{r_0,\rho}} \gamma^*\omega_{FS} \tag{3.1}$$

$$T_{\omega_\gamma}(r) := \int_{r_0}^r \frac{d\rho}{\rho}\int_{\mathbb{D}_{r_0,\rho}} \omega_\gamma$$

are both dominated (i.e. $\lesssim$) by

$$\int_{r_0}^r \frac{d\rho}{\rho}\int_{\mathbb{D}_{r_0,\rho}} dd^c\log\,(\mathbb{P}\gamma')^{-1}|\ |^2_{F_\alpha} \leq \int_{r_0}^r \frac{d\rho}{\rho}\int_{\mathbb{D}_{r_0,\rho}} dd^c\log|\gamma'(z)|^2_{F_\alpha} \tag{3.2}$$

It is elementary and classical that the asymptotic behavior of $T_{\gamma^*\omega_{FS}}(r)$ as $r \to \infty$ characterizes whether γ can be extended over ∞ (see *e.g.* [Dem97b, 2.11. cas local] or [NW14, Remark 4.7.4.(ii)]).

Lemma 3.2. *$T_{\gamma^*\omega_{FS}}(r) = O(\log r)$ if and only if γ can be extended holomorphically over ∞.* □

We will need the following Green-Jensen formula for the punctured disk.

Proposition 3.3 (Green-Jensen formula on punctured disk)**.** *Let ϕ be function on $\mathbb{D}^\circ$ such that ϕ is differentiable outside a discrete set of points disjoint from $\mathbb{D}_{r_0}$ and $dd^c\phi$ exists as a current. Then for $0 < r_0 < r$,*

$$\int_{r_0}^{r} \frac{d\rho}{\rho} \int_{\mathbb{D}_{r_0,\rho}} dd^c\phi = \int_{\partial\mathbb{D}_r} \phi \cdot d^c\log|z| - \int_{\partial\mathbb{D}_{r_0}} \phi \cdot d^c\log|z| - (\log\frac{r}{r_0}) \cdot \int_{\partial\mathbb{D}_{r_0}} d^c\phi. \tag{3.3}$$

□

By using (3.3), we have for $r_0 > 1$ fixed that,

$$\int_{r_0}^{r} \frac{d\rho}{\rho} \int_{\mathbb{D}_{r_0,\rho}} dd^c\log|\gamma'(z)|^2_{F_\alpha} = \int_{\partial\mathbb{D}_r} \log|\gamma'(z)|^2_{F_\alpha} \cdot d^c\log|z| + O(\log r). \tag{3.4}$$

We define the first term of the right hand side of (3.4) to be the *modified proximity function* $m_{\omega_\gamma}(r)$.

Lemma 3.4. *Denote by $\frac{d}{ds} := r \cdot \frac{d}{dr}$ the logarithmic derivative. Then*

$$m_{\omega_\gamma}(r) \le \log\frac{d^2T_{\omega_\gamma}(r)}{ds^2} + O(\log r).$$

Proof. Note that we can write $\omega_\gamma = |\gamma'(z)|^2_{F_\alpha} \cdot \frac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z}$. Denote by $\xi := |\gamma'(z)|^2_{F_\alpha}$ for simplicity. By direct computation one finds that

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}T_{\omega_\gamma}(r)\right) = \int_{\partial\mathbb{D}_r} \xi \cdot d^c\log|z|.$$

Using the concavity of the logarithmic function, we obtain

$$\begin{aligned} m_{\omega_\gamma}(r) &= \int_{\partial\mathbb{D}_r} \log\xi \cdot d^c\log|z| \le \log\int_{\partial\mathbb{D}_r} \xi \cdot d^c\log|z| \\ &= \log\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}T_{\omega_\gamma}(r)\right)\right\} = \log\left\{r^{-2}\cdot\frac{d^2}{ds^2}T_{\omega_\gamma}(r)\right\} \\ &= -2\log r + \log\frac{d^2T_{\omega_\gamma}(r)}{ds^2}. \end{aligned}$$

□

Applying the Calculus lemma twice, we obtain for some $\epsilon, \delta > 0$ that

$$\frac{d^2T_{\omega_\gamma}(r)}{ds^2} \le r^{2+\epsilon} \cdot T_{\omega_\gamma}(r)^{2+\delta} \quad ||,$$

where || here means that the inequality holds for r outside a set of finite Lebesque measure. Thus

$$m_{\omega_\gamma}(r) \le (2+\delta)\cdot\log T_{\omega_\gamma}(r) + O(\log r)\, ||.$$

Combining (3.2) and (3.4), we get the inequalities

$$\begin{aligned} T_{\omega_\gamma}(r) \lesssim m_{\omega_\gamma}(r) + O(\log r) \lesssim \log T_{\omega_\gamma}(r) + O(\log r)\, ||; \\ T_{\gamma^*\omega_{FS}}(r) \lesssim m_{\omega_\gamma}(r) + O(\log r) \lesssim \log T_{\omega_\gamma}(r) + O(\log r)\, ||. \end{aligned}$$

The first implies that $T_{\omega_\gamma}(r) = O(\log r)$, which then combines with the second to yield

$$T_{\gamma^*\omega_{FS}}(r) = O(\log r).$$

By Lemma 3.2, γ extends holomorphically over infinity and completes our proof of Theorem A and Theorem B via negative curvature.

Remark 3.5. From the proof one can see that the assumption on the family in Theorem B can be relaxed to a family of polarized manifolds with maximal variation of moduli (which is more common in practice), if we require the image of the holomorphic map $\gamma : \mathbb{D}^* \to U$ is not contained in a proper subvariety of U. One choose the proper subvariety Z such that the map τ^1 defined in subsection 2.1.4 is injective outside Z. Since our $\gamma(\mathbb{D}^*)$ is not contained in Z, all the arguments in section 2 and 3 go through and give us the extension of γ.

Remark 3.6. It is a classical fact that, up to addition of $O(\log r)$, the charateristic function

$$T_{(\mathbb{P}\gamma')^*\mathcal{O}(1)} := \int_2^r \frac{d\rho}{\rho} \int_{\mathbb{D}_\rho^\circ} dd^c \log\,(\mathbb{P}\gamma')^{-1}|\;|^2_{h_\alpha}$$

is independent of a *non-degenerate* Finsler metric h_α chosen on $\mathcal{O}(-1) = \mathcal{O}_{\mathbb{P}_{\mathrm{alg}}(T_Y(-\log S)^\vee)}(-1)$. So the estimate above also gives us the so called tautological inequality in our case in the form:

$$T_{(\mathbb{P}\gamma')^*\mathcal{O}(1)}(r) = O(\log r). \tag{3.5}$$

In fact, the tautological inequality as established by McQuillan is for the case of entire holomorphic curves and requires no hypothesis on curvature, albeit with extra terms on the right side of (3.5) and holds for r outside a set of finite Lebesque measure. It is not difficult to generalize it to the case of punctured disks and we state a version where the holomorphic curve lies outside the boundary divisor (see [Voj09, §29] for a precise entire curve version more amenable to arithmetic geometry).

Theorem 3.7 (Tautological Inequality for the punctured disk). *Let X be a projective manifold with a Fubini-Study metric ω_{FS} and let D be a simple normal crossing divisor on X. Let $\gamma : \mathbb{D}^\circ \to X \setminus D$ be a holomorphic map. Then $T_{(\mathbb{P}\gamma')^*O(1)}(r) \leq O(\log^+ T_{\gamma^*\omega_{FS}}(r)) + O(\log r)\ \|$.* □

Note that Theorem A derives directly from this theorem since one can choose a hermitian metric h dominating the possibly degenerate h_α. (Hence the proximity function associated to the former dominates that of the latter and so up to addition of $O(\log r)$ so does the charateristic functions.)

4. Big Picard theorem via the lemma on the logarthmic derivative

In a similar vein as that in establishing the fundamental vanishing theorem for (symmetric or jet) differentials vanishing on ample divisors, pioneered by Green-Griffiths [GG80] and completed by Siu-Yeung [SY97] and Demailly [Dem97b] via the logarithmic derivative lemma, we give another proof of Theorem A in this section. Theorem B is then a corollary as before from which follows Corollary C. Though standard fair via our methods, the latter is deduced here directly via a theorem of Siu connected to his deep result concerning the structure of currents at the end of this section.

4.1. **Preliminary in Nevannlina theory.** Let (X,ω) be a compact Kähler manifold. Consider a holomorphic map $\gamma:\mathbb{D}^*\to X$. We identify $\mathbb{D}^*$ with $\mathbb{D}^\circ := \{z\in\mathbb{C};\ |z|\geq 1\}$ via $z\mapsto \frac{1}{z}$ as before. We can extend γ to a $\gamma:\mathbb{D}^*\cup\mathbb{D}^\circ\to X$ by setting $\gamma(z)=\gamma(\frac{1}{z})$ for $z\in\mathbb{D}^\circ$.

Fix any $r_0>1$. Recall that the *characteristic function* in (3.1) is defined by

$$T_{\gamma^*\omega}(r) := \int_{r_0}^{r}\frac{d\rho}{\rho}\int_{\mathbb{D}_{r_0,\rho}}\gamma^*\omega.$$

Let us first state a couple of useful inequalities.

Lemma 4.1. *Write* $\log^+ x := \max(\log x, 0)$. *Then*

$$\log^+(\sum_{i=1}^{N} x_i) \leq \sum_{i=1}^{N}\log^+ x_i + \log N,\quad \log^+\prod_{i=1}^{N} x_i \leq \sum_{i=1}^{N}\log^+ x_i \quad \textit{for } x_i\geq 0. \tag{4.1}$$

□

The following lemma is well-known (see *e.g.* [Dem97b, Lemme 1.6]).

Lemma 4.2. *Let X be a projective manifold equipped with a hermitian metric ω and let $u: X\to\mathbb{P}^1$ be a rational function. Then for any holomorphic map $\gamma:\mathbb{D}^\circ\to X$, one has*

$$T_{(u\circ\gamma)^*\omega_{FS}}(r) \leq CT_{\gamma^*\omega}(r) + O(1) \tag{4.2}$$

where ω_{FS} is the Fubini-Study metric for $\mathbb{P}^1$. □

The following logarithmic derivative lemma for the punctured disk, see e.g. [Nog81][Lemma 2.12], is crucial in our proof.

Lemma 4.3. *Let $u:\mathbb{D}^\circ\to\mathbb{P}^1$ be any meromorphic function. Then for any $k\geq 1$, we have*

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{u^{(1)}(re^{i\theta})}{u(re^{i\theta})}\right|d\theta \leq C(\log^+ T_{u^*\omega_{FS}}(r) + \log r) \quad \|, \tag{4.3}$$

for some constant $C>0$ which does not depend on r. Here the symbol $\|$ means that the inequality holds outside a Borel subset of $(r_0,+\infty)$ of finite Lebesgue measure. □

We also need an elementary lemma (due to E. Borel), called the calculus lemma.

Lemma 4.4. *Let $\phi(r)\geq 0 (r\geq r_0\geq 0)$ be a monotone increasing function. For every $\delta>0$,*

$$\frac{d}{dr}\phi(r) \leq \phi(r)^{1+\delta} \quad \|. \tag{4.4}$$

□

4.2. **Criterion for big Picard theorem.** Now we are ready to give a new proof of Theorem A.

Proof. Via the isomorphism $\mathbb{D}^\circ \xrightarrow{\sim} \mathbb{D}^*$ by setting $z\mapsto\frac{1}{z}$ as before, we assume that $\gamma:\mathbb{D}^*\to X-D$ is a holomorphic map from $\mathbb{D}^\circ$ to $X-D$. We take a finite affine covering $\{U_\alpha\}_{\alpha\in I}$ of X and rational functions $(x_{\alpha 1},\ldots,x_{\alpha n})$ on X which are holomorphic on U_α so that

$$dx_{\alpha 1}\wedge\cdots\wedge dx_{\alpha n}\neq 0 \ \text{ on } U_\alpha$$
$$D\cap U_\alpha = (x_{\alpha,s(\alpha)+1}\cdots x_{\alpha n}=0)$$

Hence

$$(e_{\alpha 1},\ldots,e_{\alpha n}) := \left(\frac{\partial}{\partial x_{\alpha 1}},\ldots,\frac{\partial}{\partial x_{\alpha s(\alpha)}}, x_{\alpha,s(\alpha)+1}\frac{\partial}{\partial x_{\alpha,s(\alpha)+1}},\ldots,x_{\alpha n}\frac{\partial}{\partial x_{\alpha n}}\right) \tag{4.5}$$

is a basis for $T_X(-\log D)|_{U_\alpha}$. Write

$$(\gamma_{\alpha 1}(z),\ldots,\gamma_{\alpha n}(z)) := (x_{\alpha 1}\circ\gamma,\ldots,x_{\alpha n}\circ\gamma)$$

so that $\gamma_{\alpha j}:\mathbb{D}^\circ\to\mathbb{P}^1$ is a meromorphic function over $\mathbb{D}^\circ$ for any α and j. With respect to the trivialization of $T_X(-\log D)$ induced by the basis (4.5), $\gamma'(z)$ can be written as

$$\gamma'(z)=\gamma'_{\alpha 1}(z)e_{\alpha 1}+\cdots+\gamma'_{\alpha s(\alpha)}(z)e_{\alpha s(\alpha)}+(\log\gamma_{\alpha,s(\alpha)+1})'(z)e_{\alpha,s(\alpha)+1}+\cdots+(\log\gamma_{\alpha n})'(z)e_{\alpha n}$$

over U_α. Let $\{\rho_\alpha\}_{\alpha\in I}$ be a partition of unity subordinated to $\{U_\alpha\}_{\alpha\in I}$.

Since h is Finsler pseudometric for $T_X(-\log D)$ which is continuous and locally bounded from above by Definition 2.15, and I is a finite set, there is a constant $C>0$ so that

$$\rho_\alpha\circ\gamma\cdot|\gamma'(z)|_h^2\leq C\Big(\sum_{j=1}^{s(\alpha)}\rho_\alpha\circ\gamma\cdot|\gamma'_{\alpha j}(z)|^2+\sum_{i=s(\alpha)+1}^{n}|(\log\gamma_{\alpha i})'(z)|^2\Big)\quad\forall z\in\mathbb{D}^* \tag{4.6}$$

for any α. Hence

$$\begin{aligned}
T_{\gamma^*\omega}(r) &:= \int_{r_0}^{r}\frac{d\rho}{\rho}\int_{\mathbb{D}_{r_0,\rho}}\gamma^*\omega \overset{(1.1)}{\leq} \int_{r_0}^{r}\frac{d\rho}{\rho}\int_{\mathbb{D}_{r_0,\rho}}dd^c\log|\gamma'|_h^2\\
&\overset{(3.3)}{\leq} \frac{1}{2\pi}\int_0^{2\pi}\log|\gamma'(re^{i\theta})|_h d\theta+O(\log r)\\
&\leq \frac{1}{2\pi}\int_0^{2\pi}\log^+\sum_\alpha|\rho_\alpha\circ\gamma\cdot\gamma'(re^{i\theta})|_h d\theta+O(\log r)\\
&\overset{(4.1)}{\leq} \sum_\alpha\frac{1}{2\pi}\int_0^{2\pi}\log^+|\rho_\alpha\circ\gamma\cdot\gamma'(re^{i\theta})|_h d\theta+O(\log r)\\
&\overset{(4.6)+(4.1)}{\leq} \sum_\alpha\sum_{i=s(\alpha)+1}^{n}\frac{1}{2\pi}\int_0^{2\pi}\log^+|(\log\gamma_{\alpha i})'(re^{i\theta})|d\theta\\
&\quad+\sum_\alpha\sum_{j=1}^{s(\alpha)}\frac{1}{2\pi}\int_0^{2\pi}\log^+|\rho_\alpha\circ\gamma\cdot\gamma'_{\alpha j}(re^{i\theta})|d\theta+O(\log r)\\
&\overset{(4.3)}{\leq} C_1\sum_\alpha\sum_{i=s(\alpha)+1}^{n}\big(\log^+T_{\gamma_{\alpha i},\omega_{FS}}(r)+\log r\big)\\
&\quad+\sum_\alpha\sum_{j=1}^{s(\alpha)}\frac{1}{2\pi}\int_0^{2\pi}\log^+|\rho_\alpha\circ\gamma\cdot\gamma'_{\alpha j}(re^{i\theta})|d\theta+O(\log r)\quad\|\\
&\overset{(4.2)}{\leq} C_2(\log^+T_{\gamma^*\omega}(r)+\log r)+\sum_\alpha\sum_{j=1}^{s(\alpha)}\frac{1}{2\pi}\int_0^{2\pi}\log^+|\rho_\alpha\circ\gamma\cdot\gamma'_{\alpha j}(re^{i\theta})|d\theta\quad\|.
\end{aligned}\tag{4.7}$$

Here C_1 and C_2 are two positive constants which do not depend on r.

Claim. *For any $\alpha \in I$, any $j \in \{1, \dots, s(\alpha)\}$, one has*

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+|\rho_\alpha \circ \gamma \cdot \gamma'_{\alpha j}(re^{i\theta})|d\theta \leq C_3(\log^+ T_{\gamma^*\omega}(r) + \log r) + O(1) \quad \| \tag{4.8}$$

for a positive constant C_3 which does not depend on r.

Proof of Claim 4.2. The proof of the claim is borrowed from [NW14, eq.(4.7.2)]. Pick $C > 0$ so that $\rho_\alpha^2\sqrt{-1}dx_{\alpha j} \wedge d\bar{x}_{\alpha j} \leq C\omega$. Write $\gamma^*\omega := \sqrt{-1}B(t)dt \wedge d\bar{t}$. Then

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} \log^+|\rho_\alpha \circ \gamma \cdot \gamma'_{\alpha j}(re^{i\theta})|d\theta = \frac{1}{4\pi}\int_0^{2\pi} \log^+(|\rho_\alpha^2 \circ \gamma| \cdot |\gamma'_{\alpha j}(re^{i\theta})|^2)d\theta \\
&\leq \frac{1}{4\pi}\int_0^{2\pi} \log^+ B(re^{i\theta})d\theta + O(1) \leq \frac{1}{4\pi}\int_0^{2\pi} \log(1 + B(re^{i\theta}))d\theta + O(1) \\
&\leq \frac{1}{2}\log(1 + \frac{1}{2\pi}\int_0^{2\pi} B(re^{i\theta})d\theta) + O(1) = \frac{1}{2}\log(1 + \frac{1}{2\pi r}\frac{d}{dr}\int_{\mathbb{D}_{r_0,r}} rBdrd\theta) + O(1) \\
&= \frac{1}{2}\log(1 + \frac{1}{2\pi r}\frac{d}{dr}\int_{\mathbb{D}_{r_0,r}} \gamma^*\omega) + O(1) \\
&\overset{(4.4)}{\leq} \frac{1}{2}\log(1 + \frac{1}{2\pi r}(\int_{\mathbb{D}_{r_0,r}} \gamma^*\omega)^{1+\delta}) + O(1) \quad \| \\
&= \frac{1}{2}\log(1 + \frac{r^\delta}{2\pi}(\frac{d}{dr}T_{\gamma^*\omega}(r))^{1+\delta}) + O(1) \quad \| \\
&\overset{(4.4)}{\leq} \frac{1}{2}\log(1 + \frac{r^\delta}{2\pi}(T_{\gamma^*\omega}(r))^{(1+\delta)^2}) + O(1) \quad \| \\
&\leq 4\log^+ T_{\gamma^*\omega}(r) + \delta\log r + O(1) \quad \|.
\end{aligned}$$

Here we pick $0 < \delta < 1$ and the last inequality follows. The claim is proved. □

Putting (4.8) to (4.7), one obtains

$$T_{\gamma^*\omega}(r) \leq C(\log^+ T_{\gamma^*\omega}(r) + \log r) + O(1) \quad \|$$

for some positive constant C. Hence $T_{\gamma^*\omega}(r) = O(\log r)$. We apply Lemma 3.2 to conclude that γ extends to the ∞. □

Corollary C is now a direct consequence of the deep extension theorem of meromorphic maps by Siu in [Siu75] (see also [Sib85] for a more general result). The *meromorphic map* there is defined in the sense of Remmert, which generalizes the usual notion when the target space is algebraic.

Proof of Corollary C. By [Siu75, Theorem 1], any meromorphic map extends across a subvariety of codimension 2. It thus suffices to prove the extension theorem for the case of a holomorphic map of the form $\gamma : \mathbb{D}^r \times \mathbb{D}^* \to U$. For any $z \in \mathbb{D}^r$, the holomorphic map $\gamma|_{\{z\}\times\mathbb{D}^*} : \{z\} \times \mathbb{D}^* \to U$ can be extended to a holomorphic map from $\{z\} \times \mathbb{D}$ to Y. It then follows from [Siu75, p.442, $(*)$] that γ extends to a meromorphic map $\overline{\gamma} : \mathbb{D}^{r+1} \dashrightarrow Y$. □

5. Algebraic hyperbolicity for moduli spaces of polarized manifolds

In this section we will prove algebraic hyperbolicity of moduli of polarized manifolds. This question was addressed to the first named author by Erwan Rousseau in February 2019 at CIRM.

Proof of Theorem D. Since the moduli map of the polarized family $(f : V \to U, \mathcal{L}) \in \mathcal{M}_h(U)$ is quasi-finite, the family f is of maximal variation. We first take a log compactification (Y, D) of the base U. Write $X_0 := Y$, $D_0 := D$, and $U_0 := U$. Let $U^\circ \subset U$ be the Zariski open set in Theorem 2.9. Let $Z_1, \ldots, Z_m$ be all irreducible components of $X_0 - U_0^\circ$ which is not contained in D_0. Let $\mu_i : X_i \to Z_i$ be a desingularization so that $D_i := \mu_i^{-1}(D_0)$ is of simple normal crossing. For the quasi-projective manifold $U_i := X_i - D_i$, the moduli map of the new family $(f_i : V \times_U U_i \to U_i, \mathcal{L}|_{V\times_U U_i})$ is generically finite, and thus f_i is also of maximal variation. Iterating this construction using Theorem 2.9 and applying Theorem 2.19 we construct log pairs $\{(X_j, D_j)\}_{j=0,\ldots,N}$ so that the following holds.

(1) There are morphisms $\mu_i : X_i \to Y$ with $\mu_i^{-1}(D) = D_i$, so that each $\mu_i : X_i \to \mu_i(X_i)$ is a birational morphism.
(2) There are smooth Finsler pseudometrics $h_{i1}, \ldots, h_{in}$ for $T_{X_i}(-\log D_i)$.
(3) $\mu_i|_{U_i^\circ} : U_i^\circ \to \mu_i(U_i^\circ)$ is an isomorphism.
(4) There are smooth Kähler metrics $\omega_{i1}, \ldots, \omega_{in}$ on X_i such that for any curve $\gamma : C \to U_i$ with C an open set of $\mathbb{C}$ and $\gamma(C) \cap U_i^\circ \neq 0$, there exists some h_{ij} so that $|\gamma'(t)|^2_{h_{ij}} \not\equiv 0$, and

$$\mathrm{dd^c} \log |\gamma'|^2_{h_{ij}} \geq \gamma^* \omega_{ij}. \tag{5.1}$$

(5) For any $i \in \{0, \ldots, N\}$, either $\mu_i(U_i) - \mu_i(U_i^\circ)$ is zero dimensional, or there exists $I \subset \{0, \ldots, N\}$ so that

$$\mu_i(U_i) - \mu_i(U_i^\circ) \subset \cup_{j\in I} \mu_j(X_j)$$

For any irreducible and reduced curve $C \subset Y$ with $C \not\subset D$. By the above construction, there is some log (X_i, D_i) so that $C \subset \mu_i(X_i)$ and $C \cap \mu_i(U_i^\circ) \neq \varnothing$. By Item 3, C is not contained in the exceptional locus of μ_i, and let $C_i \subset X_i$ be the strict transform of C under μ_i. Denote by $\nu_i : \tilde{C}_i \to C_i \subset X_i$ the normalization of C_i, and set $P_i := (\mu_i \circ \nu_i)^{-1}(D) = \mu_i^{-1}(D_i)$. Then one has

$$d\nu_i : T_{\tilde{C}_i}(-\log P_i) \to \nu_i^* T_{X_i}(-\log D_i).$$

By Item 4, there is a Finsler pseudometric h_{ij} for $T_{X_i}(-\log D_i)$ so that (5.1) holds. Consider $\tilde{h}_i := \nu_i^* h_{ij}$, which is a complex semi-norm over $T_{\tilde{C}_i}(-\log P_i)$. By (5.1), there is a Kähler metric ω_{ij} on X_i so that the curvature current

$$\frac{\sqrt{-1}}{2\pi} \Theta_{\tilde{h}_i^{-1}}(K_{\tilde{C}_i}(\log P_i)) \geq \nu_i^* \omega_{ij}$$

Since $\mu_i \circ \nu_i : \tilde{C}_i \to C$ is the normalization of C, one has

$$2g(\tilde{C}_i) - 2 + i(C, D) = \int_{\tilde{C}_i} \frac{\sqrt{-1}}{2\pi} \Theta_{\tilde{h}_i^{-1}}(K_{\tilde{C}_i}(\log P_i)) \geq \int_{\tilde{C}_i} \nu_i^* \omega_{ij}$$

Fix a Kähler metric ω_Y on Y. Then there is a constant $\varepsilon > 0$ so that $\omega_{ij} \geq \varepsilon \mu_i^* \omega_Y$ for any $i = 0, \ldots, N$ and $j = 1, \ldots, n$. We thus have

$$2g(\tilde{C}_i) - 2 + i(C, D) \geq \varepsilon \int_{\tilde{C}_i} \mu_i^* \omega_Y = \varepsilon \deg_{\omega_Y} C$$

This shows the algebraic hyperbolicity of the base U. □

Remark 5.1. The algebraic hyperbolicity in Theorem D generalizes the Arakelov-type inequalities in [MVZ06] by Möller, Viehweg and the fourth named author, as well as the weak boundedness of moduli stacks of canonically polarized manifolds in [KL10] by Kovács-Lieblich. In [MVZ06, Theorem 0.3], the authors obtained Arakelov-type inequalities with *sharp bounds* for semistable families of projective manifolds with semi-ample canonical sheaf over $\mathbb{P}^1$. In [KL10, Definition 2.4] the authors introduced the notion of *weak boundedness* for quasi-projective varieties (which is weaker than the notion of algebraic hyperbolicity), and they proved that moduli stacks of canonically polarized manifolds are weakly bounded.

References

[BB64] W. L. Baily, Jr. and A. Borel. "On the compactification of arithmetically defined quotients of bounded symmetric domains." *Bull. Amer. Math. Soc.* (1964) vol. 70: 588–593. URL http://dx.doi.org/10.1090/S0002-9904-1964-11207-6. ↑ 3

[BBT18] Benjamin Bakker, Yohan Brunebarbe, and Jacob Tsimerman. "o-minimal gaga and a conjecture of griffiths." (2018). ArXiv:1811.12230v2. ↑ 3

[Bor97] Emile Borel. "Sur les zéros des fonctions entières." *Acta Math.* (1897) vol. 20 (1): 357–396. URL http://dx.doi.org/10.1007/BF02418037. ↑ 3

[Bor72] Armand Borel. "Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem." *J. Differential Geometry* (1972) vol. 6: 543–560. URL http://projecteuclid.org/euclid.jdg/1214430642. Collection of articles dedicated to S. S. Chern and D. C. Spencer on their sixtieth birthdays. ↑ 3

[CDK95] Eduardo Cattani, Pierre Deligne, and Aroldo Kaplan. "On the locus of Hodge classes." *J. Amer. Math. Soc.* (1995) vol. 8 (2): 483–506. URL http://dx.doi.org/10.2307/2152824. ↑ 3

[Che04] Xi Chen. "On algebraic hyperbolicity of log varieties." *Commun. Contemp. Math.* (2004) vol. 6 (4): 513–559. URL https://doi.org/10.1142/S0219199704001422. ↑ 5

[CKS86] Eduardo Cattani, Aroldo Kaplan, and Wilfried Schmid. "Degeneration of Hodge structures." *Ann. of Math. (2)* (1986) vol. 123 (3): 457–535. URL http://dx.doi.org/10.2307/1971333. ↑ 4, 7, 13

[Del87] P. Deligne. "Un théorème de finitude pour la monodromie." In "Discrete groups in geometry and analysis (New Haven, Conn., 1984)," vol. 67 of *Progr. Math.*, 1–19, (Birkhäuser Boston, Boston, MA1987). URL http://dx.doi.org/10.1007/978-1-4899-6664-3_1. ↑ 4

[Dem97a] Jean-Pierre Demailly. "Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials." In "Algebraic geometry—Santa Cruz 1995," vol. 62 of *Proc. Sympos. Pure Math.*, 285–360, (Amer. Math. Soc., Providence, RI1997). URL http://dx.doi.org/10.1090/pspum/062.2/1492539. ↑ 5

[Dem97b] ———. "Variétés hyperboliques et équations différentielles algébriques." *Gaz. Math.* (1997) (73): 3–23. ↑ 2, 15, 17, 18

[Den18a] Ya Deng. "Kobayashi hyperbolicity of moduli spaces of minimal projective manifolds of general type (with the appendix by Dan Abramovich)." *arXiv e-prints* (2018) arXiv:1806.01666. ↑ 2

[Den18b] ———. "Pseudo kobayashi hyperbolicity of base spaces of families of minimal projective manifolds with maximal variation." (2018). ArXiv:1809.05891. ↑ 9

[Den19a] ———. "Big Picard theorem for moduli spaces of polarized manifolds." *arXiv e-prints* (2019) arXiv:1912.11442. ↑ 6

[Den19b] ———. "Hyperbolicity of coarse moduli spaces and isotriviality for certain families." *arXiv e-prints* (2019) arXiv:1908.08372. ↑ 5

[Den20] ———. "Big Picard theorem and algebraic hyperbolicity for varieties admitting a variation of Hodge structures." *arXiv e-prints* (2020) arXiv:2001.04426. ↑ 4

[GG80] Mark Green and Phillip Griffiths. "Two applications of algebraic geometry to entire holomorphic mappings." In "The Chern Symposium 1979 (Proc. Internat. Sympos., Berkeley, Calif., 1979)," , (Springer, New York-Berlin1980) 41–74. ↑ 17

[GGLR19] Mark Green, Phillip Griffiths, Radu Laza, and Colleen Robles. "Period mappings and properties of the hodge line bundle." (2019). ArXiv:1708.09523v2. ↑ 3

[GK73] Phillip Griffiths and James King. "Nevanlinna theory and holomorphic mappings between algebraic varieties." *Acta Math.* (1973) vol. 130: 145–220. URL http://dx.doi.org/10.1007/BF02392265. ↑ 2, 4, 15

[Gri68a] Phillip A. Griffiths. "Periods of integrals on algebraic manifolds. I. Construction and properties of the modular varieties." *Amer. J. Math.* (1968) vol. 90: 568–626. URL http://dx.doi.org/10.2307/2373545. ↑ 4

[Gri68b] ———. "Periods of integrals on algebraic manifolds. II. Local study of the period mapping." *Amer. J. Math.* (1968) vol. 90: 805–865. URL http://dx.doi.org/10.2307/2373485. ↑ 4

[Gri70a] ———. "Periods of integrals on algebraic manifolds. III. Some global differential-geometric properties of the period mapping." *Inst. Hautes Études Sci. Publ. Math.* (1970) (38): 125–180. URL http://www.numdam.org/item?id=PMIHES_1970__38__125_0. ↑ 4

[Gri70b] ———. "Periods of integrals on algebraic manifolds: Summary of main results and discussion of open problems." *Bull. Amer. Math. Soc.* (1970) vol. 76: 228–296. URL http://dx.doi.org/10.1090/S0002-9904-1970-12444-2. ↑ 3

[JK18] Ariyan Javanpeykar and Robert A. Kucharczyk. "Algebraicity of analytic maps to a hyperbolic variety." (2018). ArXiv:1806.09338. ↑ 3

[Kaw85] Yujiro Kawamata. "Minimal models and the Kodaira dimension of algebraic fiber spaces." *J. Reine Angew. Math.* (1985) vol. 363: 1–46. URL http://dx.doi.org/10.1515/crll.1985.363.1. ↑ 7

[KL10] Sándor J. Kovács and Max Lieblich. "Boundedness of families of canonically polarized manifolds: a higher dimensional analogue of Shafarevich's conjecture." *Ann. of Math. (2)* (2010) vol. 172 (3): 1719–1748. URL http://dx.doi.org/10.4007/annals.2010.172.1719. ↑ 6, 22

[Kob98] Shoshichi Kobayashi. *Hyperbolic complex spaces*, vol. 318 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, (Springer-Verlag, Berlin1998). URL http://dx.doi.org/10.1007/978-3-662-03582-5. ↑ 2

[Kol13] János Kollár. "Moduli of varieties of general type." In "Handbook of moduli. Vol. II," vol. 25 of *Adv. Lect. Math. (ALM)*, 131–157, (Int. Press, Somerville, MA2013). ↑ 3

[Kov03] Sándor J. Kovács. "Families of varieties of general type: the Shafarevich conjecture and related problems." In "Higher dimensional varieties and rational points (Budapest, 2001)," vol. 12 of *Bolyai Soc. Math. Stud.*, 133–167, (Springer, Berlin2003). URL http://dx.doi.org/10.1007/978-3-662-05123-8_6. ↑ 2

[Lan87] Serge Lang. *Introduction to complex hyperbolic spaces*, (Springer-Verlag, New York1987). URL http://dx.doi.org/10.1007/978-1-4757-1945-1. ↑ 2

[LSZ] Xin Lu, Ruiran Sun, and Kang Zuo. "Hyperbolicity of moduli space of polarized manifolds with good minimal models and kodaira dimension one." In preparation. ↑ 10

[LSZ19] Steven Lu, Ruiran Sun, and Kang Zuo. "Nevanlinna Theory on Moduli Space and the big Picard Theorem." *arXiv e-prints* (2019) arXiv:1911.02973. ↑ 6

[Lu91] Steven Shin-Yi Lu. "On meromorphic maps into varieties of log-general type." In "Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989)," vol. 52 of *Proc. Sympos. Pure Math.*, 305–333, (Amer. Math. Soc., Providence, RI1991). ↑ 12, 13

[MVZ06] Martin Möller, Eckart Viehweg, and Kang Zuo. "Special families of curves, of abelian varieties, and of certain minimal manifolds over curves." In "Global aspects of complex geometry," 417–450, (Springer, Berlin2006). URL http://dx.doi.org/10.1007/3-540-35480-8_11. ↑ 6, 22

[Nog81] Junjiro Noguchi. "Lemma on logarithmic derivatives and holomorphic curves in algebraic varieties." *Nagoya Math. J.* (1981) vol. 83: 213–233. URL http://projecteuclid.org/euclid.nmj/1118786486. ↑ 18

[NW14] Junjiro Noguchi and Jörg Winkelmann. *Nevanlinna theory in several complex variables and Diophantine approximation*, vol. 350 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, (Springer, Tokyo2014). URL http://dx.doi.org/10.1007/978-4-431-54571-2. ↑ 15, 20

[PR07] Gianluca Pacienza and Erwan Rousseau. "On the logarithmic Kobayashi conjecture." *J. Reine Angew. Math.* (2007) vol. 611: 221–235. URL https://doi.org/10.1515/CRELLE.2007.080. ↑ 5

[PTW18] Mihnea Popa, Behrouz Taji, and Lei Wu. “Brody hyperbolicity of base spaces of certain families of varieties.” (2018). ArXiv:1801.05898. ↑ 9, 13

[Sch73] Wilfried Schmid. “Variation of Hodge structure: the singularities of the period mapping.” *Invent. Math.* (1973) vol. 22: 211–319. URL http://dx.doi.org/10.1007/BF01389674. ↑ 4, 7, 14

[Sib85] Nessim Sibony. “Quelques problèmes de prolongement de courants en analyse complexe.” *Duke Math. J.* (1985) vol. 52: 157–197. URL https://doi.org/10.1215/S0012-7094-85-05210-X. ↑ 20

[Siu75] Yum Tong Siu. “Extension of meromorphic maps into Kähler manifolds.” *Ann. of Math. (2)* (1975) vol. 102 (3): 421–462. URL http://dx.doi.org/10.2307/1971038. ↑ 20

[Som73] Andrew J. Sommese. “Some algebraic properties of the image of the period mapping.” *Rice Univ. Studies* (1973) vol. 59 (2): 123–128. Complex analysis, 1972 (Proc. Conf., Rice Univ., Houston, Tex., 1972),Vol. II: Analysis on singularities. ↑ 3

[SY97] Yum-Tong Siu and Sai-Kee Yeung. “Defects for ample divisors of abelian varieties, Schwarz lemma, and hyperbolic hypersurfaces of low degrees.” *Amer. J. Math.* (1997) vol. 119 (5): 1139–1172. URL http://muse.jhu.edu/journals/american_journal_of_mathematics/v119/119.5siu.pdf. ↑ 2, 17

[TY15] Wing-Keung To and Sai-Kee Yeung. “Finsler metrics and Kobayashi hyperbolicity of the moduli spaces of canonically polarized manifolds.” *Ann. of Math. (2)* (2015) vol. 181 (2): 547–586. URL http://dx.doi.org/10.4007/annals.2015.181.2.3. ↑ 2

[TY18] ———. “Augmented Weil-Petersson metrics on moduli spaces of polarized Ricci-flat Kähler manifolds and orbifolds.” *Asian J. Math.* (2018) vol. 22 (4): 705–727. URL http://dx.doi.org/10.4310/AJM.2018.v22.n4.a6. ↑ 2

[Vie01] Eckart Viehweg. “Positivity of direct image sheaves and applications to families of higher dimensional manifolds.” In “School on Vanishing Theorems and Effective Results in Algebraic Geometry (Trieste, 2000),” vol. 6 of *ICTP Lect. Notes*, 249–284, (Abdus Salam Int. Cent. Theoret. Phys., Trieste2001). ↑ 2

[Voj11] Paul Vojta. “Diophantine approximation and Nevanlinna theory.” In “Arithmetic geometry,” vol. 2009 of *Lecture Notes in Math.*, 111–224, (Springer, Berlin2011). URL http://dx.doi.org/10.1007/978-3-642-15945-9_3. ↑ 17

[VZ02] Eckart Viehweg and Kang Zuo. “Base spaces of non-isotrivial families of smooth minimal models.” In “Complex geometry (Göttingen, 2000),” 279–328, (Springer, Berlin2002). ↑ 6, 7, 8

[VZ03] ———. “On the Brody hyperbolicity of moduli spaces for canonically polarized manifolds.” *Duke Math. J.* (2003) vol. 118 (1): 103–150. URL http://dx.doi.org/10.1215/S0012-7094-03-11815-3. ↑ 2, 4, 6, 7, 8, 9, 11, 12, 13, 14

[Zuc84] Steven Zucker. “Degeneration of Hodge bundles (after Steenbrink).” In “Topics in transcendental algebraic geometry (Princeton, N.J., 1981/1982),” vol. 106 of *Ann. of Math. Stud.*, 121–141, (Princeton Univ. Press, Princeton, NJ1984). ↑ 7

[Zuo00] Kang Zuo. “On the negativity of kernels of Kodaira-Spencer maps on Hodge bundles and applications.” vol. 4, 279–301 (2000). URL http://dx.doi.org/10.4310/AJM.2000.v4.n1.a17. Kodaira’s issue. ↑ 4, 9, 14

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